Universität Heidelberg
Institut für Theoretische Physik
Single-shot quantum thermodynamics

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http://www.mpmueller.net/lecture2.html

Exercises 10

(to hand in: January 27, 2015)

Problem 36 (Rescaled Lorenz-curves as condition for relative majorization): (10 points)

Consider the case where we have a given Gibbs state and two diagonal density matrices, and we want to know if they are interconvertible under thermal operations. We already know that this is equivalent to the statement that there exists a stochastic matrix mapping the diagonal elements of the matrices into each other while leaving the diagonal elements of the Gibbs state invariant. However, this is not a very nice condition to check, since it's not clear how to find this matrix. In this exercise, we will see a way how to do this, and on the way learn why it is equivalent to look at *rescaled* or *thermal* Lorenz curves, which is the equivalence of statement (i) and (iii) of the theorem in Section 3.6 of the lecture notes (p. 6, Lecture 11). We will show this by an example, however in the end it should be clear how a general construction could be done.

One limitation in the end: everything presented will just work for Gibbs states with *rational* entries. Why this is not a severe restriction will be explained at the end of the exercise.

Consider the three probability vectors for which we want to check whether p is γ-majorized by q or the other way around. (That all states are comparable in two dimensions is still true for relative majorization.)

$$\gamma = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad p = \frac{1}{12}\begin{pmatrix} 10 \\ 2 \end{pmatrix}, \quad q = \frac{1}{12}\begin{pmatrix} 3 \\ 9 \end{pmatrix}.$$

(i) Fist draw the standard Lorenz curves for γ, p, q in some diagram and the rescaled (with respect to γ) Lorenz curves in another diagram, and observe that they are ordered differently. What relation under γ-majorization do you read off from the rescaled diagram? How does the diagram show that all states $\gamma-$majorize the state γ?

Now we are doing a procedure that is called Gibbs-rescaling. For this we divide the first level of the Gibbs state γ into two *fictious* levels. So we don't have probability $2/3$ anymore to find the system in level 1, but we have probability $1/3$ to find it in level 1a and the same probability to find it in level 1b, each with the same energy. This makes our Gibbs state look like the maximally mixed state $(1/3, 1/3, 1/3)$. We can do this formally by multiplying γ with some matrix G and reverse it with the matrix G^{-1} (take care that G^{-1} cannot actually be the inverse matrix since G is not a quadratic matrix; however, we have $G^{-1}G = \mathbf{1}_2$, while $GG^{-1} \neq \mathbf{1}_3$ is a projector).

$$G = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}, \qquad G^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We will do this for all our vectors and write $\tilde{x} = Gx$ which gives us

$$\tilde{\gamma} = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \tilde{p} = \frac{1}{24}\begin{pmatrix}10\\10\\4\end{pmatrix}, \quad \tilde{q} = \frac{1}{24}\begin{pmatrix}3\\3\\18\end{pmatrix}.$$

For a general γ with rational entries, we would choose another G such that $G\gamma$ is the maximally mixed state in some larger dimension.

(ii) Draw the standard Lorenz curves for $\tilde{\gamma}, \tilde{p}, \tilde{q}$ and explain why even for general γ and G this gives the same diagram as the rescaled Lorenz curves of γ, p and q.

The Gibbs-rescaling G mapped our states to some higher-dimensional states such that the Gibbs state now corresponds to the maximally mixed state. Instead of γ-majorization in two dimensions, we now have regular majorization in three dimensions. Looking at the Lorenz curves, we see that $\tilde{q} \succ \tilde{p}$ which suggest $q \succ_\gamma p$. Since q majorizes p, we know there exists some bistochastic B with $p = Bq$. One way to construct such a B is the following:

(ii) Let B the product of three matrices P, T, S, where P is some permutation matrix, and S and T are T-transforms. T-transforms are bistochastic matrices that act non-trivially in just two dimensions.

$$\tilde{p} = B\,\tilde{q} = S\,T\,P\,\tilde{q} = \begin{pmatrix}1 & 0 & 0\\0 & s & 1-s\\0 & 1-s & s\end{pmatrix}\begin{pmatrix}t & 1-t & 0\\1-t & t & 0\\0 & 0 & 1\end{pmatrix} P\,\tilde{q}$$

Now choose the matrices in the following way. Choose P such that $P\tilde{q}$ is ordered decreasingly. Choose T (and thus t) such that the first components of $P\tilde{q}$ and $\tilde{p}$ are equal. Choose (S and thus s) such that the second components of $TP\tilde{q}$ and $\tilde{q}$ are equal. Due to normalization we should now have $B\tilde{q} = \tilde{p}$.

Calculate B and $\tilde{B} = G^{-1}B\,G$:

$$B = \begin{pmatrix}0 & \frac{8}{15} & \frac{7}{15}\\ \frac{1}{8} & \frac{49}{120} & \frac{7}{15}\\ \frac{7}{8} & \frac{7}{120} & \frac{1}{15}\end{pmatrix}, \qquad \tilde{B} = G^{-1}B\,G = \begin{pmatrix}\frac{8}{15} & \frac{14}{15}\\ \frac{7}{15} & \frac{1}{15}\end{pmatrix},$$

and check that $\tilde{B}\gamma = \gamma$ and $\tilde{B}q = p$ and thus $q \succ_\gamma p$.

(iv) Explain why this is a proof of the aforementioned theorem, and describe shortly how this is done in a general setting. What would you have to do differently if the Gibbs state changes during the thermal operation (i.e. if the Hamiltonian and/or the dimension of the system changes)?

The restriction to Gibbs states with only rational components is not really a problem, since for any γ, we can always choose some γ_r with rational entries which is arbitrarily close to γ. However, the proof above can as well be done in an exact fashion for irrational Gibbs states. Then we have to convert our probability vectors to continuous probability densities over the interval $[0, 1]$ and work with integral transforms. If you are interested, have a look at:
Ruch, E., Schranner, R., & Seligman, T. H. (1978). The mixing distance. *The Journal of Chemical Physics*, 69(1), 386–392. do:10.1063/1.436364

Problem 37 (Reading some original research): (14 points)

Having followed the lecture up to this point, you should be able to understand most of the statements in the following research article:

Horodecki, M. & Oppenheim, J. Fundamental limitations for quantum and nanoscale thermodynamics. *Nat.Commun.* 4:2059 doi:10.1038/ncomms3059 (2013)

(You should be able to access the DOI link from the university network or via the VPN client; if you have problems, please write an email.)
Read this article (it's just four pages) and try to connect it to the content of the lecture. Give short (!) summaries for the different sections of the article and write down important equations or results. Relate these to your knowledge from the lecture.

It is absolutely not assumed that you understand everything in detail of the above paper. The idea is that you have a look at it just to get some impression how the content of the lecture relates to current research.

Problem 38 (Monotone functions for relative majorization continued): (6 points)

On the last sheet you already proved some properties of the relative Rényi entropies, which will be continued here.

(i) Show the following limiting property:

$$\lim_{\alpha\to 0} D_\alpha(\, p \,\|\, q\,) = -\log \sum_{p_i \neq 0} q_i.$$

The quantity on the right-hand side is therefore called $D_0(p\|q)$.

(ii) Assume that q does not contain any zero entries and show that

$$\lim_{\alpha\to\infty} D_\alpha(\, p \,\|\, q\,) = \log \max_i \left\{ \frac{p_i}{q_i} \right\}.$$

This might be a bit tricky. Start by factoring out the term $q_i(p_i/q_i)^\alpha$ from the sum, where i is chosen such that p_i/q_i is maximal.

(iii) Show that in the case when $q = \eta = (1/d, \ldots, 1/d) \in \mathbb{R}^d$ is the maximally mixed state, the relative Rényi entropy (for $\alpha \geq 0$) reduces to the already known Rényi nonuniformity I_α:

$$D_\alpha(\, p \,\|\, \eta\,) = \log d - H_\alpha(p) \quad \text{with} \quad \eta = (1/d\,, \ldots, 1/d).$$