Exercises - Week 10

In the resource theory of nonuniformity, we used majorization to check if two states are interconvertible. The resource theory of athermality makes use of a more general concept. Recall from Problem 2.4 that for probability vectors p and q we have $p \succ q$ if and only if there exists some bistochastic matrix B with $q = Bp$. Bistochastic matrices were defined as stochastic matrices that map the maximally mixed state to itself. We can generalize this and define:

Definition 1. *A stochastic matrix S is called d-stochastic if it leaves the probability vector d invariant, i.e. $Sd = d$.*

We use this to define d-majorization as

Definition 2. *Let p, q and d be probability vectors. We say that q is d-majorized by p, written as $p \succ_d q$, if there exists some d-stochastic matrix S with $q = Sp$.*

Problem 10.1: d-stochastic matrices and d-majorization (8 points)

Let us explore the set of states that are d-majorized by some vector in two dimensions.

(i) Give the set of all 2×2 bistochastic matrices, and use this to give the set of vectors majorized by some arbitrary probability vector p.

(ii) Show that the set of all 2×2 d-stochastic matrices for a general $d = (d_1, d_2)^{\mathrm{T}}$ can be written in the form

$$\begin{pmatrix} 1-t & d_1/d_2\, t \\ t & 1 - d_1/d_2\, t \end{pmatrix}. \tag{1}$$

For arbitrary probability vector d, for what range of t is the matrix a valid d-stochastic matrix?

(iii) Give the set of states d-majorized by some p. Illustrate this set with a diagram.

Problem 10.2: Relative Rényi entropies as monotones for d-majorization (6 points)

Recall Problem 3.4, where it was shown that all Rényi entropies are Schur concave. A similar set of functions with an analogous property exists for d-majorization.
Define the *relative Rényi entropies* as

$$D_\alpha(p\|q) := \frac{\operatorname{sign}(\alpha)}{\alpha - 1} \log \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha} \qquad \text{with } \alpha \in \mathbb{R}\backslash\{1\}, \tag{2}$$

where we set $0^0 := 0$ and $\operatorname{sign}\alpha := \left\{ \begin{array}{cc} 1 & \alpha \geq 0 \\ -1 & \alpha < 0 \end{array} \right\}$.

Show that $\lim_{\alpha\to 1} D_\alpha(p\|q) = D(p\|q)$, where $D(p\|q) := \sum_i^n p_i \log \frac{p_i}{q_i}$ is the *Kullback-Leibler divergence*.

Review the theorem on pages 11–12 of Lecture 10, and use this to show that

$$(p, d) \succ (p', d') \quad \Rightarrow \quad D_\alpha(p\|d) \geq D_\alpha(p'\|d') \quad \text{for all } \alpha. \tag{3}$$

Problem 10.3 (Optional): Thermal operations and off-diagonal terms (+7 points)

We define the time-averaging map (or dephasing map) on density matrices as

$$\Delta(\rho) = \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{itH} \rho\, e^{-itH} dt. \tag{4}$$

If we write in the eigenbasis of H (i.e. take H to be diagonal), this maps a matrix ρ to $\Delta(\rho)$ which has the same (block)-diagonal elements as its preimage, but contains otherwise only zeroes. That is, Δ sets all off-diagonal (or off-block-diagonal) entries of a matrix to zero.

(i) Prove this property explicitly by calculating the integral in the definition. (To do this, first expand ρ in the energy eigenbasis of H.)

(ii) Show furthermore that Δ commutes with thermal operations, i.e. that $\tau(\Delta(\rho)) = \Delta(\tau(\rho))$ for any thermal operation τ. Argue why this implies that thermal operations map block-diagonal states to block-diagonal states (i.e. cannot create coherences).

Problem 10.4: Thermal operations and non-degenerate Hamiltonians (8 points)

In the following, let us assume that we always start with the system in a diagonal state with respect to the energy eigenbasis. Furthermore, we restrict ourselves to a subset of thermal operations where the only system traced out is the heat bath, such that the Hamiltonian and dimension of the quantum system under consideration does not change.

Assume that you have a quantum system with Hamiltonian H_S and a heat bath system with Hamiltonian H_B, and look at the combined Hamiltonian $H = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_B$.

(i) Assume furthermore that H is non-degenerate. Show that unitary matrices with $[H, U] = 0$ have to be diagonal as well, and are thus of the form $U = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$. Why is the non-degeneracy of H important here?

Now let γ_B be the Gibbs state of the heat bath and ρ_S be a diagonal state of our quantum system.

(ii) Show that, under the assumptions on H as above (including non-degeneracy), we have

$$U\,(\rho_S \otimes \gamma_B)\,U^\dagger = \rho_S \otimes \gamma_B \tag{5}$$

for all U with $[U, H] = 0$, and thus as well

$$\mathrm{Tr}_B\left[U\,(\rho_S \otimes \gamma_B)\,U^\dagger\right] = \rho_S. \tag{6}$$

This shows that in the case that H is non-degenerate, the only thermal operation we can have is the identity!

Clearly, in general there are non-trivial thermal operations. For instance, let us revisit the system/environment of Problem 9.4:

$$H_S = \begin{pmatrix} 0 & 0 \\ 0 & \Delta E \end{pmatrix}, \qquad H_B = \begin{pmatrix} E & 0 \\ 0 & E+\Delta E \end{pmatrix}, \qquad \gamma_B = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \quad \text{and } \Delta E > 0. \tag{7}$$

(iii) Write out the combined Hamiltonian $H = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_B$. Why (briefly) does the result of (ii) not hold here?

Problem 10.5: Thermal operations for the qubit via the harmonic oscillator (16 points)

In this exercise, we will prove the following statement, which is a special case of the theorem discussed on pages 8–11 of Lecture 10:

A diagonal qubit state can be mapped to another diagonal qubit state via thermal operations if and only if there exists a stochastic matrix mapping the vectors of diagonal elements into each other and leaving the vector of diagonal elements of the system's Gibbs state invariant.

We argue the *only if* part of this statement first. First recall Problem 9.2, where it was shown that thermal operations cannot produce any off-diagonal terms. This means that thermal operations are linear maps from diagonal matrices to diagonal matrices, and thus, when we restrict our initial states to be diagonal, can be also seen as linear maps from vectors to vectors (whose elements are given by those diagonal elements). That is, for any thermal operation T mapping matrix with diagonal $\vec{p}$ to matrix with diagonal $\vec{q}$, we can write a linear map B_T such that $\vec{q} = B_T\vec{p}$.

(i) Argue why these linear maps B_T are stochastic matrices.

Thermal operations are also Gibbs–preserving (i.e. they the Gibbs state of the system invariant – you may have already proven this last week in Problem 9.2). Writing the Gibbs state vector as $\vec{\gamma} = (\gamma_1, \gamma_2)^{\mathrm{T}}$, it follows that the associated maps are thus γ-stochastic, ($\vec{\gamma} = B_\Phi\vec{\gamma}$ concluding the *only if* part of the statement.

Let us check this explicitly, with the qubit system and environment of Eq. 7 above. Denote by Φ the thermal operation from density matrices to density matrices:

$$\Phi(\rho_S) = \mathrm{Tr}_B\left[U\left(\rho_S \otimes \gamma_B\right)U^\dagger\right], \tag{8}$$

where

$$U := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

acts to interchange the second and third components of the combined system/bath state.

(ii) Argue why the stochastic matrix B_Φ corresponding to Φ is $B_\Phi = \left(\begin{smallmatrix} \gamma_1 & \gamma_1 \\ \gamma_2 & \gamma_2 \end{smallmatrix}\right)$, where $\gamma_S = \gamma_B = \mathrm{diag}(\gamma_1, \gamma_2)$. Check that it is stochastic and Gibbs-preserving.

Now we move onto the *if* part of the statement – namely, we must argue that for every γ-stochastic matrix, we can construct a thermal operation that implements it. Recall the generic d-stochastic matrix in (Eq. 1 in Problem 10.1). For each environment and specific unitary used in some thermal operation on a qubit, we can give some parameter t to describe to which $\vec{\gamma}$-preserving stochastic matrix this thermal operation corresponds.

(iii) To what value of t does the above stochastic matrix B_Φ correspond? Argue that for some general (possibly non-qubit) environment and unitary in the thermal operation, we just need to calculate $\mathrm{Tr}_B\left[U\left(\mathrm{diag}(1,0) \otimes \gamma_B\right)U^\dagger\right]$ to determine the corresponding value of t.

We will now choose an environment and a unitary to get close to the maximal value $t = \gamma_2/\gamma_1$. We choose H_S as in Eq. 7, but now use an n-level heat bath with Hamiltonian

$$H_{B'} = \begin{pmatrix} 0 & & & & \\ & \Delta E & & & \\ & & 2\,\Delta E & & \\ & & & \ddots & \\ & & & & (n-1)\Delta E \end{pmatrix} = \sum_{j=1}^{n} (j-1)\,\Delta E\,|j\rangle\langle j|, \tag{10}$$

where each $|j\rangle$ is the j^{th} energy eigenstate of the bath. We will write the corresponding thermal state as $\gamma_B = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_n)$ where $\delta_i = e^{-\beta(i-1)\Delta E}/Z$.

(iv) Write down the form of the combined Hamiltonian $H' = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_{B'}$, and also the state $\mathrm{diag}(1,0) \otimes \gamma_B$. Argue that there exists a unitary V such that $[V, H'] = 0$ and

$$\mathrm{Tr}_B \left[V \left(\mathrm{diag}(1,0) \otimes \gamma_B \right) V^\dagger \right] = \mathrm{diag}(\delta_1, \delta_2 + \cdots + \delta_n). \tag{11}$$

[Hint: Take inspiration from the form of Eq. 9 and note that it acts to swap the two elements of the combined state that correspond to the same energy in the combined Hamiltonian. Construct a unitary V that implements a series of similar pairwise swaps on the combined state between pairs of elements that have the same energy in the Hamiltonian H'. Show that this unitary is energy-preserving, and that it leads to Eq. 11.]

(v) Show that for the operation in Eq. 11, $t = \sum_{i=2}^{n} \delta_i$. Show that as $n \to \infty$, $t \to \gamma_2/\gamma_1 = e^{-\beta \Delta E}$.

Thus far, we can explicitly give Hamiltonians and unitaries for the environment either corresponding to $t = 0$ (the identity), the value of t supplied in part (iii), and other values arbitrarily close to $t = \gamma_2/\gamma_1$ from part (v). This helps us to construct a unitary and an environment for *all* $t \in [0, \gamma_2/\gamma_1)$, as would be required to satisfy the *if* part of the statement at the start of the question. To see this, let us return to the qubit environment case, and consider the unitaries:

$$U_\alpha := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

(vi) Show that by continuously varying the parameter $\alpha \in [0, \frac{\pi}{2}]$, we can find γ-stochastic matrices ranging from $t = 0$ to the value of t supplied in part (iii).

Clearly, a similar strategy applied to the constituent parts of the unitary V in part (iv) allows us to range $t = [0, \frac{\gamma_2}{\gamma_1})$ provided we choose large enough n. Thus, for every γ-stochastic matrix on a qubit, we have found a thermal operation that implements it, concluding the *if* part of the statement.

This exercise has thus shown that for the qubit, the problem of interconvertibility under thermal operations is reduced to the concept of d-majorization. This is still true for larger systems, but for these a more general proof is needed (as sketched in Lecture 10).