Universität Heidelberg
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Single-shot quantum thermodynamics

Winter 2014/15
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http://www.mpmueller.net/lecture2.html

Exercises 1

(to hand in: October 28, 2014)

Problem 1 (Optimality in Landauer erasure): (7 points)

In the lecture, we have computed the average work that is needed in order to reset a mixed state on a two-level system ("Landauer erasure"), given that one level is raised in steps of ΔE to infinity, with alternating thermalization steps. The result was

$$\langle W\rangle \equiv \langle W\rangle(\Delta E) = \Delta E \sum_{k=0}^{\infty} \frac{e^{-\beta k \Delta E}}{1+e^{-\beta k \Delta E}},$$

where $\beta = 1/(k_B T)$ is the inverse temperature.
Show that $\langle W\rangle(\Delta E)$ is increasing in ΔE. This proves that sending ΔE to zero, resulting in the integral $\langle W\rangle = \int_0^\infty \frac{e^{-\beta E}}{1+e^{-\beta E}}\, dE = k_B T \log 2$, is optimal for this protocol.

Problem 2 (Von Neumann entropy and thermal states): (15 points)

Given a quantum system with Hamiltonian H, the thermal state (Gibbs state) $\rho_\beta(H)$ at inverse temperature $\beta = 1/(k_B T)$ is defined as

$$\rho_\beta(H) := \frac{e^{-\beta H}}{Z}, \qquad \text{where } Z := \mathrm{tr}(e^{-\beta H}).$$

The von Neumann entropy S and quantum relative entropy (Kullback-Leibler divergence) D are defined as

$$\begin{aligned} S(\rho) &:= -\mathrm{tr}(\rho \log \rho), \\ D(\rho \| \sigma) &:= \mathrm{tr}(\rho \log \rho - \rho \log \sigma) \end{aligned}$$

where ρ and σ are density matrices. In the following, we use the fact (cf. tutorial) that $D(\rho\|\sigma) \geq 0$, with equality if and only if $\rho = \sigma$. We also define the free energy of a state ρ (at inverse temperature $\beta \neq 0$) as

$$F_\beta(\rho) := \langle H\rangle_\rho - S(\rho)/\beta,$$

where $\langle H\rangle_\rho := \mathrm{tr}(\rho H)$.

(i) Show that if $H = H_A + H_B$ is a non-interacting Hamiltonian on a bipartite quantum system AB, then $\rho_\beta(H) = \rho_\beta(H_A) \otimes \rho_\beta(H_B)$.

(ii) Show that $S\left(\rho_\beta(H)\right) = \beta \langle H\rangle_{\rho_\beta(H)} + \log Z$.

(iii) Show that $\frac{1}{\beta} D(\rho \| \rho_\beta(H)) = F_\beta(\rho) - F_\beta(\rho_\beta(H))$.

(iv) Prove that $\rho_\beta(H)$ is the unique state that minimizes the free energy.

(v) Consider the expression $D(\rho_{AB} \| \rho_A \otimes \rho_B)$, and prove that von Neumann entropy satisfies the subadditivity property $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$, with equality if and only if $\rho_{AB} = \rho_A \otimes \rho_B$. (This is sometimes just written $S(AB) \leq S(A) + S(B)$.)

Problem 3 (Subadditivity of Rényi-0 entropy): (8 points)

The Rényi entropy of order 0 of a density matrix (quantum state) ρ is defined as

$$H_0(\rho) := \log \operatorname{rank}(\rho).$$

Show that it satisfies the *subadditivity property*: if ρ_{AB} is a quantum state on a bipartite quantum system AB with Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, then

$$H_0(\rho_{AB}) \leq H_0(\rho_A) + H_0(\rho_B)$$

(which is sometimes written $H_0(AB) \leq H_0(A) + H_0(B)$ if the state is clear).
Note: It may be helpful to first prove that

$$\operatorname{rank}(\rho) = \min\{\operatorname{tr} P \mid \operatorname{tr}(P\rho) = 1, \; P = P^\dagger = P^2\},$$

that is, the rank of a quantum state ρ is minimal rank of any orthogonal projector P that has full overlap $\operatorname{tr}(P\rho) = 1$ with ρ.