Universität Heidelberg
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Single-shot quantum thermodynamics

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`http://www.mpmueller.net/lecture2.html`

Exercises 5

(to hand in: December 9, 2014)

Problem 19 (Smothed max entropy): (8 points)

We will use this exercise to get a bit more familiar with smoothed entropies by showing the following equivalence for some probability vector p:

$$H_0^\epsilon(p) = \log k \qquad \text{where } k \geq 1 \text{ is the smallest integer such that} \qquad \sum_{i=1}^{k} p_i^\downarrow \geq 1-\epsilon.$$

First of all remember the definition of the smoothed max entropy

$$H_0^\epsilon(p) = \min_{p':D(p,p')\leq\epsilon} H_0(p')$$

where $D(p,p')$ is the trace distance. Since H_0^ϵ is permutation invariant we will from now on assume that $p = p^\downarrow$. Prove the above statement by showing the two inequalities $H_0^\epsilon(p) \leq \log k$ and $H_0^\epsilon(p) \geq \log k$.

(i) Choose k as above and define $N = \sum_i^k p_i$. Now look at the state $x' = (p_1, \ldots, p_k, 0, \ldots, 0)/N$, show that $D(p,p') \leq \epsilon$ and thus $H_0^\epsilon(p) \leq \log k$.

(ii) Now we show that for all states q with $H_0(q) < \log k$ we have $D(p,q) > \epsilon$. For that look at the set supp$(q) = \{\, i \,|q_i \neq 0\}$ and use the form of the classical trace distance discussed on the last sheet $D(p,q) = \max_S \left|\sum_{i\in S} p_i - \sum_{i\in S} q_i\right|$. Use this to prove $H_0^\epsilon(p) \geq \log k$.

Problem 20 (Sufficient conditions for approximate convertability): (8 points)

We want to answer the question when two states are approximately convertible under noisy operations. First remember the definition

$$p \xrightarrow{\epsilon-\text{noisy}} q \quad :\Leftrightarrow \quad \exists\, q' : D(q,q') \leq \epsilon \quad \text{with} \quad p \xrightarrow{\text{noisy}} q'$$

Where $D(p,q)$ is the trace distance or its classical equivalent. Corresponding to the smoothed entropies there are as well smoothed version of non-uniformity measures:

$$I_0^\epsilon(p) = \max_{p':D(p,p')\leq\epsilon} I_0(p') \qquad \text{and} \qquad I_\infty^\epsilon(p) = \min_{p':D(p,p')\leq\epsilon} I_\infty(p').$$

(Note the reversed role of $\min$ and $\max$ in the definition with respect to the smoothed entropies due to the minus sign in the definition of I.) As a prerequisite we look at the exact case $\epsilon = 0$.

(i) Show geometrically via Lorenz curves that:

$$I_0(p) \;\geq\; I_\infty(q) \quad \Rightarrow \quad p \xrightarrow{\text{noisy}} q.$$

For a smoothed version of this statement we need to get a bit more technical.

(ii) Assume that $I_0^{\epsilon/2}(p) \geq I_\infty^{\epsilon/2}(q)$ and show that there exist $\bar{p}, \bar{q}$ with $D(p, \bar{p}) \leq \epsilon/2$ and $D(q, \bar{q}) \leq \epsilon/2$ and a noisy operation $\mathcal{N}$ with $\mathcal{N}(\bar{p}) = \bar{q}$.

To continue we have to rely on two properties of the trace distance. First it fulfills a triangle inequality $D(p, q) \leq D(p, r) + D(r, q)$ (for any states p, q, r) and second it is contractive under noisy operations $\mathcal{N}$. So for any $\mathcal{N}$ it holds that $D(\mathcal{N}(p), \mathcal{N}(q)) \leq D(p, q)$.

(iii) Use these two properties together with (ii) to show that for $q' = \mathcal{N}(p)$ ($\mathcal{N}$ is chosen as in (ii)) we have $D(q, q') \leq \epsilon$ and thus

$$I_0^{\epsilon/2}(p) \ \geq \ I_\infty^{\epsilon/2}(q) \quad \Rightarrow \quad p \xrightarrow{\epsilon-\text{noisy}} q.$$

Problem 21 (Smoothed non-uniformity monotones): (6 points)

A non-uniformity monotone was a function from probability vectors to the real numbers that decreases under noisy operations.

(i) By looking at p and $p \otimes (1/d, \ldots, 1/d)$ show that that I_0^ϵ is not a non-uniformity monotone. (We of course assume $\epsilon \neq 0$.)

(ii) However I_∞^ϵ is a non-uniformity monotone. Show this by using the contractivity of the trace distance and the fact that I_∞ is a non-uniformity monotone. (Hint: You want to show $I_\infty^\epsilon(\mathcal{N}(p)) \leq I_\infty^\epsilon(p)$. One step is to set $q = \mathcal{N}(p)$ and show that the set of q' with $D(q, q') \leq \epsilon$ is a superset of the elements $\mathcal{N}(p')$ with $D(p, p') \leq \epsilon$.)

Problem 22 (Landauer erasure reloaded): (8 points)

Remind yourself of the protocol that was used for Landauer erasure in the first lecture. We have two energy levels at zero energy with equal occupation probabilities. We now lift one of these levels in discrete steps with time for thermalisation in between. However now we just want to lift it up to some finite energy E_0. To these energies for the two levels corresponds some Gibbs state $\gamma = (p, 1-p)$.

(i) Show that $\gamma = (p, 1-p) = \left(e^{-\beta E_0}/(1 + e^{-\beta E_0}), 1/(1 + e^{-\beta E_0})\right)$.

(ii) Show that in the $\Delta E \to 0$ limit the average work needed is $\langle W \rangle = k_B T \, I_\infty(\gamma)$, where I_∞ is the non-uniformity of formation. Show that we get the already known result in the $E_0 \to \infty$ case.

Remark: This result gives a value for the *average* work needed to (partially) reset a bit. However it can be shown that the distribution of work needed, is sharply peaked around the average value. You can find more information on this together with a "finite time" version of the protocol considered here (that means the time for the thermalisation steps is not infinite but finite) in the paper:

Browne, C., Garner, A. J. P., Dahlsten, O. C. O., & Vedral, V. (2014, September 12) *Guaranteed energy-efficient bit reset in finite time.* arXiv:1311.7612v2