Universität Heidelberg
Institut für Theoretische Physik
Single-shot quantum thermodynamics

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http://www.mpmueller.net/lecture2.html

Exercises 3

(to hand in: November 18, 2014)

Problem 11 (Schur-Convex Functions): (8 points)

A function $F : \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-convex in the case that

$$F(p) \geq F(q) \quad \text{if} \quad p \succ q$$

where p, q are probability vectors of size n. The function is called Schur-concave if $-F$ is Schur-convex. One Schur-concave function you already know is the Shannon-entropy $H(p) = -\sum_{i=1} p_i \log(p_i)$. Show that all functions of the type

$$F(p) = \sum_{i=1}^{n} f(p_i) \quad \text{with} \quad f : \mathbb{R} \to \mathbb{R} \text{ convex}$$

are Schur-convex.

Problem 12 (Rényi entropies): (8 points)

A generalisation of the Shannon entropy are the Rényi entropies which are defined as:

$$H_\alpha(p) = -\frac{1}{\alpha - 1} \log \left(\sum_{i=1}^{n} p_i^\alpha \right) \qquad \text{with } \alpha \in \mathbb{R}_+ \backslash \{0, 1\} \tag{1}$$

$$H_0(p) = -\log \operatorname{rank} p \qquad \text{max entropy} \tag{2}$$

$$H_1(p) = -\sum_{i=1} p_i \log(p_i) \qquad \text{Shannon entropy} \tag{3}$$

$$H_\infty(p) = -\log \max_i \{p_i\} \qquad \text{min entropy} \tag{4}$$

where $\operatorname{rank} p$ is the number of non zero entries of p.

(i) Show that the definitions for the special cases are consistent with the general formula, that means $\lim_{\alpha \to 0} H_\alpha = H_0$, $\lim_{\alpha \to 1} H_\alpha = H_1$ and $\lim_{\alpha \to \infty} H_\alpha = H_\infty$.

(ii) All Rényi entropies are Schur-concave

(iii) Show that all Rényi entropies have values between 0 and $\log n$ where n is the size of the system and 0 is the value for a pure state and $\log n$ the value for the maximally mixed state.

Furthermore it holds that $H_\alpha(p)$ is non increasing for increasing α but you do not have to show that.

Problem 13 (Trace distance): (6 points)

We want to introduce a measure of distance on the set of density matricces and for that we define the trace distance $D(\rho, \sigma)$ to be

$$D(\rho, \sigma) = \frac{1}{2} \operatorname{Tr} |\rho - \sigma|$$

where the absolute value of a matrix is given as $|A| = \sqrt{A^\dagger A}$. Note that a matrix can have multiple square roots (sometimes even infinitely many) however if a matrix M is diagonalizable and thus $M = \sum_i m_i |i\rangle\langle i|$ we can choose the *principal square root* $\sqrt{M} = \sum_i \sqrt{m_i} |i\rangle\langle i|$. Thus $|\rho - \sigma| = \sum_j |\lambda_j| |j\rangle\langle j|$ where λ_i are the eigenvalues of $\rho - \sigma$.

(i) Argue that the trace distance is invariant under joint unitary transformations and thus $D(\rho, \sigma) = D(U\rho U^\dagger, U\sigma U^\dagger)$.

(ii) Choose $\rho = I/2 + a/2\, \sigma_1$ and $\sigma = I/2 + b/2\, \sigma_2$ where I is the 2×2 identity matrix and σ_i are the Pauli matrices. If you want you can check that ρ and σ fulfill the conditions for density matrices if $a, b \in [0, 1]$. Calculate $D(\rho, \sigma)$.

Problem 14 (Non-local unitaries and partial trace): (6 points)

Let $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ be a matrix with λ_i on the diagonal and zeros elsewhere. Define $\rho_A = \operatorname{diag}(1, 0)$, $\rho_B = \operatorname{diag}(p, 1-p)$ and the product state $\rho_{AB} = \rho_A \otimes \rho_B$. Choose a unitary on the composite system $U = \sin\alpha\, I \otimes I + i \cos\alpha\, \sigma_x \otimes \sigma_x$. Note that it is not a unitary of the form $U_1 \otimes U_2$ where U_i are local unitaries.

(i) Check that $U U^\dagger = I$.

(ii) Calculate the transformed state $\rho'_{AB} = U \rho_{AB} U^\dagger$.

(iii) Calculate the reduced state $\rho'_A = \operatorname{Tr}_B \rho'_{AB} = \begin{pmatrix} \sin^2\alpha & 0 \\ 0 & \cos^2\alpha \end{pmatrix}$

It could be helpful for the calculation to write σ_A and σ_B in the form $aI + b\sigma_z$ (with $a, b \in \mathbb{R}$).