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Markus Müller, Felix Binder, Andrew Garner
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Exercises - Week 6

Problem 6.1: Quantum maps and instruments (6 points)

In this exercise we get to grips with quantum operations a bit more.

(i) The depolarizing channel Λ_{dep} acts as follows on any input state ρ:

$$\Lambda_{dep}(\rho) = p\rho + (1-p)\gamma$$

where γ is the maximally mixed state. Based on this channel consider a quantum classical map whose classical register 'heralds' if the state ρ was retained ('0') or replaced by the mixed state γ ('1'):

$$\mathcal{E}_0 \otimes |0\rangle\langle 0| + \mathcal{E}_1 \otimes |1\rangle\langle 1|$$

Write down sets of Kraus operators for $\mathcal{E}_0$ and $\mathcal{E}_1$.

(ii) Here, we will explore why 'fine-grained' instruments have their name. Namely, if a fine-grained instrument is a coarse-graining of another instrument, then any set of maps that are grouped together in the process of coarse-graining must be proportional to the resulting map.
To show this, consider a fine-grained operation $\mathcal{E}(\rho) = E\rho E^\dagger$, and suppose that $\mathcal{E}(\rho) = \mathcal{F}(\rho) + \mathcal{G}(\rho)$ for all ρ. Denote the Kraus operators of $\mathcal{F}$, together with those of $\mathcal{G}$, by S_k, i.e., $\mathcal{F}(\rho) + \mathcal{G}(\rho) = \sum_k S_k \rho S_k^\dagger$. Now consider the action of $\mathcal{E}(\rho)$ on any pure state $|\psi\rangle\langle\psi|$. If $|\varphi\rangle$ is *any* state orthogonal to $E|\psi\rangle$, compute $\langle\varphi|\mathcal{E}(|\psi\rangle\langle\psi|)\,|\varphi\rangle$. What does the result tell you about $S_k|\psi\rangle\langle\psi|S_k^\dagger$?

(iii) Recall the quantum teleportation protocol from the lecture. This protocol is an example of one-way LOCC. Show this by specifying the corresponding set of maps $\{\mathcal{F}_j = \mathcal{E}_j^{(A)} \otimes \Lambda_j^{(B)}\}$.
Remark: Keep in mind that Alice has two qubits in her lab. The Hilbert space $\mathcal{H}_A$ here corresponds to that two-qubit space.

Problem 6.2: Separable states and free operations (5 points)

(i) For an N-partite Hilbert space, specify the set of separable states $\mathcal{F}_s$ and the set of separable operations ($\mathcal{O}_{sep}$).

(ii) Consider $\mathcal{O}_{c-max}$ induced by $\mathcal{F}_s$. Show that $\mathcal{O}_{sep} \subseteq \mathcal{O}_{c-max}$ – i.e., there is no operation in $\mathcal{O}_{sep}$ which is not also in $\mathcal{O}_{c-max}$.
Remark: In fact, $\mathcal{O}_{sep} = \mathcal{O}_{c-max}$ but you don't need to prove the other direction – i.e., that there is no operation in $\mathcal{O}_{c-max}$ which is not also in $\mathcal{O}_{sep}$.

(iii) Consider the case of bipartite separable states $\mathcal{F}_{2s}$. Show that any bipartite separable state can be written as a classical mixture of pure, orthonormal, bipartite product states (i.e., states of the form $|a\rangle_A |b\rangle_B$). Is any classical mixture of product states a separable state? (*Remark:* since this technically includes mixtures with a single component, note that the only separable pure states are product states).

(iv) Consider again $\mathcal{F}_{2s}$. The set $\mathcal{O}_{max}$ induced by $\mathcal{F}_{2s}$ is the set of *non-entangling operations* $\mathcal{O}_{ne}$. The swap operation $SWAP$ is implicitly defined by $SWAP(|a\rangle_A |b\rangle_B) = |b\rangle_A |a\rangle_B$ for any pair of systems and states. Show that for $\mathcal{F}_{2s}$, $\mathcal{O}_{c-max} \subsetneq \mathcal{O}_{max}$ by proofing that $SWAP \in \mathcal{O}_{ne}$ and $SWAP \notin \mathcal{O}_{c-max}$.

Problem 6.3: Pure states: maximal entanglement, and catalysis (3+4 points)

Consider bipartite pure states. For locally two-dimensional systems the maximally entangled states are the Bell states.

(i) Write down the Schmidt decomposition of the Bell state $|\phi^-\rangle$.

For two d-dimensional systems we define

$$|\phi_d\rangle := \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$$

All states $|\psi_d\rangle$ that are related to $|\phi_d\rangle$ by local unitary operations as $U_A \otimes U_B |\phi_d\rangle$ are called 'maximally entangled'.

(ii) Considering only pure states at first, argue why the term 'maximally entangled' is justified for these states (i.e., states of the form $U_A \otimes U_B |\phi_d\rangle$) and no other pure states on the same space.

(iii) Extending the argument, explain why $|\phi_d\rangle$ can be converted into any bi-partite mixed state on the same Hilbert space by LOCC.

(iv) *(optional)* Consider the following two states

$$|\chi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle + \frac{1}{\sqrt{2}}|22\rangle$$
$$|\psi\rangle = \frac{1}{\sqrt{10}}|00\rangle + \sqrt{\frac{2}{5}}|11\rangle + \sqrt{\frac{2}{5}}|22\rangle + \frac{1}{\sqrt{10}}|33\rangle$$

Explain why neither one of these be converted into the other by LOCC.
Interestingly, there are bipartite states $|\gamma\rangle$ such that $|\psi\rangle \otimes |\gamma\rangle$ can be converted to $|\chi\rangle \otimes |\gamma\rangle$ by LOCC – even though $|\psi\rangle$ cannot be converted into $|\chi\rangle$ on its own. This effect is called *catalysis* and the enabling state $|\gamma\rangle$ is called a *catalyst*. (Its dimension, or rather that of its Hilbert space, is finite but completely arbitrary.) Like in chemistry, the catalyst enables a transition that is not possible without it, but it is not consumed in the process.
Is there any catalyst $|\gamma'\rangle$ that allows us to convert $|\chi\rangle$ into $|\psi\rangle$, i.e. $|\chi\rangle \otimes |\gamma'\rangle \to |\psi\rangle \otimes |\gamma'\rangle$? Substantiate your answer with a proof.
Remark: Here, $|\psi\rangle, |\chi\rangle \in \mathcal{H_A} \otimes \mathcal{H_B}$ and $|\gamma\rangle \in \mathcal{H}_C \otimes \mathcal{H}_D$. Consider the bipartition $AC|BD$.

(v) *(optional)* For the states in the previous question, find a catalyst $|\gamma\rangle$ that enables the LOCC conversion from $|\psi\rangle \otimes |\gamma\rangle$ to $|\chi\rangle \otimes |\gamma\rangle$.
Hint: There are solutions where the catalyst is a qubit. Even so, you might find it helpful to use a computer to find $|\gamma\rangle$.

Problem 6.4: Multipartite entanglement, GHZ and W states (5 points)

Consider the following two tri-partite qubit states:

$$|W\rangle := \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$
$$|GHZ\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

(i) Show that both states are entangled (i.e. not separable) across all possible bi-partitions of the tri-partite space.
Remark: The total Hilbert space is tri-partite, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. In a bi-partition of

this space two of the local Hilbert spaces are grouped together. For instance, Bob may have access to systems B and C: $\mathcal{H}_{B'} = \mathcal{H}_B \otimes \mathcal{H}_C$ and we consider $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{B'}$.
Digression: Entanglement and free operations are always with respect to a given Hilbert space structure. For instance, when we are studying the state of a many-partite system such as a spin chain the question 'how much entanglement does the state (or 'system') have?' is only meaningful if the Hilbert space structure is clear from context (e.g., bipartite entanglement between a single spin and the rest of the spin chain, the left and the right side of the chain with respect to the centre, etc).

(ii) For any bi-partition, can $|W\rangle$ be converted to $|GHZ\rangle$ by LOCC? Is the inverse transformation possible?

(iii) For both states, consider the bipartition $AB|C$. For each state argue if there exists an LOCC protocol which results in a state $|a\rangle_A |\psi\rangle_{BC}$ where ψ is a Bell state. If yes, specify the protocol. If no, explain why the transformation is not possible with LOCC.
Hint: For one possible solution for $|GHZ\rangle$, you may start by re-expressing it such that the state of the first qubit is written in the Pauli-X eigenbasis.

(iv) Returning to the tri-partite scenario, it can be shown that $|GHZ\rangle$ cannot be converted to $|W\rangle$ by (tri-partite) LOCC (you are not expect to proof this here). State why the inverse transformation $|W\rangle \to |GHZ\rangle$ is also impossible under LOCC.