An Introduction To Resource Theories (Example: Nonuniformity Theory)

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Seminar: *Selected topics in Mathematical Physics: Quantum Information Theory*

Abstract

This document corresponds to the presentation *An Introduction To Resource Theories (Example: Nonuniformity Theory)* given by Marius Krumm in the master studies seminar course *Selected topics in Mathematical Physics: Quantum Information Theory* at the University of Heidelberg.
The aim of the talk is to offer an introduction to the young field of resource theories and their application to statistical mechanics and quantum information theory. Such theories take an agent-based approach to physics: One considers an experimenter, who only has limited access to a physical system, and asks in which ways can that experimenter influence that system. Because of the restrictions, there are resources which have to be consumed to make some operations possible. This presentation introduces resource theories by explaining some of the main results of a particular resource theory, the theory of informational non-equilibrium (or short: non-uniformity).

1 Introduction

Very often in physics, one takes the **dynamicist**'s point of view: The physicist analyzes a physical system and its time evolution, but does not change the system. There is also another, **agent-based** approach to physics: There is an experimenter (the agent) who has limited access to a physical system. In which ways can that physicist influence the system? What state conversions are possible? How can that agent take advantage of the system? This agent-based point of view is a natural approach to thermodynamics: For example one of the oldest questions is how much work can be performed by an engine. A similar, old example for the agent-based point of view is the Kelvin-Planck formulation of the second law of thermodynamics: " It is impossible to devise a cyclically operating device, the sole effect of which is to absorb energy in the form of heat from a single thermal reservoir and to deliver an equivalent amount of work."
In a resource theory, one characterizes the agent's limitations by defining some operations that are considered to be free. The idea is that the experimenter uses a lab which grants excellent control of some of the systems properties. Then one asks how can that experimenter influence the physical system: What state conversions are possible? What are the resources that need to be consumed to allow transitions which are not free? This last question is the reason why these theories are called resource theories.

A famous example is the resource theory of entanglement. Here the free operations are **LOCC** (Local operations and classical communication): The physical quantum system consists of two parts A and B. There are two physicists: One can perform any operation on system A, while the other physicist can perform any operation on system B. This means the physicists can perform local operations. They can also talk to each other by using classical communication such that they can synchronize their actions. However, even together, they do not have full control of the whole composite quantum system: Thus entanglement between both parts is a resource in this setting.

2 Defining Noisy Operations

In this document we analyze a typical resource theory called *resource theory of informational non-equilibrium* (or shorter: *resource theory of nonuniformity*). The free operations are the *noisy operations*. This analysis is completely based on [1]: The idea is to discuss some of the main results and to give some intuition for what a resource theory is and what one can learn from such theories. To provide easy access to [1], the theorems and definitions are labeled with the same number like in [1]. However, the numbers will grow fast and some of the proofs are very long and technical; thus for the exact proofs, the reader should take a look at [1] to find the proofs presented in a well-written and exact way. Also we cannot consider all results in this document.

Now we finally turn to noisy operations:

1. The physicist is allowed to add noise for free. The noise is represented by the density operator $\rho_{\text{noise}} = \frac{\mathbb{1}_d}{d}$. The noise's Hilbert space is called the ancilla space and denoted by $\mathcal{H}_a$.
2. The physicist has full control of the composite system $\mathcal{H}_{\text{input}} \otimes \mathcal{H}_a$, i.e. any unitary operation on the composite system can be performed for free.
3. The physicist can focus on subsystems, i.e. marginalize over subsystems.

Definition 1 *A* ***noisy quantum operation*** *f is one that admits of the following decomposition.*[1] *There must exist a finite-dimensional ancilla space $\mathcal{H}_a$, and a unitary U on $\mathcal{H}_{input} \otimes \mathcal{H}_a$ such that, for all input states ρ_{input},*

$$f(\rho_{input}) = Tr_{a'}\left(U\left(\rho_{input} \otimes \frac{\mathbb{1}_a}{d_a}\right)U^\dagger\right), \tag{2.1}$$

where $\mathcal{H}_{a'}$ is the the space complementary to $\mathcal{H}_{output}$ in the total Hilbert space, that is $\mathcal{H}_{output} \otimes \mathcal{H}_{a'} = \mathcal{H}_{input} \otimes \mathcal{H}_a$, and $d_a := dim(\mathcal{H}_a)$.

[1] To be closer to the experimental reality, one should define the noisy operations such that they can be arbitrarily well approximated by operations of the form (2.1). Consider an operation f which is not of form (2.1), but which can be arbitrarily well approximated by operations of the form (2.1). Then it is natural to say that an experimenter who can perform any operation of the form (2.1) (for free) can also perform the operation f (for free).

One important property of noisy operations is, that they are unital (see Lemma 5 of [1] for more details):

$$f\left(\frac{\mathbb{1}_{\text{input}}}{d_{\text{input}}}\right) = \frac{\mathbb{1}_{\text{output}}}{d_{\text{output}}} \tag{2.2}$$

Thus one can already guess, that **non-uniformity** will be the key-resource of this resource theory.

Next we consider the corresponding classical noisy operations. We consider discrete physical state spaces Ω (e.g. $\Omega = \{1,2,3,4,5,6\}$ for a 6-sided die). We define $S(\Omega)$ to be the set of probability distributions over Ω and represent them as vectors (e.g. $(1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$ would be the probability vector corresponding to a fair 6-sided die). We will refer to these vectors as (statistical) states, and call $V(\Omega)$ the smallest vector space into which $S(\Omega)$ can be embedded. An intuitive analogy is as follows: just like quantum states $|\phi\rangle$ generalize to density operators ρ (also called states), the physical states from Ω generalize to the states from $S(\Omega)$.
By analogy, we replace $U(\cdot)U^\dagger$ with a permutation π, $\frac{\mathbb{1}_d}{d}$ with the uniform distribution m_a, and replace the partial trace with marginalizing.

Definition 4 *A noisy classical operation D is one that admits of the following decomposition*[2]*: There must exist an ancilla system with a discrete physical state space Ω_a and a permutation on $\Omega_{input} \times \Omega_a$ with an induced representation π on $V(\Omega_{input}) \otimes V(\Omega_a)$ such that, for all input states x_{input},*

$$Dx_{input} = \sum_{\Omega_{a'}} \pi(x_{input} \otimes m_a) \tag{2.3}$$

where m_a is the normalized uniform distribution on Ω_a and $\Omega_{a'}$ is the physical state space complementary to Ω_{output}, that is, $\Omega_{input} \times \Omega_a = \Omega_{output} \times \Omega_{a'}$.

3 Exact State Conversion

In this chapter we try to find rules, that tell us if a specific state conversion is possible.

It is important to remember that for noisy quantum operations, all unitaries are free. Thus wlog we can assume that all density operators are diagonal. Also arbitrary unitaries can be applied to the full composite system, entanglement will not lead to problems. Thus it is intuitive, that all questions about quantum state conversion reduce to their classical analogues (define $\lambda(\rho)$ to be the probability vector defined by the eigenvalues of ρ, where the eigenvalues are listed in non-increasing order):

Lemma 7 *There exists a noisy quantum operation that achieves the quantum state conversion $\rho \to \sigma$ if and only if there is a noisy classical operation that achieves the classical state conversion $\lambda(\rho) \to \lambda(\sigma)$.*

[2]Just like in the quantum case, one should also include operations that can be arbitrarily well approximated by operations of the form (2.3).

Definition 11 *We write* $x \xrightarrow{noisy} y$ *if there exists a noisy classical operation* D *such that* $y = Dx$.

Next, we introduce a powerful graphical tool to determine if a state conversion is possible. But we need some definitions first: For a probability distribution vector x, we define $x^{\downarrow}$ to be probability vector for which the components are that of x ordered such, that they are decreasing in size: $(x_1^{\downarrow}, x_2^{\downarrow}, ..., x_{d_x}^{\downarrow})$ is a permutation of $(x_1, x_2, ..., x_{d_x})$ with:

$$x_1^{\downarrow} \geq x_2^{\downarrow} \geq ... \geq x_{d_x}^{\downarrow} \tag{3.4}$$

Now we can define the **Ky Fan k-norm** to be the sum of the k largest components:

$$S_k(x) := \sum_{j=1}^{k} x_j^{\downarrow} \tag{3.5}$$

where $k \in \{1, 2, ..., d_x\}$. Furthermore we set $S_0(x) = 0$.
Finally we define the **Lorenz curve** $L_x(u)$ of x to be the linear interpolation of the points $\left(\frac{k}{d_x}, \frac{S_k(x)}{S_{d_x}(x)}\right)$ for all $k \in \{0, 1, 2, ..., d_x\}$. Here, $u \in [0, 1]$ and $S_{d_x}(x) = 1$ as we only consider normalized distributions.
As a first example we consider $x = (1/8, 1/2, 1/4, 0, 1/8)$. Then $x^{\downarrow} = (1/2, 1/4, 1/8, 1/8, 0)$. The corresponding Lorenz curve is shown in figure (3.1).

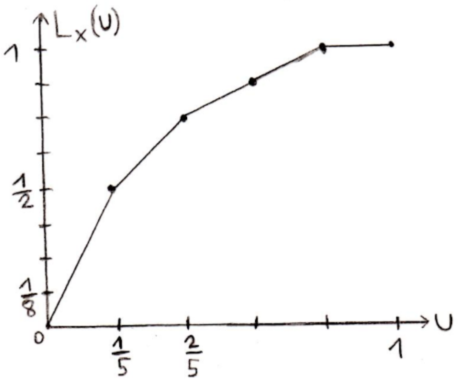

Figure 3.1: The Lorenz curve $L_x(u)$ for $x = (1/8, 1/2, 1/4, 0, 1/8)$.

Now, without proof, we state one on the main results from [1]:

Proposition 14 (ii) $x \xrightarrow{noisy} y$ *if and only if the Lorenz curve of* x *is everywhere greater or equal to the Lorenz curve of* y*:*

$$L_x(u) \geq L_y(u) \quad \forall u \in [0, 1]. \tag{3.6}$$

In this case, we say that x ***noisy-majorizes*** y.[3]

[3]Indeed, for equal dimension $d_x = d_y$, noisy-majorization reduces to majorization: $x \succ y :\Leftrightarrow x$ majorizes $y :\Leftrightarrow \sum_{j=1}^{k} x_j^{\downarrow} \geq \sum_{j=1}^{k} y_j^{\downarrow} \;\; \forall k \Leftrightarrow L_x(u) \geq L_y(u) \;\; \forall u \Leftrightarrow x \xrightarrow{noisy} y$

With this powerful rule, the question if a noisy state conversion is possible reduces to plotting and comparing some simple graphs. Once more we consider some examples:

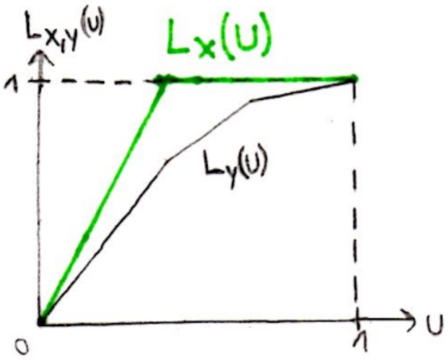

Figure 3.2: In this example the noisy state conversion of x to y is possible, while the conversion of y to x is impossible.

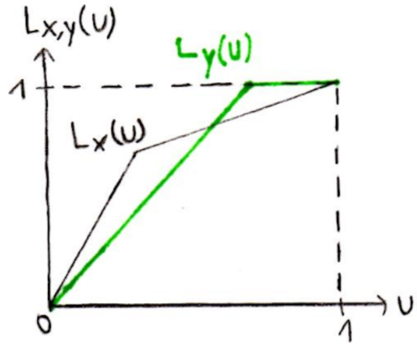

Figure 3.3: In this example x cannot be transformed to y by noisy operations, and also y cannot be transformed to x.

With proposition 14(ii), it is also easy to find **resource monotones**:

Definition 2 *A function M, mapping density operators to real numbers, is a **nonuniformity monotone** if for any two states ρ and σ (possibly of different dimensions), $\rho \to \sigma$ by noisy operations implies that $M(\rho) \geq M(\sigma)$.*

The classical nonuniformity monotones are defined in a similar way. Before we construct any monotones, at first we interpret them:
The monotones cannot get larger if a noisy operation is performed. That behaviour reminds of the second law of thermodynamics, often simply stated as "$\Delta S \geq 0$" . In a general resource theory, one cannot use the entropy to determine if a state conversion is possible. One has to replace the entropy with resource monotones characteristic for the resource theory under consideration. Furthermore, one has to take to take figure (3.3) into account: There are states x and y for which neither $x \to y$ nor $y \to x$ is possible. This shows that noisy-majorization is not a total order; there are states that cannot be compared. This means that a single nonuniformity monotone is not enough: If there was a statement of the form $x \xrightarrow{noisy} y \Leftrightarrow I(x) \geq I(y)$ for a resource monotone I, then always at least one of the statements $x \xrightarrow{noisy} y$ or $y \xrightarrow{noisy} x$ would be true (because at least one of the statements $I(x) \geq I(y)$ or $I(y) \geq I(x)$ is always true). Thus to formulate an adequate second law (which gives both necessary and sufficient conditions), one has to consider a whole family of monotones instead. In the case of noisy operations, from proposition 14(ii) we can deduce a complete set of such monotones to be $M_{\text{Lorenz},u}(x) := L_x(u)$, i.e. we have:

$$x \xrightarrow{noisy} y \quad \Leftrightarrow \quad M_{\text{Lorenz},u}(x) \geq M_{\text{Lorenz},u}(y) \quad \forall u \in [0,1] \tag{3.7}$$

But as the Lorenz curves are just a linear interpolation of few points, it is enough to consider finitely many monotones:

Lemma 17 *For x and y of finite dimension, x noisy-majorizes y if and only if $L_x(u) \geq L_y(u)$ at the points $u = k/d_y$ for all $k = 1, 2, ..., d_y - 1$. In other words, it suffices to consider the $d_y - 1$ monotones*

$$M_{\text{Lorenz},k/d_y}(x) = L_x(k/d_y), \quad k = 1, 2, ..., d_y - 1. \tag{3.8}$$

Note that the need for a whole family of nonuniformity monotones means, that we will need more than a single function/measure to characterize how much nonuniformity a particular state has (in the sense that if that measure says that one state has more nonuniformity than the other, then the state conversion is possible).
Long story short: One cannot measure non-uniformity with just one number.

Next, we consider two very important nonuniform monotones:

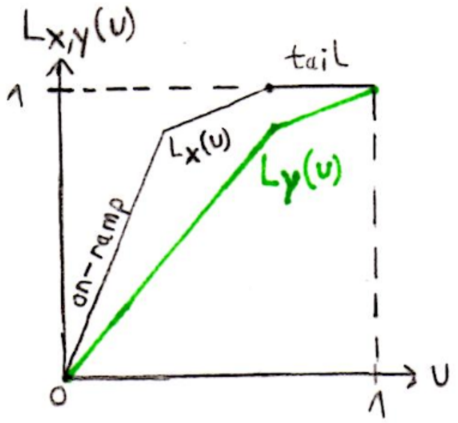

Figure 3.4: This plot can be used to visualize why $I_0(x)$ and $I_\infty(x)$ are nonuniformity monotones.

We take a look at figure (3.4). The flat, horizontal plateau is called the **tail** of the Lorenz-curve. The first straight-line is called the **on-ramp** part of the Lorenz curve. It is clear, that the on-ramp slope $m_{\text{on-ramp}} := \frac{x_1^\downarrow}{1/d_x}$ and the tail length $l(x) = \frac{d_x - |\text{supp(x)}|}{d_x}$ are nonuniformity monotones. Now if we define $I_0(x) := \log(d_x) - \log(|\text{supp}(x)|)$ and $I_\infty(x) := \log(d_x) + \log(x_1^\downarrow)$, we find $m_{\text{on-ramp}}(x) = 2^{I_\infty(x)}$ and $l(x) = 1 - 2^{-I_0(x)}$. Thus $I_\infty(x)$ and $I_0(x)$ are also nununiformity monotones. They are two examples of the **Rényi order-p nonuniformities** I_p:

$$-H_p(x) := \text{sign}(p)\frac{p}{p-1}\log(||x||_p) \tag{3.9}$$

$$I_p(x) := \text{sign}(p)\log(d_x) - H_p(x) \tag{3.10}$$

$-H_p(x)$ is called the Rényi order-p negentropy. One obtains $I_0(x)$ for $p \to 0^+$, and $I_\infty(x)$ for $p \to \infty$.
Using Lorenz curves one can also find a simple sufficient condition.

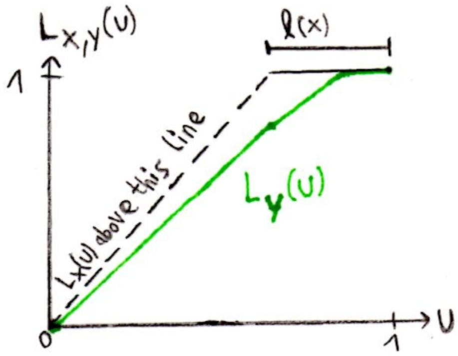

Figure 3.5: This figure is used to prove Lemma 32.

We take a look at figure (3.5). For a distribution x, only the tail of the Lorenz-curve is shown. A dashed line connects the left end of the tail with $(0, 0)$. If one takes a closer look at the definition of $L_x(u)$, one sees that $L_x(u)$ is a concave function. In fact, as $L_x(u)$ is defined to be the linear interpolation of $\left(\frac{k}{d_x}, \sum_{j=1}^{k} x_j^{\downarrow}\right)$, it is clear that the slope cannot increase with u. This means, that the Lorenz curve of x is found on or above the dashed line: If the real curve had a point below the dashed line, then the slope in that point must be smaller than that of the dashed line (because the slope cannot increase, but it has to be smaller than that of the dashed line somewhere). But as the slope cannot get larger after that point, it would be impossible for the Lorenz curve to be connected to its own tail. This of course is a contradiction. Thus, the Lorenz curve of x is indeed above the dashed line. The figure also shows a random distribution y whose on-ramp slope is smaller than the slope of the dashed line. From the figure it is clear, that such a distribution y must be noisy-majorized by x, again because the slope cannot become larger and because $L_y(u) \leq 1 \quad \forall u$. Thus we conclude: If $m_{\text{on-ramp}}(y) \leq \frac{1}{1-l(x)}$, i.e. $2^{I_\infty(y)} \leq 2^{I_0(x)}$, then $x \xrightarrow{noisy} y$:

Lemma 32 *If x and y are states such that $I_0(x) \geq I_\infty(y)$ then $x \xrightarrow{noisy} y$.*

4 Approximate State Conversion

So far we have only considered exact state conversions. But in experimental reality, one cannot expect to implement perfect state conversions. This is why one should consider approximate state conversions instead. To do so, we need a metric that tells us how similar two states (of equal dimension) are. For sake of simplicity, we will take the trace distance:

$$\mathcal{D}_{\text{tr}}(\rho, \sigma) := \frac{1}{2}\text{tr}|\rho - \sigma| = \frac{1}{2}||\rho - \sigma||_1 \tag{4.11}$$

The corresponding classical metric is:

$$\mathcal{D}_{\text{tr}}(x, y) := \frac{1}{2}\sum_{j=1}^{d_x} |x_j - y_j| \tag{4.12}$$

So from now on, we will assume $\mathcal{D} = \mathcal{D}_{\text{tr}}$.
At first we generalize the Rényi-nonuniformities I_0 and I_∞:

Definition 57 *We define the* ***smooth order-*0 *and order-*∞ *Rényi nonuniformities*** *as*

$$I_\infty^\epsilon(x) := \min_{x':\mathcal{D}(x,x')\le\epsilon} I_\infty(x') \tag{4.13}$$

$$I_0^\epsilon(x) := \max_{x':\mathcal{D}(x,x')\le\epsilon} I_0(x') \tag{4.14}$$

A lot of the results for the exact state conversion generalize to the approximate case. At first, just like Lemma 7, the approximate noisy quantum state conversion can be reduced to the approximate classical state conversion:

Lemma 55 *Let $\epsilon > 0$, let ρ and σ be two quantum states. Then the following statements are equivalent:*

1. *There exists a quantum state σ' with $\mathcal{D}(\sigma, \sigma') \le \epsilon$ such that noisy quantum operations can transform ρ to σ'.*
2. *There exists a distribution s with $\mathcal{D}(\lambda(\sigma), s) \le \epsilon$ such that noisy classical operations can transform $\lambda(\rho)$ to s.*

Here, $\lambda(\rho)$ again is the distribution defined by the eigenvalues of ρ. So once more we only need to focus on the classical case:

Definition 56 *We write $x \xrightarrow{\epsilon-noisy} y$ if there exists a noisy classical operation taking x to a state that is ϵ-close to y relative to $\mathcal{D}$.*

Lemma 32 generalizes to:

Lemma 63 *If x and y are states such that $I_0^{\epsilon/2}(x) \ge I_\infty^{\epsilon/2}(y)$, then $x \xrightarrow{\epsilon-noisy} y$.*

The property of I_∞ being a nonuniformity monotone now generalizes to:

Lemma 64 *If x and y are states such that $x \xrightarrow{\epsilon-noisy} y$, then $I_\infty^{\epsilon+\delta}(y) \le I_\infty^\delta(x)$ for every $\delta \ge 0$.*

5 Asymptotic State Conversion

So far we have only considered *single-shot* state conversion, i.e. the questions if one can transform one copy of a state into one copy of another state. While this is a interesting question on its one, one might also be interested in *asymptotic* state conversions: Here, in an industrial-like fashion, one wants to transform many copies of one state into many copies of another state. And of course, one wants this transformation to be as efficient as possible. The results from the last chapter will directly lead to the answer for what the best rate is that can be achieved. However, before we can use them, we need one statement more:

Lemma 65 *We have for every* $0 < \epsilon < 1$

$$\lim_{n\to\infty} \frac{1}{n} H_\infty^\epsilon(x^{\otimes n}) = H(x) \tag{5.15}$$

$$\lim_{n\to\infty} \frac{1}{n} H_0^\epsilon(x^{\otimes n}) = H(x) \tag{5.16}$$

for all finite probability distributions x*, where* $H(x) = -\sum_j x_j \log x_j$ *is the* ***Shannon entropy****.*

Here $I_\infty^\epsilon(x) = \log(d_x) - H_\infty^\epsilon(x)$ and $I_0^\epsilon(x) = \log(d_x) - H_0^\epsilon(x)$. Important for us is, that Lemma 65 directly generalises to the nonuniformities:
We define the **Shannon nonuniformity** as $I(x) := \log(d_x) - H(x)$. Then

$$\frac{1}{n} I_{0,\infty}^\epsilon(x^{\otimes n}) = \frac{1}{n}\left(\log(d_x^n) - H_{0,\infty}^\epsilon(x^{\otimes n})\right) = \log(d_x) - \frac{1}{n} H_{0,\infty}^\epsilon(x^{\otimes n}) \tag{5.17}$$

Thus we have:

$$\lim_{n\to\infty} \frac{1}{n} I_\infty^\epsilon(x^{\otimes n}) = I(x) \tag{5.18}$$

$$\lim_{n\to\infty} \frac{1}{n} I_0^\epsilon(x^{\otimes n}) = I(x) \tag{5.19}$$

The basic idea is that in the limit of many copies, we can use Lemma 63 and 64 with $I_{0,\infty} \to I$.

So we consider now the following situation:
We have two states x and y, and we are interested in conversions of the form $x^{\otimes n} \xrightarrow{\epsilon-noisy} y^{\otimes m}$. Especially, we are interested in the largest integer m_n for which this conversion if possible. For that we consider the sufficient and necessary conditions as given by Lemma 63 and 64:

1. **sufficient:**
   From Lemma 63 we find that if $I_0^{\epsilon/2}(x^{\otimes n}) \geq I_\infty^{\epsilon/2}(y^{\otimes m})$ then the conversion is possible for sure. With Lemma 65, we rewrite this result in a sloppy way: If $nI(x) \gtrsim mI(y)$ then $x^{\otimes n} \xrightarrow{\epsilon-noisy} y^{\otimes m}$. Thus we can assume (in the limit of many copies) that the conversion is possible for $m \lesssim n\frac{I(x)}{I(y)}$.

2. **necessary:**
   From Lemma 64 we find that if the conversion is possible, then $I_\infty^\delta(x^{\otimes n}) \geq I_\infty^{\epsilon+\delta}(y^{\otimes m})$. With Lemma 65, we rewrite this result in a sloppy way: If $x^{\otimes n} \xrightarrow{\epsilon-noisy} y^{\otimes m}$ then $nI(x) \gtrsim mI(y)$. Thus we can assume (in the limit of many copies) that the conversion is impossible for $m \gtrsim n\frac{I(x)}{I(y)}$.

We conclude that the largest integer is $m_n \approx n\frac{I(x)}{I(y)}$. While our reasoning is quite sloppy, the rigorous proof follows the same idea and leads to:

Lemma 66 *Let x and y be two states of possibly different dimensionalities that are not both uniform, let $0 < \epsilon < 1$. For every $n \in \mathbb{N}$, let m_n be the largest integer such that*

$$x^{\otimes n} \xrightarrow{\epsilon-noisy} y^{\otimes m_n} \tag{5.20}$$

Then

$$\lim_{n\to\infty} \frac{m_n}{n} = \frac{I(x)}{I(y)} \tag{5.21}$$

Similarly, let k_n be the smallest integer such that

$$x^{\otimes k_n} \xrightarrow{\epsilon-noisy} y^{\otimes n} \tag{5.22}$$

Then

$$\lim_{n\to\infty} \frac{k_n}{n} = \frac{I(y)}{I(x)} \tag{5.23}$$

At first we notice that the optimal rate does not depend on the allowed error ϵ. Thus it makes sense, that one can make the error n-dependent ϵ_n and slowly send it to 0 without getting a worse rate:

Lemma 67 *There is a protocol for transforming n copies of x into m_n copies of y with asymptotically vanishing error at rate*

$$r := \lim_{n\to\infty} \frac{m_n}{n} = \frac{I(x)}{I(y)}; \tag{5.24}$$

however, no higher rate is achievable by any protocol of this kind.

A second important observation is, that in the limit of many copies, all state conversion questions can be answered by the Shannon nonuniformity - no need for whole families of monotones anymore. Especially if we use $I(x^{\otimes n}) = nI(x)$, then we obtain a second law of the form:

$$x^{\otimes n} \xrightarrow{\epsilon-noisy} y^{\otimes m} \quad \Leftrightarrow \quad I(x^{\otimes n}) \geq I(y^{\otimes m}) \textit{ for large } n, m.$$

Note that this formulation is still sloppy. The correct formulation instead says $1 = \lim_{n\to\infty} \frac{I(y^{\otimes m_n})}{I(x^{\otimes n})}$, or more generally[4] for $m'_n \leq m_n$: $1 \geq \limsup_{n\to\infty} \frac{I(y^{\otimes m'_n})}{I(x^{\otimes n})}$.
Now we can make more assumptions: Assume that x and y belong to the very same system, especially $d_x = d_y$. Also assume that in the industrial limit of many copies, we want to convert each x to y, i.e. we demand $\lim_{n\to\infty} \frac{m_n}{n} = 1$. With these assumptions we can write in a sloppy way:

$$x^{\otimes n} \xrightarrow{\epsilon-noisy} y^{\otimes n} \quad \Leftrightarrow \quad H(x) \leq H(y) \textit{ for large } n.$$

And we can write this finding even more suggestive in the form:

$$x \to y \quad \Leftrightarrow \quad H(x) \leq H(y) \textit{ in the limit of many copies.}$$

[4]The transition is possible for $m'_n \leq m_n$ because marginalization is a noisy operation.

We have just re-“derived” the common formulation of the second law of thermodynamics as necessary and sufficient condition for state conversion!
Indeed this is a typical property of resource theories: They reproduce some standart results from information theory and thermodynamics. Of course, this is a first test for a resource theory - a theory that contradicts well-established results does something wrong. But even more important is that resource theories give exact and well-defined conditions for when the standart results from information theory and thermodynamics actually are true. And resource theories also tell us what happens if these conditions are not fulfilled. For example, we could analyze single-shot state conversions within the framework of our resource theory. Ironically, we needed much more effort to arrive at the standart results than to discuss the cases that are far away from the typical assumptions of thermodynamics or information theory: This means that resource theories can be a very good approach to discuss thermodynamics and information theory in limits, for which the standart approaches find no answers!

6 Conclusion And Outlook

In this document, we have analyzed a typical example of a resource theory. We have assumed an agent-based point of view, asking: What can that agent do, if only a limited class of operations is free? This is a powerful framework to investigate state conversions. We have seen that resource monotones can be used to formulate necessary and sufficient conditions for state conversions in the spirit of the second law of thermodynamics. The simplest case to discuss was that of single-shot state conversion. As this case is very far away from the assumptions of standart thermodynamics, resource theories promise to be a good framework to discuss such conversions. But resource theories can also be used to discuss approximate or asymptotic state conversions - especially they can be used to give exact conditions for standart results of thermodynamics and information theory. Furthermore, the agent-based approach promises to be close to real application: In that sense, one important mission of the field is to identify and analyze those resource theories, that come closest to experimental reality. One example for such a theory is given by the resource theory of entanglement (LOCC), the resource theory that discusses two quantum labs that share entangled quantum objects. Another important mission of the field is to find reciepes and tools to efficiently analyze resource theories. This is because the correct resource theory of course depends on the conditions of the lab and the experiment. This means there might be many resource theories, each describing different experimental circumstances. Thus it is important to find efficient methods to make predictions in the framework of a new resource theory. We have already seen, that the identification of resource monotones are an important step, as they can be used to formulate necessary or sufficient conditions.

In this document, many things could not be discussed. The reader has surely noticed, that the number labeling the definitions and theorems cited from [1] has grown very fast - we have left out many results and proofs. Probably most important is that we have not discussed catalysts. Inspired by chemistry, catalysts are states that can be used in state conversions, but have to be returned: $x \otimes z \to y \otimes z$. In the context of thermodynamics, catalysts could be considered to be cyclically operating engines - that alone is more than enough reason to think about catalysts, because cyclically operating engines of course are an important part of thermodynamics. Furthermore, while noisy operations (i.e. the theory that tells you if noise can be useful) are interesting on their own, the corresponding resource theory can be considered as a toy model for athermality theory, where the free states are thermal and one is interested in the extraction of work: Indeed for a thermal state of a system with trivial Hamiltion $\hat{H} = 0$, we find $\rho_{\text{thermal}} = \frac{1}{Z}e^{-\beta\hat{H}} = \frac{\mathbb{1}_d}{d} = \rho_{\text{noise}}$. For more details, one should take a look at [2].

References

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[3] M. Horodecki, P. Horodecki, and J. Oppenheim, *Reversible transformations from pure to mixed states, and the unique measure of information*, Phys. Rev. A **67**, 062104 (2003)