Exercises 8

(to hand in: January 13, 2015)

Problem 30 (d-stochastic matrices and d-majorization): (10 points)

In the resource theory of nonuniformity, we used the concept of majorization to check if two states are interconvertible. This is not appropriate anymore for the resource theory of athermality where we will use a more general concept. Remember exercise sheet 2 with the definition of majorization and the result of Problem 8. For probability vectors p and q we have $p \succ q$ if and only if there exists some bistochastic matrix B with $q = Bp$. Bistochastic matrices were defined as stochastic matrices that map the maximally mixed state to itself. We can generalize this and define:

Definition 1. *A stochastic matrix S is called d-stochastic if it leaves the probability vector d invariant, i.e. $Sd = d$.*

We use this to define d-majorization as

Definition 2. *Let p, q and d be probability vectors. We say that q is d-majorized by p, written as $p \succ_d q$, if there exists some d-stochastic matrix S with $q = Sp$.*

Due to the result of Problem 8, this just defines ordinary majorization in the case that d is the maximally mixed state. We could then use the inequalities on top of exercise sheet 2 to check this definition of majorization. However for general d we need a more general set of inequalities which will be given later. Now we want to explore the set of states that is d-majorized by some vector in two dimensions.

(i) Give the set of all 2×2 bistochastic matrices (they are parametrized by one parameter), and use this to give the set of vectors majorized by some arbitrary p.

(ii) Give the set of all 2×2 d-stochastic matrices for a general d (again they are parametrized by one parameter) and give the set of states d-majorized by some p. Feel free to draw a small picture.

Problem 31 (Thermal operations and off-diagonal terms): (10 points)

We define the time-averaging map (or dephasing map) on density matrices as

$$\Delta(\rho) = \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{itH} \rho\, e^{-itH} dt.$$

If we assume H to be diagonal, this maps a matrix ρ to $\Delta(\rho)$ which has the same (block)-diagonal elements as its preimage, but contains otherwise only zeroes. That is, Δ sets all off-diagonal (or off-block-diagonal) entries of a matrix to zero. Prove this property by explicitly calculating the integral in the definition. (To do that, first expand ρ in the energy eigenbasis of H.) Show furthermore that Δ commutes with thermal operations, i.e. that $\tau(\Delta(\rho)) = \Delta(\tau(\rho))$ for some thermal operation τ.

Problem 32 (Gibbs-preserving maps can create coherence): (10 points)

Let us consider a qubit system with the following Hamiltonian H and Gibbs state γ

$$H = \Delta E\, |1\rangle\langle 1| \qquad \gamma = p_0\, |0\rangle\langle 0| + p_1\, |1\rangle\langle 1| \qquad p_0 = \frac{1}{1+e^{-\beta\Delta E}} \qquad p_1 = \frac{e^{-\beta\Delta E}}{1+e^{-\beta\Delta E}},$$

where $\beta \geq 0$. We have seen in Problem 29 that thermal operations preserve the Gibbs state of the system, but cannot map diagonal states to states with off-diagonal terms (in the case of a non-degenerate Hamiltonian). We will now explicitly construct a map from qubit density matrices to qubit density matrices that preserves the Gibbs state, but creates off-diagonal terms. Thus the thermal operations can achieve less state transitions than the Gibbs-preserving maps. This is in contrast to the case where we have a zero Hamiltonian, and thus are dealing with the resource theory of nonuniformity.

We define the map

$$\Phi(\cdot) = \langle 0|\cdot|0\rangle\, \sigma + \langle 1|\cdot|1\rangle\, \rho$$

where ρ and σ are two fixed qubit density matrices.

(i) Show that Φ is linear and trace-preserving.

(ii) Show that Φ maps positive matrices to positive matrices and so is a positive map.

(iii) Fix some arbitrary density matrix ρ. How do we have to choose σ that $\Phi(\gamma) = \gamma$? Show that this choice of σ has trace one and is a positive matrix. You can use the following fact: If A and B are positive matrices and all eigenvalues of B are smaller than (or equal to) all eigenvalues of A, than $A - B$ is a positive matrix.

(iv) Show that Φ can map a diagonal density matrix to a density matrix with off-diagonal terms.

Bonus: Draw a picture of the Bloch ball (set of all qubit density matrices) and explain what Φ does geometrically.