Exercises - Week 4

Problem 4.1: Sharp states as measure of non-uniformity (8 points)

Let Δ_n be the set of all probability vectors of size n and consider the set $\mathcal{P} = \cup_{i\in\mathbb{N}}\Delta_i$ of probability vectors of arbitrary size. We know that for elements p, q in this set (with possibly different dimensions) we can reach q from p with noisy operations if their Lorenz curves satisfy $L_p \geq L_q$. In the space $\mathcal{P}$ we can introduce *sharp states* as reference states to answer questions about *non-uniformity of formation* and *extractable non-uniformity*.

(i) Show that a sharp state s_I with $I = \log(l/k)$ corresponds to a set of noisy equivalent states – that is, if you combine a sharp state with a maximal mixture of size n you get a sharp state with the same index I.

We define the generalized non-uniformity measure as

$$I_\alpha(p) = \log d_p - H_\alpha(p) \qquad (\alpha \geq 0), \tag{1}$$

where H_α are the Rényi entropies (recall their definition in Problem 3.4).

(ii) Show that $I_\alpha(s_I) = I$ for all $\alpha \geq 0$ for every sharp state s_I.

Problem 4.2: Distinguishability and the classical trace distance (6 points)

In the lecture the *trace distance* and its classical analogue were introduced as a distance measure in the space of quantum states or classical probability vectors. For the classical case, we will prove an operational interpretation of the l_1 distance between distributions p and q, defined:

$$D(p,q) = \frac{1}{2}\sum_{i=1}^{n} |p_i - q_i|.$$

The index set of outcomes described by p (or q) is $\Omega := \{1, \ldots, n\}$, and a measurement on the system is given by a subset $S \subseteq \Omega$. For example, on a six–sided die this measurement could correspond to one of the questions: "Is the result a number larger than 3?" ($S = \{4,5,6\}$), "Is the result a six?" ($S = \{6\}$) or "Is the result an even number?" ($S = \{2,4,6\}$). The probability that this question is answered in the positive is $p(S) := \sum_{i\in S} p_i$.

Now consider two arbitrary p and q of the same dimension, and define $S := \{i \mid p_i - q_i \geq 0\}$ and $S^\perp := \Omega \setminus S$ (i.e. all the elements of Ω not in S). Prove for all p and q that

$$\sum_{i\in S} p_i - \sum_{i\in S} q_i = \left|\sum_{i\in S} p_i - \sum_{i\in S} q_i\right| = \left|\sum_{i\in S^\perp} p_i - \sum_{i\in S^\perp} q_i\right| = D(p,q).$$

Next, show that any other choice of S gives a smaller value of the quantity $\sum_{i\in S} p_i - \sum_{i\in S} q_i$, and argue why this indicates that S is the measurement that best distinguishes between p and q.

Hence, we have shown the l_1 distance is equivalently given as:

$$D(p,q) := \frac{1}{2}\sum_{i=1}^{n} |p_i - q_i| = \max_{S\subset\Omega} \left(p(S) - q(S)\right).$$

(A similar interpretation and proof exists for the quantum case, which you can find in Chapter 9.2 of Nielsen & Chuang.)

Problem 4.3: Smothed max entropy (8 points)

We will use this exercise to get a bit more familiar with smoothed entropies by showing the following equivalence for some probability vector p:

$$H_0^\epsilon(p) = \log k \qquad \text{where } k \geq 1 \text{ is the smallest integer such that} \qquad \sum_{i=1}^{k} p_i^\downarrow \geq 1 - \epsilon. \qquad (2)$$

First of all remember the definition of the smoothed max entropy

$$H_0^\epsilon(p) = \min_{p' : D(p,p') \leq \epsilon} H_0(p')$$

where $D(p, p')$ is the trace distance. Since H_0^ϵ is permutation invariant we will from now on assume that $p = p^\downarrow$. We will now prove equation (2) by showing the two inequalities $H_0^\epsilon(p) \leq \log k$ and $H_0^\epsilon(p) \geq \log k$:

(i) Choose k as above and define $N = \sum_i^k p_i$. Now look at the state $p' = (p_1, \ldots, p_k, 0, \ldots, 0)/N$, show that $D(p, p') \leq \epsilon$ and thus $H_0^\epsilon(p) \leq \log k$.

(ii) Now we show that for all states q with $H_0(q) < \log k$, we have $D(p, q) > \epsilon$. To do this, look at the set $\mathrm{supp}(q) = \{\, i \,|q_i \neq 0\}$ and use the classical trace distance $D(p, q) = \max_S \left|\sum_{i\in S} p_i - \sum_{i\in S} q_i\right|$. Use this to prove $H_0^\epsilon(p) \geq \log k$.

Problem 4.4: Sufficient conditions for approximate convertability (8 points)

We want to address whether two states are approximately convertible under noisy operations. First remember the definition

$$p \xrightarrow{\epsilon-\text{noisy}} q \quad :\Leftrightarrow \quad \exists\, q' : D(q, q') \leq \epsilon \quad \text{with} \quad p \xrightarrow{\text{noisy}} q'$$

where $D(p, q)$ is the trace distance or its classical equivalent. Much like with smoothed entropies, there are also smoothed version of non-uniformity measures:

$$I_0^\epsilon(p) = \max_{p' : D(p,p') \leq \epsilon} I_0(p') \qquad \text{and} \qquad I_\infty^\epsilon(p) = \min_{p' : D(p,p') \leq \epsilon} I_\infty(p').$$

(Note the reversed role of $\min$ and $\max$ in the definition with respect to the smoothed entropies.) We first look at the exact case $\epsilon = 0$.

(i) Show geometrically via Lorenz curves that:

$$I_0(p) \;\geq\; I_\infty(q) \quad \Rightarrow \quad p \xrightarrow{\text{noisy}} q.$$

Now we move to the smoothed version of this statement.

(ii) Assume that $I_0^{\epsilon/2}(p) \geq I_\infty^{\epsilon/2}(q)$. Show that there exists $\bar{p}$, $\bar{q}$ with $D(p, \bar{p}) \leq \epsilon/2$ and $D(q, \bar{q}) \leq \epsilon/2$ and a noisy operation $\mathcal{N}$ such that $\mathcal{N}(\bar{p}) = \bar{q}$.

Now, we must use two properties of the trace distance. First, that it fulfills a triangle inequality $D(p, q) \leq D(p, r) + D(r, q)$ (for any states p, q, r) and second that it is contractive under noisy operations $\mathcal{N}$ (that is, for any noisy operation $\mathcal{N}$ it holds that $D\left(\mathcal{N}(p), \mathcal{N}(q)\right) \leq D(p, q)$).

(iii) Use these two properties together with (ii) to show that for $q' = \mathcal{N}(p)$ ($\mathcal{N}$ is chosen as in (ii)) we have $D(q, q') \leq \epsilon$ and thus

$$I_0^{\epsilon/2}(p) \;\geq\; I_\infty^{\epsilon/2}(q) \quad \Rightarrow \quad p \xrightarrow{\epsilon-\text{noisy}} q.$$

Problem 4.5: Smoothed non-uniformity monotones (6 points)

A non-uniformity monotone is a function from probability vectors to the real numbers that is non-increasing under noisy operations.

(i) By looking at p and $p \otimes (1/d, \ldots, 1/d)$, show that I_0^ϵ is not a non-uniformity monotone (assume $\epsilon \neq 0$).

(ii) However I_∞^ϵ is a non-uniformity monotone. Show this by using the contractivity of the trace distance and the fact that I_∞ is a non-uniformity monotone. (Hint: You want to show $I_\infty^\epsilon(\mathcal{N}(p)) \leq I_\infty^\epsilon(p)$. One step is to set $q = \mathcal{N}(p)$ and show that the set of q' with $D(q, q') \leq \epsilon$ is a superset of the elements $\mathcal{N}(p')$ with $D(p, p') \leq \epsilon$.)

Problem 4.6 (Optional): Extractable non-uniformity (+5 points)

Consider a physicist who prepares an n-qubit state in the laboratory. She has two devices at hand: one gives pure states as output, and the other maximally mixed states. She just uses one device to prepare the state and hands it afterwards to one of her co-workers. This co-worker does not know which device was used to prepare the state, and assigns equal probabilities to the two possible cases.

(i) Give the density matrix that he would use to describe the quantum state.

(ii) What is the extractable non-uniformity and the non-uniformity of formation of this state?

Problem 4.7 (Optional): Landauer erasure revisited (+5 points)

Remind yourself of the protocol that was used for Landauer erasure in the first lecture. We initially have two energy levels at zero energy with equal occupation probabilities. We lift one of these levels in discrete steps (of ΔE), and allow time for complete thermalization in between. However, rather than raising this level to ∞, we instead lift it up to some finite energy E_0. Let $\gamma := (p, 1-p)$ be the Gibbs state corresponding to the final state of these levels.

(i) Show that $\gamma := (p, 1-p) = \big(e^{-\beta E_0}/(1 + e^{-\beta E_0}), 1/(1 + e^{-\beta E_0})\big)$.

(ii) Show that in the $\Delta E \to 0$ limit, the average work needed is $\langle W \rangle = k_B T \, I_\infty(\gamma)$, where I_∞ is the non-uniformity of formation. Show that this recovers the well-known result in the $E_0 \to \infty$ case.

Remark: This result gives a value for the *average* work needed to (partially) reset a bit. However it can be shown that the distribution of work needed, is sharply peaked around the average value. You can find more information on this together with a "finite time" version of the protocol considered here (that means the time allowed for thermalization is finite, and so the system is not perfectly in the Gibbs state) in the paper by Browne et al. (2014) *Phys. Rev. Lett.* **133**(10):100603 (arXiv:1311.7612v2).