Selected topics in Mathematical Physics: Quantum Information Theory

Talk 5: Averages over the unitary group

Yuyan Li

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Abstract

In this talk, it is shown how to calculate averages over the Unitary group by applying mathematical tools presented in the previous talk. Particulary, *Schur's Lemma* is utilized to two useful examples. Eventually, the results are used to show that a random pure state on a finite-dimensional bipartite Hilbert space is close to maximally entangled.

1 Representations of the Unitary group and Schur's Lemma

There are two common representations of the Unitary group:

- **The standard representation.** In the standard representation of $U(n)$ on $\mathbb{C}^n$ every unitary matrix $U \in U(n)$ is mapped onto itself, i.e. $U \mapsto U$. This is a irreducible representation.

- **The representation on $\mathbb{C}^n \otimes \mathbb{C}^n$.** A very useful representation when considering bipartite systems is the map $U \mapsto U \otimes U$. This is a reducible representation of $U(n)$. The group-module $\mathbb{C}^n \otimes \mathbb{C}^n$ can be decomposed into the following two subspaces

$$\mathbb{C}^n_{\text{sym}} := \{|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n : \mathbb{F}(|\psi\rangle) = |\psi\rangle\} \tag{1}$$

$$\mathbb{C}^n_{\text{antisym}} := \{|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n : \mathbb{F}(|\psi\rangle) = -|\psi\rangle\} \tag{2}$$

  They are defined through the "flip operator" $\mathbb{F}$ on $\mathbb{C}^n \otimes \mathbb{C}^n$, with:

$$\mathbb{F}(|\psi\rangle \otimes |\varphi\rangle) = |\varphi\rangle \otimes |\psi\rangle \tag{3}$$

  These two spaces are invariant and cannot be decomposed into more subspaces.

One important result of group theory is *Schur's Lemma*. There are different formulations depicting similar ideas that are called by the same name. The version presented here will be the most useful for this talks purpose.

Schur's Lemma. Let $g \mapsto U_g \in U(X)$ be a unitary representation of the group G on the finite-dimensional complex space X. Then there is a unique decomposition of X into subspaces X_i, so that these subspaces are mutually orthogonal. I.e.:

$$X = \bigotimes_{i=1}^{m} X_i, \quad X_i \perp X_j (i \neq j) \tag{4}$$

In addition, the representation is irreducible on every subspace X_i.

Furthermore, given a diagonalizable matrix A that commutes with all group representations:

$$[A, U_g] = 0 \quad \forall g \in G, \tag{5}$$

then A can be decomposed as follows:

$$A = \sum_{i=1}^{m} \lambda_i \pi_i. \tag{6}$$

The π_i are the orthogonal projectors onto the subspaces X_i and λ_i are comple numbers.

2 Averages over the unitary group

The previous talk introduced the *Haar measure* and utilized it as a way to calculate expectation values over the unitary group. This technique in combination with *Schur's Lemma* will be applied here in two examples.

Random unitary transformation of a hermitian matrix. Let ρ be a hermitian matrix. The following integral gives the expected resulting matrix M after performing a random unitary transformation:

$$M := \int_{U(n)} U\rho U^\dagger dU \tag{7}$$

To compute M, it will first be shown that M commutes with all unitaries. Let $V \in U(n)$ and consider $VM = \int_{U(n)} VU\rho U^\dagger dU$. Apply a substitution with $W := VU$, thus $U = V^\dagger W$ and $U^\dagger = W^\dagger V$ to obtain:

$$VM = \int_{U(n)} VU\rho U^\dagger dU = \int_{U(n)} W\rho W^\dagger V d(V^\dagger W) = \int_{U(n)} W\rho W^\dagger V dW = MV \tag{8}$$

In the last step, the invariance of the *Haar measure* is utilized. Thus $[M, V] = 0$ for all unitaries V which are also the group representaiotns of $U(n)$ on $\mathbb{C}^n$. Since ρ is hermitian, M is also hermitian and therefore diagonalizable. Consequently, *Schur's Lemma* can be applied on the problem. The standard representation is irreducible, so $\mathbb{C}^n$ cannot be decomposed into further invariant subspaces and the only orthogonal projector is the identity, $\pi = \mathbb{1}_n$. Therefore,

$$M = \lambda \mathbb{1}_n \tag{9}$$

with a constant $\lambda \in \mathbb{C}$ which can be computed by taking the trace of M:

$$\lambda n = \lambda \operatorname{tr}(\mathbb{1}_n) = \operatorname{tr}(M) = \int_{U(n)} \operatorname{tr}(U\rho U^\dagger) dU = \operatorname{tr}(\rho) \tag{10}$$

The cyclicity of the trace and normalization of the Haar measure are used in the last step. The result is thus:

$$M = \int_{U(n)} U\rho U^\dagger dU = \frac{\operatorname{tr}(\rho)}{n} \mathbb{1}_n \tag{11}$$

If ρ is a density matrix, in particular $\operatorname{tr}(\rho) = 1$, the outcome describes a random unitary transformation applied to an initally pure state. The expectation of the resulting density matrix is then ${}^{\mathbb{1}_n}/_n$, which corresponds to the maximally mixed state, i.e. the state with the least information about the system.

Random unitary transformation on a bipartite system. Let ρ be hermitian matrix on $\mathbb{C}^n \otimes \mathbb{C}^n$. In a similar way to before, the integral

$$N := \int_{U(n)} (U \otimes U)\rho(U^\dagger \otimes U^\dagger) dU \tag{12}$$

is evaluated. This time, the representation $U \mapsto U \otimes U$ of the unitary group is used. Let $V \otimes V$ be the group representation of the unitary matrix V. Then the same substitution $W := VU$ can be applied and one gets:

$$\begin{aligned}(V \otimes V)N &= \int_{U(n)} (V \otimes V)(U \otimes U)\rho(U^\dagger \otimes U^\dagger) dU \\ &= \int_{U(n)} (W \otimes W)\rho(W^\dagger \otimes W^\dagger)(V \otimes V) dW = N(V \otimes V)\end{aligned} \tag{13}$$

Utilizing the unitary invariance of the Haar measure again, it is shown that N commutes with every group representation of $U(n)$, i.e. $[N, U \otimes U] = 0$ for all $U \in U(n)$. Considering the decomposition of $\mathbb{C}^n \otimes \mathbb{C}^n$ shown in the previous section, the matrix N can therefore be written as

$$N = \lambda_1 \pi_{\text{sym}} + \lambda_2 \pi_{\text{antisym}}, \tag{14}$$

where π_{sym} and π_{antisym} are the orthogonal projectors onto the symmetric and antisymmetric subspaces, respectively.

To compute the two constants λ_1 and λ_2, one first needs a few properties of the *flip operator* $\mathbb{F}$. Using the *spectral theorem*, it is clear that:

$$\mathbb{F} = \pi_{\text{sym}} - \pi_{\text{antisym}} \tag{15}$$

$$\mathbb{1}_{n^2} = \pi_{\text{sym}} + \pi_{\text{antisym}} \tag{16}$$

By addition and subtration of these two equation, the following forms of the orthogonal projector are found:

$$\pi_{\text{sym}} = \frac{1}{2}(\mathbb{1}_{n^2} + \mathbb{F}) \tag{17}$$

$$\pi_{\text{antisym}} = \frac{1}{2}(\mathbb{1}_{n^2} - \mathbb{F}) \tag{18}$$

Furthermore, the trace of the flip operator can be easily calculated. Let $|\varphi\rangle_i$ be an orthonormal basis of $\mathbb{C}^n$, then

$$\text{tr}(\mathbb{F}) = \sum_{i,j=0}^{n} \langle \varphi_i \varphi_j | \, \mathbb{F} \, | \varphi_i \varphi_j \rangle = \sum_{i,j=0}^{n} \langle \varphi_i \varphi_j | \varphi_j \varphi_i \rangle = \sum_{i,j=0}^{n} \delta_{ij}\delta_{ij} = n \tag{19}$$

and, with $\text{tr}(\mathbb{1}_{n^2}) = n^2$, the traces of the projectors π_{sym} and π_{antisym} become

$$\text{tr}(\pi_{\text{sym}}) = \frac{n(n+1)}{2} \quad \text{and} \quad \text{tr}(\pi_{\text{antisym}}) = \frac{n(n-1)}{2}. \tag{20}$$

Now consider the traces of $\mathbb{F}N = \int_{U(n)} U \otimes U \mathbb{F} \rho U^\dagger \otimes U^\dagger dU$ (note that $\mathbb{F}$ commutes with all $U \otimes U$) and $\mathbb{1}_{n^2} N$ while using the projectors to express $\mathbb{F}$ and $\mathbb{1}_{n^2}$:

$$\text{tr}(\mathbb{F}N) = \int_{U(n)} \text{tr}(U \otimes U(\pi_{\text{sym}} - \pi_{\text{antisym}})\rho U^\dagger \otimes U^\dagger dU = tr(\rho\pi_{\text{sym}}) - tr(\rho\pi_{\text{antisym}}) \tag{21}$$

$$\text{tr}(\mathbb{1}_{n^2} N) = \int_{U(n)} \text{tr}(U \otimes U(\pi_{\text{sym}} + \pi_{\text{antisym}})\rho U^\dagger \otimes U^\dagger) dU = \text{tr}(\rho\pi_{\text{sym}}) + \text{tr}(\rho\pi_{\text{antisym}}) \tag{22}$$

Also evaluate the same traces while considering the right hand side in eq. (14):

$$\mathrm{tr}(\mathbb{F}N) = \mathrm{tr}(\mathbb{F}(\lambda_1 \pi_{\mathrm{sym}} + \lambda_2 \pi_{\mathrm{antisym}})) = \lambda_1 \,\mathrm{tr}(\pi_{\mathrm{sym}}) - \lambda_2 \,\mathrm{tr}(\pi_{\mathrm{antisym}}) \tag{23}$$

$$\mathrm{tr}(\mathbb{1}_{n^2} N) = \mathrm{tr}(\lambda_1 \pi_{\mathrm{sym}} + \lambda_2 \pi_{\mathrm{antisym}}) = \lambda_1 \,\mathrm{tr}(\pi_{\mathrm{sym}}) + \lambda_2 \,\mathrm{tr}(\pi_{\mathrm{antisym}}) \tag{24}$$

Thus one gets the following two equations:

$$tr(\rho \pi_{\mathrm{sym}}) - tr(\rho \pi_{\mathrm{antisym}}) = \lambda_1 \,\mathrm{tr}(\pi_{\mathrm{sym}}) - \lambda_2 \,\mathrm{tr}(\pi_{\mathrm{antisym}}) \tag{25}$$

$$tr(\rho \pi_{\mathrm{sym}}) + tr(\rho \pi_{\mathrm{antisym}}) = \lambda_1 \,\mathrm{tr}(\pi_{\mathrm{sym}}) + \lambda_2 \,\mathrm{tr}(\pi_{\mathrm{antisym}}) \tag{26}$$

Finally, by addition and subtraction and plugging in eq. (20), the results for the two constants λ_1 and λ_2 are found to be:

$$\lambda_1 = \frac{2}{n(n+1)} \,\mathrm{tr}(\rho \pi_{\mathrm{sym}}) \quad \text{and} \quad \lambda_2 = \frac{2}{n(n-1)} \,\mathrm{tr}(\rho \pi_{\mathrm{antisym}}) \tag{27}$$

3 Average entanglement of random pure quantum states on a bipartite system

The previous talk introduced the concept of a random pure state. In particular, the *Haar measure* was used to define the following expectation value:

$$\mathbb{E}_\psi f(|\psi\rangle) = \int f(|\psi\rangle) d\psi := \int_{U(n)} f(U |\psi_0\rangle) dU \tag{28}$$

Here, $|\psi_0\rangle \in \mathbb{C}^n$ is an arbitrary fixed pure state. Thanks to the unitary invariance of the *Haar measure*, the above definiton is indenpendent of the initial state ψ_0. This concept will now be used to examine the entanglement of random bipartite pure quantum states.

Consider two quantum systems A and B which are described by the finite-dimensional Hilbert space $A \otimes B$. Assume that the dimension of A is not larger than the dimension of B, i.e. $|A| \leq |B|$ (the value signs are used to denote the dimension of a Hilbert space in this talk). This is a typical situation in statistical physics, where A is the quantum system of interest embedded in a bigger enviroment B. The question for this talk is: How entangled is a randomly drawn state $|\psi\rangle \in A \otimes B$?

One way to quantify the entanglement of a pure state is by considering the reduced density matrix $\rho_A = \mathrm{tr}_B(\psi_{AB}) = \mathrm{tr}_B(|\psi\rangle \langle\psi|)$, where $\psi_{AB} := |\psi\rangle \langle\psi|$ denotes the density matrix corresponding to the random state. The reason for adding the Hilbert space as an index will be seen later. Computing the trace $\mathrm{tr}(\rho_A^2)$ provides information about the amount of mixing of the state with the following possible outcomes:

- $\mathrm{tr}(\rho_A^2) = 1 \Leftrightarrow \rho_A$ is pure $\Leftrightarrow |\psi\rangle$ is unentangled
- $\mathrm{tr}(\rho_A^2) < 1 \Leftrightarrow \rho_A$ is mixed $\Leftrightarrow |\psi\rangle$ is entangled
- $\mathrm{tr}(\rho_A^2) = \frac{1}{|A|} \Leftrightarrow |\psi\rangle$ is maximally entangled

Thus the generic entanglement can be estimated by computing the expectation value:

$$\mathbb{E}_\psi \,\mathrm{tr}\, \rho_A^2 = \int \mathrm{tr}(\rho_A^2) d\psi = \mathrm{tr}\left[\int (\mathrm{tr}_B |\psi\rangle \langle\psi|)^2 d\psi\right] \tag{29}$$

A computational trick is needed to simplify this seemingly complicated integral.

Trick for computing the trace of a squared matrix. Consider the "flip operator" $\mathbb{F}$ as defined in eq. (3) and any matrix ρ on $\mathbb{C}^n$, then the following statement holds:

$$\operatorname{tr}(\rho^2) = \operatorname{tr}((\rho \otimes \rho)\mathbb{F}) \tag{30}$$

Proof. To prove this, consider an orthonormal basis ϕ_i of $\mathbb{C}^n$, then:

$$\begin{aligned}\operatorname{tr}((\rho \otimes \rho)\mathbb{F}) &= \sum_{i,j=0}^{n} \langle \varphi_i \varphi_j | (\rho \otimes \rho)\mathbb{F} | \varphi_i \varphi_j \rangle = \sum_{i,j=0}^{n} \langle \varphi_i \varphi_j | \rho \otimes \rho | \varphi_j \varphi_i \rangle \\ &= \sum_{i,j=0}^{n} \langle \varphi_i | \rho | \varphi_j \rangle \langle \varphi_j | \rho | \varphi_i \rangle = \sum_{i=0}^{n} \langle \varphi_i | \rho^2 | \varphi_i \rangle = \operatorname{tr}(\rho^2)\end{aligned}$$ □

In order to use this trick, the Hilbert space at hand, $A \otimes B$ has to be extended by introducing an auxiliary copy $A' \otimes B'$. Combining these yields a quadripartite Hilbert space on which the random state becomes $\psi = \psi_{AB} \otimes \psi A'B'$ with the reduced density matrix $\rho_A \otimes \rho_{A'} = \operatorname{tr}_{BB'}(\psi_{AB} \otimes \psi A'B')$. Thus one obtains:

$$\operatorname{tr}(\rho_A^2) = \operatorname{tr}[(\rho_A \otimes \rho_{A'})\mathbb{F}_{AA'}] = \operatorname{tr}[(\psi_{AB} \otimes \psi_{A'B'})(\mathbb{F}_{AA'} \otimes \mathbb{1}_{BB'})] \tag{31}$$

Inserting this identity into eq. (29) yields:

$$\begin{aligned}\mathbb{E}_\psi \operatorname{tr}(\rho_A^2) &= \mathbb{E}_\psi \operatorname{tr}[(\psi_{AB} \otimes \psi_{A'B'})(\mathbb{F}_{AA'} \otimes \mathbb{1}_{BB'})] \\ &= \operatorname{tr}\left[(\mathbb{F}_{AA'} \otimes \mathbb{1}_{BB'}) \int |\psi\rangle \langle\psi| \otimes |\psi\rangle \langle\psi| \, d\psi\right]\end{aligned} \tag{32}$$

With the definition of the expectation value and the result from the previous section, the integral becomes:

$$\begin{aligned}\int |\psi\rangle \langle\psi| \otimes |\psi\rangle \langle\psi| \, d\psi = &\int_{U(n)} (U \otimes U)(|\psi_0\rangle \langle\psi_0| \otimes |\psi_0\rangle \langle\psi_0|)(U^\dagger \otimes U^\dagger) dU \\ = &\frac{2}{n(n+1)} \operatorname{tr}[\pi_{\text{sym}}^{AB:A'B'}(|\psi_0\rangle \langle\psi_0| \otimes |\psi_0\rangle \langle\psi_0|)]\pi_{\text{sym}}^{AB:A'B'} \\ &+ \frac{2}{n(n-1)} \operatorname{tr}[\pi_{\text{antisym}}^{AB:A'B'}(|\psi_0\rangle \langle\psi_0| \otimes |\psi_0\rangle \langle\psi_0|)]\pi_{\text{antisym}}^{AB:A'B'} \\ = &\frac{2}{n(n+1)} \pi_{\text{sym}}^{AB:A'B'}\end{aligned} \tag{33}$$

here, $n := |A||B|$ denotes the dimension of the initial Hilbert space and $\pi_{\text{sym}}^{AB:A'B'}$ is the orthogonal projector on the symmetric subspace, when considering the "flip" between the two spaces AB and $A'B'$. The antisymmetric term vanishes because $|\psi_0\rangle \langle\psi_0| \otimes |\psi_0\rangle \langle\psi_0|$ is obviously symmetric, causing the corresponding trace to disappear. The same argument leads to the trace in the symmetric part to become 1, since $|\psi_0\rangle \langle\psi_0|$ is a density matrix.

Inserting this result back into eq. (32) and using the known properties of the symmetric projector produces:

$$\begin{aligned}&\frac{2}{n(n+1)} \operatorname{tr}\left[(\mathbb{F}_{AA'} \otimes \mathbb{1}_{BB'})\frac{1}{2}(\mathbb{1}_{ABA'B'} + \mathbb{F}_{ABA'B'})\right] \\ = &\frac{1}{n(n+1)} \operatorname{tr}\left[(\mathbb{F}_{AA'} \otimes \mathbb{1}_{BB'}) + (\mathbb{1}_{AA'} \otimes \mathbb{F}_{BB'})\right] \\ = &\frac{|A||B|^2 + |A|^2|B|}{|A||B|(|A||B| + 1)}\end{aligned} \tag{34}$$

Here, the fact that $\mathbb{F}^2 = \mathbb{1}$ is used and one finally obains the result:

$$\mathbb{E}_\psi \operatorname{tr}(\rho_A^2) = \frac{|A| + |B|}{|A||B| + 1} \tag{35}$$

If "the enviroment becomes very big", i.e. $|B| \to \infty$, then this term approaches $1/|A|$ and $|\psi\rangle$ is almost maximally entangled. This means, if $|A| >> |B|$ then pure states are expected to be close to maximally entangled on average.

This result is mathematically quite interesting but doesn't say much about physics. The calculated value $\operatorname{tr}(\rho_A^2)$ doesn't have an actual physical meaning. Furthermore, the result is only a statement about the average entanglement of a pure state but dosen't provide any information about the distribution. Both these topics will be discussed in further talks.