Universität Heidelberg
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Single-shot quantum thermodynamics

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`http://www.mpmueller.net/lecture2.html`

Exercises 4

(to hand in: November 25, 2014)

Problem 15 (Sharp states as measure of non-uniformity): (8 points)

Let Δ_n be the set of all probability vectors of size n and let us look at the set $\mathcal{P} = \cup_{i\in\mathbb{N}}\Delta_i$ of probability vectors of arbitrary size. We know that for elements p, q in this set (with possibly different dimensions) we can reach q from p with noisy operations if $L_p \geq L_q$. In the space $\mathcal{P}$ we introduced the so called *sharp states* as reference states to answer questions about *non-uniformity of formation* and *extractable non-uniformity*.

(i) Show that a sharp state s_I with $I = \log(l/k)$ actually corresponds to a set of noisy equivalent states. That means, if you multiply a sharp state with a maximal mixture you get a sharp state with the same index $I = \log(l/k) = \log(nl/nk)$.

(ii) We define the generalized non-uniformity measure as

$$I_\alpha(p) = \log d_p - H_\alpha(p) \qquad (\alpha \geq 0),$$

where H_α are the Rényi entropies. Show that for sharp states s_I we have $I_\alpha(s_I) = I$. (And note the different uses of the letter I here.)

Problem 16 (Exctractable non-uniformity and non-uniformity of formation): (10 points)

Consider a physicist who prepares an n-qubit state in the laboratory. She has two devices at hand, one gives pure states as output and the other one just maximally mixed states. She just uses one device to prepare the state (wich is a density matrix on $(\mathbb{C}^2)^{\otimes n}$) and hands it afterwards to one of her co-workers. This co-worker does not know which device was used to prepare the state, and assigns equal probabilities to the two possible cases.

(i) Give the density matrix that he would use to describe the quantum state.

(ii) What is the extractable non-uniformity and the non-uniformity of formation of this state?

Problem 17 (Distinguishability measures): (12 points)

In the lecture the *trace distance* and its classical analogue were introduced as a distance measure in the space of quantum states or classical probability vectors. For the classical case we will prove a nice operational interpretation of this l_1 distance which is defined as:

$$d(p,q) = \frac{1}{2}\sum_{i=1}^{n} |p_i - q_i| \ .$$

The index set of outcomes described by p (or q) is $\{1, \ldots, n\}$. We say a measurement on the system is given by a set $S \subset \{1, \ldots, n\}$. If you look for example at a six sided die, this measuremens could correspond to one of the questions: "Is the result a number larger than 3", "Is the result a six" or "Is the result an even number". The probability that this question is answered in the positive is $p(S) := \sum_{i\in S} p_i$. Now the l_1 distance is equivalently given as:

$$d(p,q) = \max_{S\subset\{1,\dots n\}} |p(S) - q(S)| = \max_{S\subset\{1,\dots n\}} \big(p(S) - q(S)\big)\,.$$

To show this, choose $S = \{i \mid p_i - q_i \geq 0\}$ and define $S^C = \{1,\dots,n\} \setminus S$. Prove that

$$\left|\sum_{i\in S} p_i - \sum_{i\in S} q_i\right| = \left|\sum_{i\in S^c} p_i - \sum_{i\in S^c} q_i\right| = \frac{1}{2}\sum_i^n |p_i - q_i| = d(p,q).$$

Show furthermore that any other choice of S gives a smaller value for the quantity $|\sum_{i\in S} p_i - \sum_{i\in S} q_i|$.

We thus showed that S is the measurement that does the best job in distinguishing the distributions p and q. A similar interpretation and proof exists for the quantum case, which you can find in Chapter 9.2 in the book by Nielsen & Chuang.

Problem 18 (Bonus: Non-orthogonal states cannot be perfectly distinguished): (+4 points)

In the lecture the following question appeared: Is it possible to build a measuring device that always gives a "yes" as output if the measured quantum system is in a specific state $|\psi\rangle$ and at all other times a "no"? This is not possible due to the fact that two states that can be perfecly distinguished must be orthogonal. Prove this for pure states and projective measurments by looking at $\langle\psi| P |\psi\rangle$ and $\langle\phi| P |\phi\rangle$ (for some state $|\phi\rangle \neq |\psi\rangle$ and some projector P) and/or using some standard linear algebra.