Exercises - Week 7

Remark: Throughout this problem set we only consider bi-partite entanglement. That is, a Hilbert space structure $\mathcal{H}_A \otimes \mathcal{H}_B$ is implied in all problems that deal with entanglement.

Problem 7.1: Entanglement monotones and exact, single-shot transformations (5 points)

Let $\{\gamma_i(\psi)\}$ be the decreasingly ordered Schmidt coefficients of a state $|\psi\rangle$ (*Remark:* i takes values from 1 to the Schmidt rank d_s. All γ_k with $k > d_s$ are zero). Recall Nielsen's theorem for state transformations $|\psi\rangle \to |\chi\rangle$ and use it to answer the following for LOCC and pure states:

(i) All of the following are entanglement monotones for pure states:

$$E_k(|\psi\rangle) := \sum_{i=k}^{\infty} \gamma_i(\psi)$$

Proof that an LOCC conversion $|\psi\rangle \to |\chi\rangle$ is possible if and only if $E_k(|\psi\rangle) \geq E_k(|\chi\rangle)$ for all k. Which of the E_k are entanglement measures (i.e. weakly discriminant in addition to being monotonic)? Are any of them faithful entanglement measures (i.e., strictly discriminant)?
Hint: Note that the sum can be computed with the upper limit capped at $d = \max(\dim_A, \dim_B)$. All summands for $i > d$ (or in fact d_S) vanish.

(ii) Proof that for all Renyi entropies S_α the following are entanglement monotones for pure states:

$$E_{S_\alpha}(|\psi\rangle) := S_\alpha(\mathrm{tr}_B\, |\psi\rangle\langle\psi|)$$

Hint: The results of Problem 3.4 will be useful here.

Problem 7.2: Asymptotic state conversion (4 points)

(i) Use the fact that $E_C = E_D$ for pure states to show that pure state LOCC conversion is always asymptotically reversible in the sense that if $|\psi\rangle$ can be asymptotically converted to $|\chi\rangle$ at a rate $r_f = k/n$ than the inverse conversion $|\chi\rangle \to |\psi\rangle$ is asymptotically possible at a rate $r_r = n/k$.

(ii) Then use the uniqueness theorem to show that the state conversion $|\psi\rangle \to |\chi\rangle$ is asymptotically possible at a rate

$$r = \frac{EoE(|\psi\rangle)}{EoE(|\chi\rangle)}.$$

Hint: r is the optimal rate k/n for $|\psi\rangle^{\otimes n} \to |\chi\rangle^{\otimes k}$ in the asymptotic limit ('the most k per n'). Recall that E_C is defined 'inversely' to E_D ('the least n per k').

Problem 7.3: Regularisation (2 points)

A resource monotone f that is not extensive (i.e., $f(\rho^{\otimes n}) \neq nf(\rho)$) can often be made extensive by introduction of its regularised variant f^∞:

$$f^\infty(\rho) := \lim_{n\to\infty} \frac{f(\rho^{\otimes n})}{n}$$

Proof that for any f for which f^∞ exists, f^∞ is indeed extensive.

Problem 7.4: Properties of entanglement monotones (12 points)

In this exercise we will examine some of properties and desiderata for entanglement monotones more closely by focusing on specific examples.

(i) For pure states, the **entropy of entanglement (EoE)** is defined as the von Neumann entropy S of one of the reduced states:

$$EoE(|\psi\rangle) := S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$$

The EoE is an entanglement measure for pure states. We have already shown its monotonicity (Problem 7.1).

Proof the following:

a) $S(\mathrm{tr}_A |\psi\rangle\langle\psi|) = S(\mathrm{tr}_B |\psi\rangle\langle\psi|)$.

b) EoE is normalised.

c) EoE is a faithful for pure states.

d) For mixed states, $S(\mathrm{tr}_B \rho)$ is neither monotonic nor even weakly discriminant.

e) EoE is additive (and hence extensive).

(ii) Show that the **Schmidt rank** is an entanglement monotone for pure states. How can we turn the Schmidt rank from a monotone into a measure (i.e., by making it weakly discriminant)? Is this derived measure faithful?

(iii) We can formally construct a **resource monotone from a distance function d** by defining:

$$E_d(\rho) := \inf_{\varphi \in \mathcal{F}} d(\rho, \varphi)$$

(a) Show that any E_d defined in this way is faithful.

(b) Show that if d obeys a data processing inequality $d(\rho,\sigma) \geq d(\Lambda(\rho),\Lambda(\sigma))$ for all quantum channels Λ (not just free ones) then E_d is monotonic under all non-entangling operations.

Remark: Recall the requirements that a distance function has to fulfill:

- $d(x,y) \geq 0$ *(non-negativity)*
- $d(x,y) = 0 \Leftrightarrow x = y$ *(identity of indiscernibles)*
- $d(x,y) = d(y,x)$ *(symmetry)*
- $d(x,z) \leq d(x,y) + d(y,z)$ *(triangle inequality)*

(iv) Recall the **relative entropy** $R(\rho||\sigma) := -S(\rho) - \mathrm{tr}[\rho \log \sigma]$. R is not a distance on state space because it is neither symmetric $R(\rho||\sigma) \neq R(\sigma||\rho)$ nor does it satisfy the triangle inequality. This is equally true of its classical variant (called Kullback-Leibler divergence).

a) Show that R is not symmetric by finding two states, ρ and σ as an example.

Regardless of this, R can still be used as an entanglement monotone E_R using the same construction as for E_d as above.

b) The quantum relative entropy obeys joint convexity in its arguments, i.e., for $p+q=1$ and states ρ,σ,α,β::

$$R(p\rho + q\sigma||p\alpha + q\beta) \leq pR(\rho||\alpha) + qR(\sigma||\beta)$$

Use this to show that E_R is convex.

Remark: We have previously labelled both the relative entropy and the trace distance with D. On this problem set, E_D is the distillable entanglement. d stands for any distance function on state space that fulfils the requirements listed above. This includes the trace distance among others. In fact, as we can see from the last part of the problem, some of the conditions are not even necessary for the present purpose of constructing an entanglement monotone.