8.3.iv

(decreasingly) ordered eigenvalues   energy eigenstates

$$\rho \succ \sigma \Rightarrow \sum_{j=1}^{k} r_j \geq \sum_{j=1}^{k} s_j \quad \forall k$$

$$\Pi_\rho := \sum r_i |i\rangle\langle i|$$
$$\Pi_\sigma := \sum s_i |i\rangle\langle i|$$
$$H := \sum \varepsilon_i |i\rangle\langle i|$$

increasingly ordered

$$\Rightarrow \sum_{j=1}^{k} (r_j - s_j) \geq 0 \quad \forall k \qquad (0)$$

$E_\rho = E_\sigma$

$$\Delta W \equiv W(\rho, H) - W(\sigma, H) = \cancel{E_\rho} - E_{\Pi_\rho} - \cancel{E_\sigma} + E_{\Pi_\sigma}$$

$$= \sum_{i}^{d} \varepsilon_i (s_i - r_i)$$

$$\overset{(1)}{=} \sum_{i=1}^{d} \sum_{j=2}^{i} \overbrace{(\varepsilon_j - \varepsilon_{j-1})}^{\equiv \Delta_j} \overbrace{(s_i - r_i)}^{\equiv \delta_i} \overset{(2)}{=} \sum_{i=1}^{d} \delta_i \sum_{j=1}^{i} \Delta_j$$

(1) $\varepsilon_k = \sum_{i=0}^{k} \Delta_k$

(2) $\Delta_1 := 0$

$$= \underbrace{\delta_1 \Delta_1}_{=0} + \delta_2 (\Delta_1 + \Delta_2) + \delta_3 (\Delta_1 + \Delta_2 + \Delta_3) + \ldots$$

$$= \underbrace{\Delta_1}_{=0} \underbrace{\Big(\sum_{i=1}^{d} \delta_i\Big)}_{=0} + \Delta_2 \underbrace{\Big(\sum_{i=2}^{d} \delta_i\Big)}_{=0-\delta_1} + \ldots + \Delta_d \underbrace{\delta_d}_{= -\sum_{i=1}^{d-1} \delta_i}$$

$$= \sum_{j=2}^{d} \Delta_j \sum_{i=1}^{j-1} (-\delta_i) = \sum_{j=2}^{d} \underbrace{(\varepsilon_j - \varepsilon_{j-1})}_{\geq 0} \underbrace{\sum_{i=1}^{j-1} (r_i - s_i)}_{\geq 0 \;\Leftarrow\; (0)} \geq 0 \qquad \square$$