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Exercises - Week 3

Problem 3.1: Quantum operations (10 points)

A key use of resource theories is to determine whether a particular transformation on a state is permissible. In this exercise, we will first look at the most general class of physical transformation on quantum states. These *quantum operations* are the **completely–positive trace–preserving** (CPTP) maps on density matrices. A map $\mathcal{E} : \rho \mapsto \rho'$ is positive (semidefinite) if for all ρ with nonnegative eigenvalues ρ' also has nonnegative eigenvalues. *Complete positivity* means that when $\mathcal{E}$ is applied on a subsystem of any larger (possibly entangled) quantum state, the whole larger state is not transformed into a state with negative eigenvalues. Trace–preserving means $\mathrm{Tr}\,\rho' = \mathrm{Tr}\,\rho$.

Briefly argue why unitary transformations are CPTP maps.

The (partial) transpose is a linear map $\mathcal{T}$ that takes $\rho \to \rho^{\mathrm{T}}$. Argue briefly why this map is positive and trace-preserving. By finding a two qubit state that is not mapped onto a valid quantum state when $\mathcal{T}$ is applied to just one of qubit, show that $\mathcal{T}$ is not completely positive.

The set of CPTP maps is characterized as exactly the maps of the form

$$\rho'_A = \mathcal{E}(\rho_A) := \mathrm{Tr}_E \left(U \rho \otimes \sigma_E U^\dagger \right), \tag{1}$$

where σ_E is a choice of "environmental" state (i.e. a density matrix with $\mathrm{Tr}\,\sigma_E = 1$), and U is a unitary operation that acts jointly on the input system and environment. A CPTP map can equivalently be represented by a (nonunique) choice of *Kraus operators* $\{B_i\}$ where $\rho'_A = \sum_i^N B_i \rho_A B_i^\dagger$ and $\sum_i B_i^\dagger B_i = \mathbb{1}$. (Conversely, every set of Kraus operators can be "dilated" into the form of eq. (1)).

Write down a set of Kraus operators and describe also a U and σ_E in the form of eq. (1) for the following quantum operations:

(i) The preparation of a pure state $|\psi\rangle$ (from any initial state of the same dimension).

(ii) The decoherence of a qubit from $\rho = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ to $\rho = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$.

(iii) The "partial swap" model of thermalization where with probability p, the input state is replaced with the diagonal state $\gamma = \mathrm{diag}\,(\gamma_1, \dots \gamma_n)$, and otherwise it is left unchanged. Here, the γ_i are arbitrary non-negative real numbers that add up to one (so that γ is a density matrix).

Problem 3.2: Unital maps and noisy operations (6 points)

For simplicity, we shall assume that the input and output dimensions are the same throughout this question: $d_{\text{in}} = d_{\text{out}} = d$.

Quantum case. A quantum noisy operation $\mathcal{E}$ has the form $\mathcal{E} : \rho \mapsto \sigma := \mathrm{Tr}_B\big[U\rho \otimes \gamma_B U^\dagger\big]$ (where U is a unitary and γ_B is a maximally mixed state). Meanwhile, a **unital map** is a transformation that preserves the maximally mixed state $\frac{1}{d}\mathbb{1}_d \equiv \mathrm{diag}\big(\frac{1}{d}, \ldots, \frac{1}{d}\big)$.

Argue why the quantum noisy operations are unital maps. *(Note: it can be shown that the converse is not always true.)*

Classical case. Let $\vec{r}, \vec{s} \in \mathbb{R}^d$ be probability vectors, and denote the d–outcome uniform distribution as $\vec{\gamma}_d := \big(\frac{1}{d}, \ldots, \frac{1}{d}\big)^{\mathrm{T}} \in \mathbb{R}^d$. A classical noisy map $\mathcal{D}$ has the form $\mathcal{D} : \vec{r} \mapsto \vec{s} := [\pi(\vec{r} \otimes \vec{\gamma}_{d_1})]_A$, π is a dd_1–dimensional permutation (for $d_1 \in \mathbb{N}$), and $[\ldots]_A$ denotes the marginal distribution with dimension d. Meanwhile, a **uniform–preserving** map preserves $\vec{\gamma}_d$.

Argue that classical noisy operations must be also be uniform–preserving stochastic maps. Conversely, show that every uniform–preserving stochastic map D can be implemented to arbitary accuracy as a noisy operation. *(Hint: Show that marginalizing a permutation of some system of dimension dd_1 corresponds to taking a mixture of permutations of a system of size d, then use the results in Problem 2.4.)*

[**Bonus (+4 points):** Write down definitions of classical noisy operations and uniform–preservation when $d_{\text{in}} \neq d_{\text{out}}$. Show that the above equivalence also holds in this case.]

Problem 3.3: Classical and quantum noisy operations (6 points)

We now have the tools to prove a lemma stated without proof in the lecture notes. Let ρ, σ be normalized quantum states of dimension d_{in} and d_{out} respectively, written in terms of their eigenbases as $\rho := \sum_{i=1}^{d_{\text{in}}} \lambda_i(\rho) \, |\phi_i\rangle\langle\phi_i|$ and $\sigma := \sum_{j=1}^{d_{\text{out}}} \lambda_j(\sigma) \, |\psi_j\rangle\langle\psi_j|$.

Theorem: There exists an noisy quantum operation $\rho \to \sigma$ (up to arbitray accuracy) if and only if there exists a noisy classical operation $\lambda(\rho) \to \lambda(\sigma)$ (up to arbitray accuracy).

Only if. Suppose noisy operation $\mathcal{E} : \rho \mapsto \sigma$. By expanding $\sigma = \mathcal{E}(\rho)$, find (in terms of $\mathcal{E}$, $\{\lambda_i(\rho)\}_i$, $\{|\phi_i\rangle\}_i$, $\{\lambda_j(\sigma)\}_j$, and $\{|\phi_j\rangle\}_j$) the matrix elements D_{ji} of D such that $\lambda(\sigma)_j = \sum_j D_{ji}\lambda(\rho)_i$. Argue why the form of D indicates that it can be implemented to arbitrary accuracy as a noisy operation.

If. Suppose D is a noisy classical operation satisfying $\lambda(\sigma) = D\lambda(\rho)$, and define the map $\mathcal{E}(\rho) := \sum_{k,j} D_{kj} |\psi_k\rangle\langle\phi_j|\rho|\phi_j\rangle\langle\psi_k|$. Verify that $\mathcal{E}(\rho) = \sigma$. Recall (from the previous problem) that classical noisy operations can be implemented by marginalizing over a permutation R of the input system augmented with uniform distribution $\vec{1}_{d_1}$. Hence, we can express D in terms of this permutation: $D_{kj} = \frac{1}{d_1}\sum_{l,i} R_{kl,ji}$. Use this to show that $\mathcal{E}$ has the form of a noisy quantum operation $\mathrm{Tr}_{B'}\left(U\rho \otimes \gamma U^\dagger\right)$. What is U (in terms of R), and what is γ?

Problem 3.4: Schur–Convex functions and Rényi entropies (16 points)

Recall that the function $F : \mathbb{R} \to \mathbb{R}$ is *convex* if $F(ta + (1-t)\,b) \leq tF(a) + (1-t)\,F(b)$ for all $a, b \in \mathbb{R}$, $t \in [0,1]$. A function $G : \mathbb{R}^n \to \mathbb{R}$ is **Schur–convex** when

$$G(\vec{p}) \geq G(\vec{q}) \quad \text{if} \quad \vec{p} \succ \vec{q}, \tag{2}$$

where $\vec{p}$ and $\vec{q}$ are probability vectors of size n. G is Schur–concave if $-G$ is Schur–convex.

Show that all functions of the type

$$F(\vec{p}) = \sum_{i=1}^{n} f(p_i) \quad \text{with} \quad f : \mathbb{R} \to \mathbb{R} \text{ convex} \tag{3}$$

are Schur-convex. *(Hint: the statements in Problem 2.4 might be useful.)*

One Schur-concave function you are already familiar with is the *Shannon entropy*. The Shannon entropy can be generalized to the Rényi entropies, defined:

$$H_\alpha(\vec{p}) := -\frac{1}{\alpha - 1} \log\left(\sum_{i=1}^{n} p_i^\alpha\right) \qquad \text{with } \alpha \in \mathbb{R}_+ \backslash \{0, 1\}, \tag{4}$$

with a few special cases corresponding to $\alpha = 0, 1$ or ∞:

$$H_0(\vec{p}) := \log \operatorname{rank} p \qquad \text{max entropy}, \tag{5}$$

$$H_1(\vec{p}) := -\sum_{i=1} p_i \log(p_i) \qquad \text{Shannon entropy}, \tag{6}$$

$$H_\infty(\vec{p}) := -\log \max_i \{p_i\} \qquad \text{min entropy}, \tag{7}$$

where $\operatorname{rank} \vec{p}$ is the number of non zero entries of $\vec{p}$.

Show that:

(i) The definitions for the three special cases are consistent with the general formula, in the sense that $\lim_{\alpha \to 0} H_\alpha = H_0$, $\lim_{\alpha \to 1} H_\alpha = H_1$ and $\lim_{\alpha \to \infty} H_\alpha = H_\infty$.

(ii) All Rényi entropies are Schur-concave.

(iii) All Rényi entropies have values between 0 and $\log n$ where n is the size of the system and 0 is the value for a pure (i.e. $p_1^\downarrow = 1$) distribution and $\log n$ the value for the maximally mixed distribution.

Note that $H_\alpha(p)$ is non-increasing for increasing α (you do not need to show this).

Problem 3.5 (Optional): The data-processing inequality (+6 points)

A series of random variables $X_1 \to X_2 \to \ldots$ forms a *Markov chain* if each variable X_{n+1} is independent of $X_1 \ldots X_{n-1}$ given X_n. That is:

$$\mathrm{P}(X_{n+1} = x_{n_1} \,|\, X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \mathrm{P}(X_{n+1} = x_{n_1} \,|\, X_n = x_n)\,.$$

Consequently, the (Shannon) entropies satisfy $\mathrm{H}(X_{n+1} \,|\, X_1 X_2 \ldots X_n) = \mathrm{H}(X_{n+1} \,|\, X_n)$.

Show that if $X \to Y \to Z$ is a Markov chain, then $Z \to Y \to X$ is also a Markov chain. *(Hint: consider expanding* $\mathrm{P}(X = x, Y = y, Z = z)$*.)*

For a Markov chain $X \to Y \to Z$, the (classical) **data-processing inequality** (DPI) is:

$$\mathrm{H}(X) \geq \mathrm{I}(X\,;\,Y) \geq \mathrm{I}(X\,;\,Z)\,, \tag{8}$$

where $\mathrm{I}(A\,;\,B) := \mathrm{H}(A) - \mathrm{H}(A \,|\, B)$ is the *mutual information* between A and B. In words: the information Z contains about X cannot exceed the information Y contains about X if Z is obtained by "processing" Y.

Prove the second inequality of ineq. (8). (You may use the *strong subadditivity* property of Shannon entropy that $\mathrm{H}(ABC) + \mathrm{H}(B) \leq \mathrm{H}(AB) + \mathrm{H}(BC)$.)