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http://www.mpmueller.net/resource.html

Exercises - Week 9

In the following, we say that a state ρ is *diagonal* for a quantum system S with Hamiltonian H_S if and only if $[\rho, H_S] = 0$. Equivalently, there is some (energy) eigenbasis of H_S such that the matrix representation of ρ in this basis is a diagonal matrix. The *Gibbs state at inverse temperature* β is written $\gamma_\beta := \exp(-\beta H_S)/Z$, where $Z \in \mathbb{R}$ is normalized such that $\mathrm{tr}(\gamma_\beta) = 1$.

Problem 9.1: Passive and completely passive states in low dimensions (8 points)

(i) Argue that every density matrix of full rank can be seen as Gibbs state if you choose the Hamiltonian correctly. (I.e. show that every full rank density matrix can be written as $\rho = e^{-\beta H}/Z$ for appropriate choice of H and $\beta \geq 0$).

(ii) Show that for a *fixed* diagonal Hamiltonian H of dimension two, *every* diagonal density matrix of full rank is a Gibbs state for some appropriate inverse temperature $\beta \in \mathbb{R}$. For what states is β non-negative? What does this imply about the relationship between the sets of passive and completely passive states in two dimensions?

(iii) In Lecture 8, we saw an example of a three-level state that is passive but not completely passive, with the Hamiltonian $H = 0|0\rangle\langle 0| + 2|2\rangle\langle 2| + 3|3\rangle\langle 3|$. Find a *full-rank* state that is passive under this Hamiltonian, but is 2-active (and hence not completely passive).

Problem 9.2: General properties of thermal operations (6 points)

Consider thermal operations $\Phi : S \to S$, where S is a finite-dimensional quantum system.

(i) Argue why if $H = 0$ the set of thermal operations reduces to the set of *noisy operations* (as defined in Lecture 2).

(ii) Show that all thermal operations are *Gibbs preserving* – i.e. they leave the Gibbs state of the system invariant, $\Phi(\gamma_S) = \gamma_S$.

(iii) Show that the image of any diagonal state under thermal operations is diagonal. Thus, no superpositions of energy eigenstates can be created by thermal operations.

Problem 9.3: Transition rates (5 points)

The optimal rate of interconversion $R(A \to B)$ between two resources A and B is defined as the largest asymptotic value $R := \lim_{n\to\infty} \frac{m_n}{n}$ for integers n,m_n for which the state conversion $A^{\otimes n} \xrightarrow{\epsilon} B^{\otimes m_n}$ is possible via the allowed operations of the resource theory, where $\xrightarrow{\epsilon}$ indicates a transition up to some error $\epsilon > 0$ (assume here that the rate does not depend on the choice of ϵ).

Suppose A, B and C are three resources (e.g. quantum states), and all transitions between A, B and C are possible and finite (i.e. $0 < R(X \to Y) < \infty$ for all $X, Y \in \{A, B, C\}$). We call a resource theory *asymtotically reversible* if $R(A \to B)R(B \to A) = 1$.

(i) Explain briefly what this condition means. In particular, what would it mean if there were A and B such that $R(A \to B)R(B \to A) < 1$? What could go wrong with the resource theory if $R(A \to B)R(B \to A) > 1$?

(ii) Argue why rates in an asymptotically reversible resource theory satisfy

$$R(A \to B) = \frac{R(A \to C)}{R(B \to C)}. \tag{1}$$

This relation allows us to calculate the transition rates between A and B in terms of *distillation to* and *formation from* some "standard" resource state C. For example, in the resource theory of athermality, C could be a pure state in an excited energy level. The optimal distillation procedure in this case is outlined in Lecture 9.

(iii) Write down the expression for asymptotic rates in the resource theory of non-uniformity (recall Lecture 4) – and show that this implies the theory is asymptotically reversible. What would make a good choice of standard resource state C?

Problem 9.4: Transitions with a small environment (8 points)

Consider a two-dimensional quantum system S and a two-dimensional heat bath B with respective Hamiltonians

$$H_S = \begin{pmatrix} 0 & 0 \\ 0 & \Delta E \end{pmatrix}, \qquad H_B = \begin{pmatrix} E & 0 \\ 0 & E + \Delta E \end{pmatrix}. \tag{2}$$

Let $\beta \geq 0$ be some arbitrary but fixed inverse temperature.

(i) Show that the Gibbs states of the system and the bath are identical, i.e. $\gamma_S = \gamma_B$.

(ii) Let the system be in the diagonal state $\rho_S = \mathrm{diag}(p, 1-p)$, where $0 \leq p \leq 1$. Show that under thermal operations the transition $\rho_S \mapsto \gamma_S$ is possible, and give a unitary that (together with a partial trace) achieves this transition.

(iii) What other state transitions are possible with this environment? That is, give the set of states reachable through thermal operations as a subset of the set of two-component probability vectors. (Recall from Problem 9.2 that since the states are all diagonal, we can represent them as probability vectors.)

Hint: Argue that we obtain transitions from $\mathrm{diag}(p, 1-p)$ to $\mathrm{diag}(p', 1-p')$, where

$$\begin{pmatrix} p' \\ 1-p' \end{pmatrix} = B \begin{pmatrix} p \\ 1-p \end{pmatrix} \tag{3}$$

for some matrix B. Use known properties of B to determine all possible p'.

Problem 9.5: Gibbs-preserving maps can create coherence (10+3 points)

Let us consider a qubit system with the following Hamiltonian H and Gibbs state γ

$$H = \Delta E|1\rangle\langle 1|, \qquad \gamma = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1|, \qquad p_0 = \frac{1}{1+e^{-\beta\Delta E}}, \qquad p_1 = \frac{e^{-\beta\Delta E}}{1+e^{-\beta\Delta E}}, \tag{4}$$

where $\beta \geq 0$. We have seen in Problem 9.2 that thermal operations preserve the Gibbs state of the system, but cannot map diagonal states to states with off-diagonal terms (in the case of a non-degenerate Hamiltonian). We will now explicitly construct a map from qubit density matrices to qubit density matrices that preserves the Gibbs state, but creates off-diagonal terms. Thus the thermal operations can achieve less state transitions than the Gibbs-preserving maps.

We define the map

$$\Phi(\cdot) := \langle 0| \cdot |0\rangle\, \sigma + \langle 1| \cdot |1\rangle\, \rho \tag{5}$$

where ρ and σ are two fixed, normalized qubit density matrices.

We first establish that Φ is indeed a valid quantum operation.

(i) Show that Φ is linear and trace-preserving.

(ii) Show that Φ maps positive matrices to positive matrices and so is a positive map.

(ii*) **Bonus (+3):** Show moreover that Φ is completely positive. [One possible path: find a prescription for a Kraus operator expression of eq. 5 in terms of the eigenvalues/vectors of σ and ρ.]

(iii) Give a (rough, qualitative!) description of how such a map might be implemented on a system, in terms of measurement and preparation.

Now we show that such a map can be Gibbs preserving, yet still create coherence.

(iv) Fix some arbitrary density matrix ρ. How do we have to choose σ that $\Phi(\gamma) = \gamma$? Show that this choice of σ has trace one and is a positive matrix. [You can use the following fact: If A and B are positive matrices and all eigenvalues of B are smaller than (or equal to) all eigenvalues of A, then $A - B$ is a positive matrix.]

(v) Show that Φ can map a diagonal density matrix to a density matrix with off-diagonal terms.

(vi) Draw a picture of the Bloch ball (set of all qubit density matrices) and explain what Φ does geometrically.