Universität Heidelberg
Institut für Theoretische Physik
Single-shot quantum thermodynamics

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http://www.mpmueller.net/lecture2.html

Exercises 2

(to hand in: November 11, 2014)

In the lecture we already heared about majorisation and saw a definition. The concept of majorisation is concerned with probability distributions and introduces an order on them. Writing $x \succ y$ means that y is "closer" to the uniform distribution than x or equivalently "more uniform" than x. On this exercise sheet x, y, z will be probability vectors of size n, thus they have non-negative components and all their elements sum up to one. To every x we define the vector $x^\downarrow$ wich has the the same elements as x but is ordered decreasingly $x_1^\downarrow \geq \ldots \geq x_n^\downarrow$. We say x *majorizes* y and write

$$x \succ y \quad \text{if} \quad \sum_{i=1}^{k} x_i^\downarrow \geq \sum_{i=1}^{k} y_i^\downarrow \quad \text{for} \quad k = 1, \ldots, n$$

This definition is easy to check if you have two given vectors but it is a bit abstract and not really clear why it captures the notion of "uniformness" or "non-uniformness". It is the aim of this exercise sheet to get a better understanding of majorisation and to present some equivalent statements to the definition above.
First convince yourself that $x \succ x$ and if $x \succ y$ than as well $Px \prec P'y$ for arbitray permutations P and P'.

Problem 5 (Drawing some Lorenz curves): (6 points)

If we have a probability vector x we can define the vector $s(x)$ of partial sums of x as

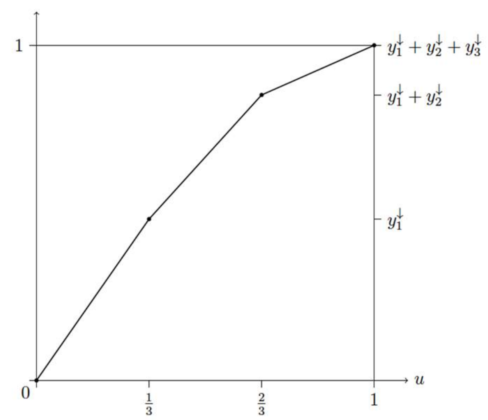
$$s(x)_i = \sum_{j=1}^{i} x_j.$$

which is just a n component positive vector. If we draw the vector $s(x^\downarrow)$ as a piecwise linear function we get what we call a *Lorenz curve* (after the economist Max O. Lorenz). How this is done is most easyly described in a picture.

On the right we see the Lorenz curve for a three component probabiliy vector. It is a piecwise linear function on the interval $[0, 1]$ which has the value $s(x^\downarrow)_i$ at the point i/n. The majorisation condition can be expressed in terms of Lorenz curves in the following way

$$x \succ y \quad \text{if} \quad s(x^\downarrow)_i \geq s(y^\downarrow)_i \quad \forall i$$

Now choose n to be 3 or 4 and write down some explicit vector. Find two more probability distributions x and z such that $x \succ y \succ z$ and draw the Lorenz curves for all three in one diagram. Please don't use $(1, 0, \ldots, 0)$ or $(1/n, \ldots, 1/n)$.



Furthermore find a probability vector y' such that neither $y \succ y'$ nor $y' \succ y$ and draw the corresponding Lorenz curves. Proove and try to argue graphically that the following holds for arbitrary x:

$$(1, 0, \dots, 0) \succ x \succ (1/n, \dots, 1/n).$$

Problem 6 (The convex set of majorized states): (8 points)

In this exercise we will prove that the set of probability vectors majorized by some x is convex. That means

$$\text{If } x \succ y \text{ and } x \succ z \quad \text{than} \quad x \succ ty + (1-t)z \qquad \text{for } t \in [0, 1].$$

First remember the definition of majorisation in terms of Lorenz curves

$$x \succ y \quad \text{if} \quad s(x^{\downarrow})_i \geq s(y^{\downarrow})_i \quad \forall i$$

and show that

(i) $s(x^{\downarrow})_i \geq s(Px^{\downarrow})_i \quad \forall i$ where $Px^{\downarrow}$ is some permutation of the vector $x^{\downarrow}$.

(ii) For real numbers $a, b, c \in \mathbb{R}$ with $a \geq b$ and $a \geq c$ it holds that $tb + (1-t)c \leq a$ for $t \in [0, 1]$.

Now use this to prove the convexity statement. Afterwards convince yourself that the following slightly more general statement holds as well

$$\text{If } x \succ y^i \quad \text{than} \quad x \succ p_1 y^1 + \dots + p_k y^k$$

for probability vectors y^i and all probability distribution $p = (p_1, \dots, p_k)$. For a given x we always know a collection of such y^i majorized by x and this are its permutations. So what we have shown is that the set of elements majorized by some x contains all convex mixtures of the permutations of x. Furthermore it can be shown that it does not contain any more states.

Theorem: The vector y is majorized by x if and only if there exist a probability distribution p and some permutations P_i such that

$$y = \sum_i p_i \, P_i x \tag{1}$$

(You do not have to prove this.)

Problem 7 (Drawing the set of probability vectors): (6 points)

For arbitrary dimensions the set of probability vectors is quite difficult to visualize but for $n = 3$ we can get some nice insightful pictures. This set is the intersection of the positive octant of $\mathbb{R}^3$ with the plane defined by $x + y + z = 1$. Draw this set and indicate the points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ and $(1/3, 1/3, 1/3)$. Since the set is two dimensional it is convenient for drawing to forget about its embedding in $\mathbb{R}^3$ and just draw it in the plane. Now pick a probability distribution x with $x_1 \neq x_2 \neq x_3$ and indicate it in the set together with all its permutations. Use what you know from the previous exercise to draw the set of all vectors that x majorizes.

Bonus: We know that majorisation does not really care about the order of the vector components so lets restrict ourselves to the part of the probability distributons which are ordered decreasingly $x_1 \geq x_2 \geq x_3$. Find out which part of the whole set this is and draw it. In this subset now choose again a point x and draw in addition to the set that x is majorizing as well the set of elements that majorize x. To do that you can try to look systematically at some Lorenz curves. In the end you should end up with three different regions: the points majorizing x, the points majorized by x and the points that are not comparable to x.

Problem 8 (Permutations and other bistochastic matrices): (10 points)

A stochastic matrix B of size n is a $n \times n$ real matrix whose matrix elements satisfy the first two of the following properties

$$(i): \quad B_{ij} \geq 0 \qquad (ii): \quad \sum_{i=1}^{n} B_{ij} = 1 \qquad (iii): \quad \sum_{j=1}^{n} B_{ij} = 1.$$

If in addition the matrix elements obeys (iii) we call B a *bistochastic matrix.*
Prove the following statements

(i) The stochastic matrices are exactly the linear maps that map probability vectors into probability vectors.

(ii) Bistochastic matrices are exactly the subset of stochastic matrices that leave the maximally mixed state $(1/n, \dots, 1/n)$ invariant.

(iii) Permutation matrices are the only bistochastic matrices that are *reversible*. That means that they are invertible and the inverse is still a bistocastic matrix.

(iv) The set of bistochastic matrices is convex. Thus if B_i are bistochastic and p is a probaility distribution than $\sum_i p_i B_i$ is bistochastic.

The converse of the last statement holds as well:

Theorem (Birkhoff & von Neumann): A matrix B is bistochastic if and only if there exist permutations P_i and and a probability distribution p such that

$$B = \sum_i p_i P_i \ .$$

(Again, you don't have to prove that.) Use this together with the theorem of Exercise 8 to show:

$$x \succ y \quad \text{if and only if} \quad \exists\ B \text{ bistochastic, with} \quad Bx = y$$