Summer Semester 2020
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Exercises - Week 11

Problem 11.1: Relative Rényi entropies as monotones for d-majorization (part 2) (7 points)

Recall the definition of the *relative Rényi entropy* as

$$D_\alpha(p\|q) := \frac{\operatorname{sign}(\alpha)}{\alpha-1}\log\sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha} \qquad \text{with } \alpha\in\mathbb{R}\backslash\{1\}, \tag{1}$$

where we set $0^0 := 0$ and $\operatorname{sign}\alpha := 1$ if $\alpha \geq 0$ and -1 otherwise. In Problem 10.2, you may have already proven some properties of the relative Rényi entropies. Here, you will prove some more properties.

(i) Show that

$$\lim_{\alpha\to 0^+} D_\alpha(\,p\,\|\,q\,) = -\log\sum_{p_i\neq 0} q_i. \tag{2}$$

(Note: $\lim\alpha\to 0^+$ *means* α *approaching* 0 *from above).* Under what conditions is this expression finite?

The quantity on the right-hand side is thus called $D_0(p\|q)$.

(ii) Assume that q does not contain any zero entries and show that

$$\lim_{\alpha\to\infty} D_\alpha(\,p\,\|\,q\,) = \log\max_i\left\{\frac{p_i}{q_i}\right\}. \tag{3}$$

(Hint: begin by factoring out the term $q_j(p_j/q_j)^\alpha$ from the sum, where j is chosen so as to maximize p_j/q_j.)

(iii) Show that in the case when $q=\eta=(1/d,\,\ldots,1/d)\in\mathbb{R}^d$ is the maximally mixed state, the relative Rényi entropy (for $\alpha\geq 0$) reduces to the already known Rényi nonuniformity I_α :

$$D_\alpha(\,p\,\|\,\eta\,) = \log d - H_\alpha(p) \quad \text{with} \quad \eta = (1/d\,,\ldots,1/d). \tag{4}$$

Problem 11.2: Rescaled Lorenz-curves as condition for relative majorization (14 points)
Consider the case where we have a system with two diagonal density matrices ρ and σ and a given Gibbs state γ, and we want to know if ρ and σ are interconvertible under thermal operations. We already know that this is equivalent to the statement that there exists a stochastic matrix mapping the diagonal elements of the matrices into each other while leaving the diagonal elements of the Gibbs state invariant (i.e. a γ-stochastic matrix). However, this is not a straightforward condition to check, since it is not clear how to find this matrix. In this exercise, we will see one way how to do this, and learn why it is equivalent to looking at the *thermal* Lorenz curves (i.e. see the equivalence of statements (i) and (iii) of the theorem on pages 11–12 of Lecture 10). We will show this for an example, but ultimately it should be clear how this construction can be generalized.

Consider the three probability vectors:

$$\gamma = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad p = \frac{1}{12}\begin{pmatrix} 10 \\ 2 \end{pmatrix}, \quad q = \frac{1}{12}\begin{pmatrix} 3 \\ 9 \end{pmatrix}. \tag{5}$$

(i) Fist draw a diagram of the standard Lorenz curves for γ, p, q, and another diagram for the rescaled (with respect to γ) Lorenz curves (see page 12 of Lecture 10 for instructions about how to do this). If drawn correctly, these should imply different orders. What relation under γ-majorization does the rescaled diagram show? How does this diagram show that all states $\gamma-$majorize the state γ?

Next, we will uase a procedure that is called *Gibbs-rescaling*. For this we divide the first level of the Gibbs state γ into two *fictious* levels "$1a$" and "$1b$" each with the same energy. That is, rather than having probability $2/3$ to find the system in level 1, we instead have probability $1/3$ to find it in level 1a and the same probability to find it in level 1b. As a result of this splitting, we have a new Gibbs state $\tilde{\gamma}$ that is maximally mixed $(1/3, 1/3, 1/3)^{\mathrm{T}}$. Formally, this procedure can be expressed as multiplying γ by some rectangular matrix G with *left–inverse* G^{-1}.

$$G = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}, \qquad G^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

(ii) Verify that $G^{-1}G = \mathbb{1}$ (the identity). What is GG^{-1} ?

(iii) Calculate $\tilde{p} = Gp$ and $\tilde{q} = Gq$. On a new diagram, draw the (standard) Lorenz curves for $\tilde{\gamma}$, $\tilde{p}$, $\tilde{q}$.

(iv) Outline briefly how for a general choice of γ with rational entries, some G such that $G\gamma$ is maximally mixed could be chosen. (You do not need to provide a full formal proof.) Explain why (even for general γ and G) the Lorenz curves drawn give the same diagram as the γ–rescaled Lorenz curves of γ, p and q.

Thus, the Gibbs-rescaling map G takes our states to some higher-dimensional states such that the Gibbs state now corresponds to the maximally mixed state. Returning to our specific example, instead of γ-majorization in two dimensions, we now have regular majorization in three dimensions. Looking at the Lorenz curves, we see that $\tilde{q} \succ \tilde{p}$ which suggests $q \succ_\gamma p$. Since $\tilde{q}$ majorizes $\tilde{p}$, we know there exists some bistochastic B with $\tilde{p} = B\tilde{q}$.

We can construct B in the following way: Let B be the product of three matrices P, T, S, where P is some permutation matrix, and S and T are T-transforms (bistochastic matrices that act non-trivially in just two dimensions):

$$\tilde{p} = B\,\tilde{q} = STP\,\tilde{q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s & 1-s \\ 0 & 1-s & s \end{pmatrix} \begin{pmatrix} t & 1-t & 0 \\ 1-t & t & 0 \\ 0 & 0 & 1 \end{pmatrix} P\,\tilde{q} \tag{7}$$

(v) Now choose the matrices S, T, P in the following way: Write down a P such that $P\tilde{q}$ is ordered decreasingly. Find a T (and thus t) such that the first components of $P\tilde{q}$ and $\tilde{p}$ are equal. Find a S (and thus s) such that the second components of $TP\tilde{q}$ and $\tilde{q}$ are equal. (Due to normalization we should now have $B\tilde{q} = \tilde{p}$). Calculate explicit forms of B and $\tilde{B} = G^{-1}B\,G$. Check that $\tilde{B}\gamma = \gamma$ and $\tilde{B}q = p$ and thus $q \succ_\gamma p$.

(vi) Describe briefly how this is done in a general setting, and explain why this is a proof of the aforementioned theorem in Lecture 10. What would steps would need to be changed if the Gibbs state changes during the thermal operation (i.e. if the Hamiltonian and/or the dimension of the system changes)?

The restriction to Gibbs states with only rational components is not really a problem, since for any γ, we can always choose some γ_r with rational entries which is arbitrarily close to γ. However, the proof above can also be done in an exact fashion for irrational Gibbs states. Then we have to convert our probability vectors to continuous probability densities over the interval $[0,1]$ and work with integral transforms (see, for instance, Ruch et al. (1978), doi:10.1063/1.436364).

(Additional reading on the topic of Gibbs–rescaling may be found in arXiv:1207.0434.)

Problem 11.3 (Optional): The work cost of creation using a battery (10 points)

Before attempting this question, first review Section 3.8 of Lecture 11. You here will prove the more general statement for a system with a battery – namely, you will show: *There is a thermal operation that transforms a system from thermal state γ_S to some target diagonal state ρ_S, accompanied by a drop in the battery's energy from state E to 0,*

$$\gamma_S \otimes |E\rangle\langle E| \overset{\text{thermal}}{\longrightarrow} \rho_s \otimes |0\rangle\langle 0|, \tag{8}$$

if and only if

$$E \geq F_\infty(\rho_S) - F(\gamma_S). \tag{9}$$

(Note that $F(\gamma_S) = F_0(\gamma_S) = F_\infty(\gamma_S)$.)

As usual, we will consider finite-dimensional diagonal states, so that states of the system and battery may be written as probability vectors. We denote the system and battery's respective Gibbs states as γ_S and γ_B. For combined states of the joint system, it is helpful to adopt a two-index notation "x_{ij}" to denote the joint probability of the system being in state i and the battery in state j for some state x.

(i) Write down an expression for the elements s_{ij} of the initial state $s := \gamma_S \otimes |E\rangle\langle E|$, and show that s's γ-Lorenz curve rises at a constant gradient of $\frac{1}{(\gamma_B)_{j'}}$ from $(0,0)$ to $((\gamma_B)_{j'}, 1)$, and is then subsequently flat.

(ii) Now, we choose some arbitrary final state $r := \rho_S \otimes |0\rangle\langle 0|$, formed from some general choice of ρ_S with diagonal elements p_i, and fixed choice of battery $|0\rangle\langle 0|$. Defining $i', j' := \operatorname{argmax}_{i,j} \frac{r_{ij}}{\gamma_{ij}}$ show that the initial gradient of r's γ-Lorenz curve G is given by

$$G := \frac{p_{i'}}{(\gamma_S)_i(\gamma_B)_1}. \tag{10}$$

(iii) Explain why

$$\frac{1}{(\gamma_B)_{j'}} \geq \frac{p_{i'}}{(\gamma_S)_i(\gamma_B)_1} \tag{11}$$

is a necessary and sufficient condition for $s \succ_\gamma r$.

Since thermo–majorization is a necessary and sufficient condition for the existence of a thermal operation implementing Eq. 8, Eq. 11 is also such a condition.

(iv) Show that Eq. 11 is equivalent to

$$E \geq F_\infty(\rho_S) - F(\gamma_S). \tag{12}$$