# What is the Navascués-Pironio-Acín hierarchy, and what is a semidefinite program

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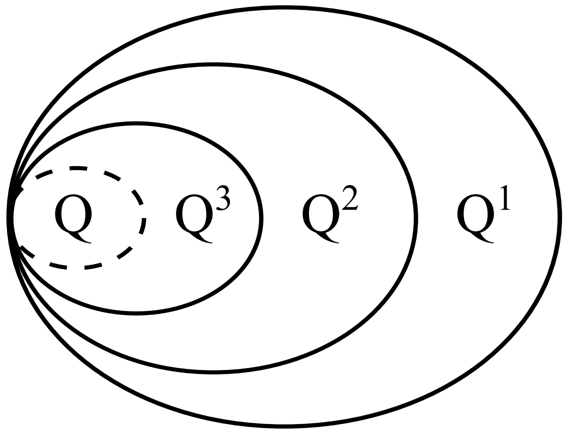
in the lecture

Quantum correlations and generalized probabilistic theories: an introduction.




## Contents

# 1 Introduction

Miguel Navascués, Stefano Pironio and Antonio Acín described in their paper *A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations* (Navascués et al., 2008) the problem of characterizing the correlations that arise when local measurements are performed on separate quantum systems. They introduced an infinity hierarchy of necessary conditions that has to be satisfied by any set of quantum correlations. A major role plays semidefinite programming (SDP) to test each of this steps.
The whole paper discusses the results mainly in the bipartite case. Alice and Bob represents the standard bipartite scenario in quantum information science. They share a quantum system in a joint state $\rho$, are separated and non-communicating. Both of them can make a measurement on the shared system. Figure 1.1 shows a sketch of the described bipartite case with the idea of a black box. Alice inputs a measurement X in the box and gets a measurement outcome $a \in X$. The same way Bob is carrying out his measurement Y with an output $b \in Y$, thus we can characterize the whole system by the joint detection probability $P(a, b)$, where the set $P = \{P(a, b)\}$ of all probabilities is called a *behavior*.
In the scenario described above there might be a non-trivial correlation between Alice's and Bob's outcomes $a$ and $b$. The main problem the Navascués-Pironio-Acín hierarchy tries to solve is the following:

**If a behavior $P$ is given, does there exist a quantum state $\rho$ and local measurements $X$ and $Y$ that can reproduce the outcome behavior $P$?**

We want to archive the understanding of the hierarchy by introducing in the following sections semidefinite programming (SDP), mainly based on Navascués et al. (2008) and Vandenberghe and Boyd (1996), then some fundamental definitions and theorems concerning the topic and at the end the hierarchy itself.

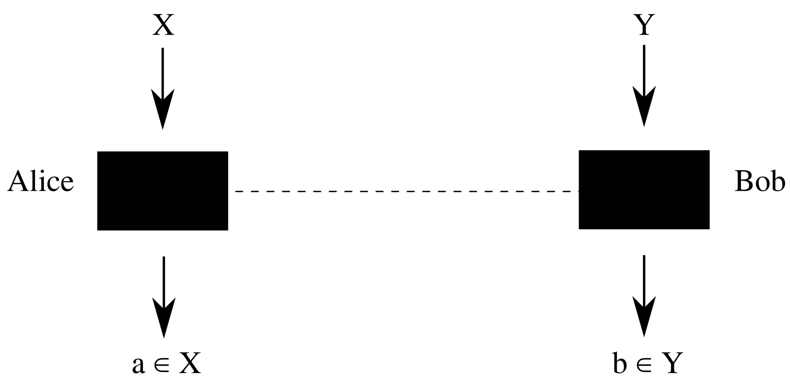



**Figure 1.1:** Bipartite scenario of a shared quantum system with two non-communicating parties. Alice inputs a measurement $X$ in the box and obtains a measurement output $a \in X$. Similarly, Bob with measurement input $Y$ and output $b \in Y$. The joint detection probabilities $P(a, b)$ characterize the whole system. (Navascués et al., 2008)

# 2 Semidefinite Programming

Semidefinite programming (SDP) is a class of convex optimization problems, where the problem variable is some $n \times n$ matrix and the extremum of the linear objective function is sought under the constraint of positive semidefinite matrices. This constraint yields sets which can sometimes be nonlinear and nonsmooth, but are always convex (Vandenberghe and Boyd, 1996).

The primal problem is given by

$$\begin{aligned} \text{maximize} \quad & \operatorname{tr}(GZ) \\ \text{subject to} \quad & \operatorname{tr}(F_i Z) = c_i, \quad i = 1, ..., p \\ & Z \succeq 0, \end{aligned} \tag{2.1}$$

where $Z, G$ and $F_i$ are $n \times n$ matrices and $c_i$ are scalars. The variable $Z$, which satisfies theses conditions, is called prime feasible (Navascués et al., 2008).

There exists a associated dual feasible variable, solving the problem

$$\begin{aligned} \text{minimize} \quad & c^{\mathrm{T}} x \\ \text{subject to} \quad & F(x) = \sum_{i=1}^{p} x_i F_i - G \succeq 0 \end{aligned}$$

for each prime feasible equations. Here the problem variable $x \in \mathbb{R}^n$ is a $p$-dimensional vector. The problem data are the p-dimensional vector $c$ and the positive semidefinite symmetric $n \times n$ matrices $F_i$. These conditions for the second feasible variable yield constraints for the extremal result of the primal variable, since it holds for the primal feasible point Z and the dual feasible point x:

$$c^{\mathrm{T}} x - \operatorname{tr}(GZ) = \sum_{i=1}^{p} \operatorname{tr}(ZF_i) x_i - \operatorname{tr}(GZ) = \operatorname{tr}(ZF(x)) \geq 0.$$

Hence, the optimal primal value $p^*$ and the optimal dual value $d^*$ fulfill

$$d^* \leq p^*.$$

Figure 2.1 shows an illustrative example for the case $x \in \mathbb{R}^2$ and $F_i \in \mathbb{R}^{7\times 7}$ to illustrate the basic idea of semidefinite programming (Vandenberghe and Boyd, 1996).

Although semidefinite programming is quite general (unifies standard problems like linear, quadratic programming), it has at worst polynomial complexity like the linear case. This means that there exist algorithms that output the value of the SDP up to an additive error $\epsilon$ in time that is polynomial in the program description size and in the order of $\ln(1/\epsilon)$ (Vandenberghe and Boyd, 1996).

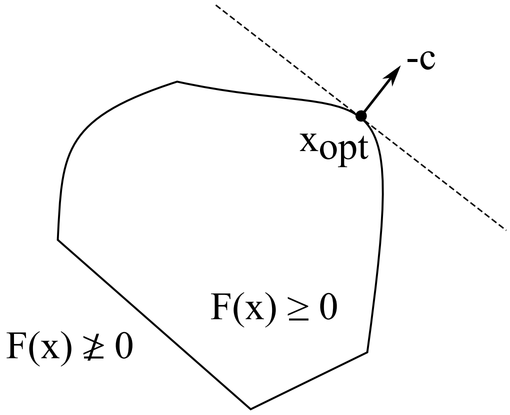



**Figure 2.1:** Example of a semidefinite programming problem with $x \in \mathbb{R}^2$ and $F_i \in \mathbb{R}^{7\times 7}$. The goal of SDP is basically to move as far as possible in $-c$ direction without leaving the convex feasible region $F(x) \geq 0$. $x_{opt}$ denotes the optimal point. (Vandenberghe and Boyd, 1996)

# 3 Basic Definitions and Theorems

To introduce the Navascués-Pironio-Acín hierarchy and to understand the mathematics behind the discussion in Navascués et al. (2008), we first have to look at some important definitions and theorems adapted from Navascués et al. (2008) (proofs are given in Navascués et al. (2008)).

### Definition: Quantum bahaviour

The behavior $P$ is a quantum behavior if there exists a pure (normalized) state $|\Psi\rangle$ in a Hilbert space $\mathcal{H}$, a set of measurement operators $\{E_a : a \in \tilde{A}\}$ for Alice and a set of measurement operators $\{E_b : b \in \tilde{B}\}$ for Bob such that for all $a \in \tilde{A}$ and $b \in \tilde{B}$

$$P(a) = \langle\Psi|E_a|\Psi\rangle$$
$$P(b) = \langle\Psi|E_b|\Psi\rangle$$
$$P(a,b) = \langle\Psi|E_aE_b|\Psi\rangle$$

with the measurement operators satisfying

1. $E_a^\dagger = E_a$ and $E_b^\dagger = E_b$ (hermiticity),
2. $E_aE_{\bar{a}} = \delta_{a\bar{a}}E_a$ if $X(a) = X(a')$ and $E_bE_{\bar{b}} = \delta_{b\bar{b}}E_b$ if $Y(b) = Y(b')$ (orthogonality)
3. $[E_a, E_b] = 0$ (commutativity),

whereas $\tilde{A}$ denotes the reduced output sets[1] $\tilde{X} = \{a : a \in X, a \neq a_X\}$ and $\tilde{A} = \cup_X \tilde{X}$. $a_X \in X$ is a selected output for each input $X$. Analogously it holds for the sets $\tilde{Y}$ and $\tilde{B}$. The set of all quantum behaviors will be denoted by Q.

[1] This reduction holds due to the single overdetermination because of the completeness $\sum_{a\in X} E_a = \mathbb{1}$ (i.e. $\sum_{a\in\tilde{A}} E_a = 1 - E_{a_X}$) and $\sum_{b\in Y} E_b = \mathbb{1}$.

**Definition and Theorem: Projectors, Operators, Set of Operators $\mathcal{S}_n$**

Let $\tilde{\mathcal{E}}$ denote the set of projectors of the above definition plus the identity ($\tilde{\mathcal{E}} = \mathbb{1} \cup \{E_a : a \in \tilde{A}\} \cup \{E_b : b \in \tilde{B}\}$).
Let $\mathcal{O} = \{O_1, \ldots, O_n\}$ be a set of $n$ operators, where each $O_i$ is a linear combination of products of projectors in $\tilde{\mathcal{E}}$. Thus $\mathcal{O}$ is a finite subset of the algebra generated by $\tilde{\mathcal{E}}$. Define $\mathcal{F}(\mathcal{O})$ as the set of all independent equalities of the form:

$$\sum_{ij} (F_k)_{ij} \langle \Psi | O_i^\dagger O_j | \Psi \rangle = g_k(P), \quad k = 1, \ldots, m, \tag{3.2}$$

which are satisfied by the operator $O_i$, where the coefficients $g_k(P)$ are linear functions of the probabilities $P(a, b)$:

$$g_k(P) = (g_k)_0 + \sum_{a,b} (g_k)_{ab} P(a, b) \tag{3.3}$$

and where $|\Psi\rangle$ is the state appearing in the above definition.
We can define $\mathcal{S}_n$ as follows:

$$\begin{aligned} &\mathcal{S}_0 = \{\mathbb{1}\}, \\ &\mathcal{S}_1 = \mathcal{S}_0 \cup \{E_a : a \in \tilde{A}\} \cup \{E_b : b \in \tilde{B}\}, \\ &\mathcal{S}_2 = \mathcal{S}_1 \cup \{E_a E_{a'} : a, a' \in \tilde{A}\} \cup \{E_b E_{b'} : b, b' \in \tilde{B}\} \cup \{E_a E_b : a \in \tilde{A}, b \in \tilde{B}\}, \\ &\mathcal{S}_3 = \ldots \end{aligned}$$

It is clear that $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots \subseteq \mathcal{S}_n \subseteq \ldots$, and that any operator $O_i \in \mathcal{O}$ can be written as a linear combination of operators in $\mathcal{S}_n$ for $n$ sufficiently large.

**Theorem: certificate $\Gamma$**

Let $\mathcal{O}$ be a set of operators and $\mathcal{F}(\mathcal{O})$ the set of equations of the form (3.2) satisfied by operators in $\mathcal{O}$. Then, a necessary condition for a behavior $P$ to be quantum is that there exists a complex Hermitian $n \times n$ positive semidefinite matrix $\Gamma \succeq 0$ whose entries $\Gamma_{ij}$ satisfy

$$\sum_{ij} (F_k)_{ij} \Gamma_{ij} = g_k(P), \quad k = 1, \ldots, m. \tag{3.4}$$

Moreover, if the coefficients $F_k$ and $g_k$ in (3.2) are real, we can take $\Gamma$ to be real as well.
We will call $\Gamma$ a *certificate* associated to $\mathcal{O}$ if it satisfied the above conditions.
The existence of a certificate $\Gamma$ can be cast as a SDP. Therefore we have to maximize $\lambda$, subject to equation (3.4) and $\Gamma - \lambda \mathbb{1} \succeq 0$. This problem can be put in the known form (2.1) of a SDP problem.

**Lemma: linear combinations of operators $\mathcal{O}$**

Let $\mathcal{O}$ and $\mathcal{O}'$ be two sets of operators such that every operator in $\mathcal{O}'$ is a linear combination of operators in $\mathcal{O}$. Then, the existence of such a certificate $\Gamma$ associated to $\mathcal{O}$, for a given behavior $P$, implies the existence of a certificate $\Gamma'$ associated to $\mathcal{O}'$.

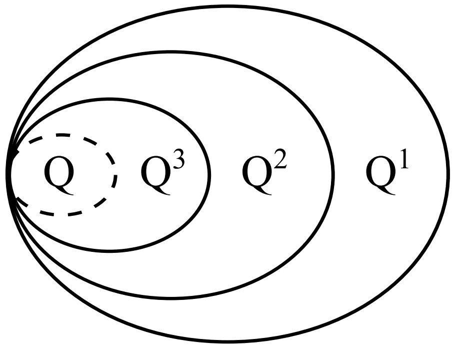



**Figure 4.1:** Geometrical interpretation of the Navascués-Pironio-Acín Hierarchy. $Q$ is the set of all quantum behaviors. If a given behavior $P$ satisfies a certificate $\Gamma^n$ of order $n$ it belongs to $Q^n$. (Navascués et al., 2007)

**Theorem: completness** $\lim_{n\to\infty} Q^n = Q$

Let $P$ be a behavior such that there exists a certificate $\Gamma^n$ of order $n$ for all $n \geq 1$. Then $P$ belongs to $Q$.

# 4 Navascués-Pironio-Acín Hierarchy

The Navascués-Pironio-Acín hierarchy is mainly based on the lemma of linear combinations of operators. If we define a certificate of order $n$ by $\Gamma^n$ as a certificate associated to the set of operators $\mathcal{S}_n$, the certificate is thus a $|\mathcal{S}_n| \times |\mathcal{S}_n|$ matrix. With our knowledge about SDP and equation (3.4), we can conclude that $\Gamma^n$ is a real positive semidefinite matrix. As mentioned before, it holds that $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots \subseteq \mathcal{S}_n \subseteq \ldots$, and thus the family of certificates $\Gamma^1$, $\Gamma^2$,... , $\Gamma^n$,... , represents a hierarchy of conditions satisfied by quantum probabilities, where each condition in the hierarchy is stronger than the previous ones.
If we now want to solve the question in the introduction part we have to carry out the following strategy (Navascués et al., 2008):
First we have to check if there exist a certificate $\Gamma^1$ of order 1 for our given behavior $P$. If there is no such certificate, we can conclude that the behavior $P$ is not a quantum behavior. If there exist such a behavior $\Gamma^1$ we check the existence of the next order certificate $\Gamma^2$. We can repeat the procedure with certificates of increasing order as long as the behavior $P$ satisfies the previous test.
Figure 4.1 shows a geometrical interpretation of the Navascués-Pironio-Acín Hierarchy. There $Q^n$ denotes the set of all behavior $P$ for which there exists a certificate of order $n$. For a non quantum behavior there exists an order $n$ above there does not exists a certificate $\Gamma^n$.
The theorem about completness shows that any non-quantum behavior $P$ has to fail at some step one of the conditions in the Navascués-Pironio-Acín Hierarchy. For the sake of completeness we want to mention that Navascués et al. (2008) introduced a cancellation condition, and showed that if this condition is fulfilled we do not have to prove higher order certificates to show the existence of a quantum behavior.

# References

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