Universität Heidelberg
Institut für Theoretische Physik
Single-shot quantum thermodynamics

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Markus Müller, Jakob Scharlau
`http://www.mpmueller.net/lecture2.html`

Exercises 7

(to hand in: January 7, 2015 in the lecture)

In the following, we say that a state ρ is *diagonal* on a quantum system S with Hamiltonian H_S if and only if $[\rho, H_S] = 0$. This is equivalent to the fact that there is some (energy) eigenbasis of H_S such that the matrix representation of ρ in this basis is a diagonal matrix. The *Gibbs state at inverse temperature* β is $\gamma_\beta := \exp(-\beta H_S)/Z$, where $Z \in \mathbb{R}$ is such that $\mathrm{tr}(\gamma_\beta) = 1$.

Problem 26 (Gibbs states are completely passive): (8 points)

In the lecture, we have learned that a state ρ_S is passive (with respect to some Hamiltonian H_S) if and only if $\rho_S = \mathrm{diag}(p_1, \ldots, p_n)$ in some eigenbasis of H_S, and $E_i > E_j \Rightarrow p_i \leq p_j$. A state ρ_s is called completely passive if $\rho_S^{\otimes n}$ is passive for all n. (In this statement, the Hamiltonian on the system $S^{\otimes n}$ is taken to be $H_S \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \mathbb{1} \otimes H_S \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \ldots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes H_S$.) Show that all Gibbs states γ_β (for all inverse temperatures $\beta \geq 0$) are passive and completely passive.

Problem 27 (Passive and completely passive states in small dimensions): (8 points)

(i) Argue that every density matrix of full rank can be seen as Gibbs state if you choose the Hamiltonian correctly. Or stated differently, every density matrix can be written as $\rho = e^{-\beta H}/Z$ if H and $\beta \geq 0$ are chosen appropriately. (You don't have to be totally mathematically rigorous here.)

(ii) Show that for a *fixed* diagonal Hamiltonian H of size two, *every* diagonal density matrix of full rank is a Gibbs state for some appropriate inverse temperature $\beta \geq 0$. Use Exercise 26 to conclude that all passive states in two dimensions are completely passive.

(iii) Find an example in three dimensions of a state (and a corresponding Hamiltonian) that is passive but not completely passive.

Problem 28 (General properties of thermal operations): (6 points)

Consider thermal operations $\Phi : S \to S$, where S is a finite-dimensional quantum system.

(i) Show that all thermal operations leave the Gibbs state of the system invariant, i.e. $\Phi(\gamma_S) = \gamma_S$.

(ii) Show that the image of any diagonal state under thermal operations is diagonal. Thus, no superpositions of energy eigenstates can be created by thermal operations.

Problem 29 (Transitions with a small environment): (8 points)

Consider a two-dimensional quantum system S and a two-dimensional heat bath B with respective Hamiltonians

$$H_S = \begin{pmatrix} 0 & 0 \\ 0 & \Delta E \end{pmatrix} \qquad H_B = \begin{pmatrix} E & 0 \\ 0 & E+\Delta E \end{pmatrix}.$$

Let $\beta \geq 0$ be some arbitrary but fixed inverse temperature.

(i) Show that the Gibbs states of the system and the bath are identical, i.e. $\gamma_S = \gamma_B$.

(ii) Let the system be in the diagonal state $\rho_S = \mathrm{diag}(p, 1-p)$, where $0 \leq p \leq 1$. Show that under thermal operations the transition $\rho_S \mapsto \gamma_S$ is possible, and give a unitary that (together with a partial trace) achieves this transition.

(iii) What other state transitions are possible with this environment? Give the set of achievable states as a subset of the set of two-component probability vectors. (Of course the states are all matrices, but since they are all diagonal (cf. Exercise 28), we can represent them as probability vectors.)

Hint: Argue that we obtain transitions from $\mathrm{diag}(p, 1-p)$ to $\mathrm{diag}(p', 1-p')$, and that

$$\begin{pmatrix} p' \\ 1-p' \end{pmatrix} = B \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

for some matrix B. Use known properties of B to determine all possible p'.