Quantum correlations and generalized probabilistic theories: an introduction

Exercises to prepare for the oral exam in February 2019
(Dated: **Final version** of January 30, 2019)

I will list a few exercises here that you should solve as a preparation for the oral exam. These exercises are *not* meant as a kind of homework; you will not have to hand them in, and they will not be corrected or graded. Instead, you should sit down, solve them (either alone or in groups, as you prefer) and make sure that you understand what is going on, both mathematically and conceptually.

Exercise 1 (PR boxes, no signalling, and randomness). *Consider a two-party behavior $P(x,y|a,b)$ with settings $a,b\in\{0,1\}$ and outcomes $x,y\in\{-1,+1\}$. Suppose that the outcomes are*

- *perfectly correlated if $(a,b)\in\{(0,0),(0,1),(1,0)\}$, and*
- *perfectly anticorrelated if $(a,b)=(1,1)$.*

Prove that no-signalling implies that both parties must locally see perfectly random outcomes for every setting. Conceptually, how does that relate to what we have learned in Lecture 4?

Exercise 2 (Filters in quantum mechanics). *Suppose that an $n\times n$ density matrix ρ impinges on an n-slit arrangement with the kth slit blocked, where $1\leq k\leq n$. As a consequence, the state after the slits is described by a subnormalized density matrix ρ' with $\langle k|\rho'|k\rangle=0$, i.e. a (positive semidefinite) matrix with a zero at the kth diagonal entry.*

Prove that we must then have $\langle j|\rho'|k\rangle=0$ for all j, i.e. the kth row and the kth column of ρ' must be identically equal to zero. Where did this appear in Lecture 6?

Exercise 3 (Transposition). *Let $A=\mathbb{C}^n$, and consider the map $T:\mathcal{B}(A)\to\mathcal{B}(A)$ defined by $T(A)=A^\top$ (the transposition). Show that T is a positive map (i.e. $A\geq 0\Rightarrow T(A)\geq 0$, i.e. positive semidefiniteness is preserved) and that it is trace-preserving, but that it is* not *a quantum operation.*

Hint: see the notes for lecture 7.

Exercise 4 (Sets of quantum states). *For $n\in\mathbb{N}$, consider the set of $n\times n$ density matrices,*

$$\Omega_n=\{\rho\in\mathcal{B}(\mathbb{C}^n)\mid\rho\geq 0,\mathrm{tr}(\rho)=1\}.$$

Show that Ω_n is a compact convex set, and determine its extreme points. Show that for $n\geq 3$, there are states in the boundary of Ω_n which are not pure.

Exercise 5. *Suppose that $E_1,E_2,\ldots,E_n$ is a POVM (positive operator-valued measure) on B, describing a measurement M with n outcomes. Consider any quantum operation $T:\mathcal{B}(A)\to\mathcal{B}(B)$. Show that there is another POVM $E'_1,E'_2,\ldots,E'_n$ on A which describes the composite measurement that consists of first applying T and then performing M.*

Hint: argue that $\langle A,B\rangle:=\mathrm{tr}(A^\dagger B)$ is an inner product on $\mathcal{B}(\mathcal{H})$ if $\mathcal{H}$ is a finite-dimensional Hilbert space (the "Hilbert-Schmidt inner product"). Then consider the adjoint $T^\dagger$ of T (where the adjoint is taken with respect to the Hilbert-Schmidt inner product).

Exercise 6 (Some details on lecture 8).

- *Show that every reversible transformation must be normalization-preserving.*
- *For the gbit as defined in the lecture notes, check that A_+ has the form as claimed, given the definition of Ω_A.*

Two state space (A,Ω_A) and (B,Ω_B) are equivalent *if there is an invertible linear map $L:A\to B$ such that $\Omega_B=L\Omega_A$.*

- *Show that equivalent state spaces have the same dimension.*
- *Show that the qubit state space (A,Ω_A) with $A=\mathbb{C}^{2\times 2}_{\mathrm{sa}}$ and $\Omega_A=\{\rho\in A\mid\rho\geq 0,\mathrm{tr}(\rho)=1\}$ is equivalent to the Bloch ball state space (B,Ω_B), where $B=\mathbb{R}^4$ and $\Omega_B=\{(1,\vec{r})^\top\mid\|\vec{r}\|\leq 1\}$.*

- *Consider the transpose map, $T : A \to A$, $T(\rho) = \rho^\top$. Determine how it acts in the Bloch ball picture, i.e. compute $\tilde{T} : B \to B$ with $\tilde{T} = L \circ T \circ L^{-1}$. (Here and in the following, "$\circ$" denotes the composition of (linear) maps.)*

Exercise 7 (Equivalence of dynamical state spaces)**.** *Two dynamical state spaces $(A, \Omega_A, \mathcal{T}_A)$ and $(B, \Omega_B, \mathcal{T}_B)$ are* equivalent *if there exists an invertible linear map $L : A \to B$ such that $\Omega_B = L\Omega_A$ and $\mathcal{T}_B = L \circ \mathcal{T}_A \circ L^{-1}$.*

- *Show that the sets of effects of A and B satisfy $\mathcal{E}_B = \mathcal{E}_A \circ L^{-1}$; in particular, $u_B = u_A \circ L^{-1}$.*
- *Explain why equivalent (dynamical) state spaces agree in all their physical predictions. (Hint: consider laboratory setups consisting of preparations, transformations and measurements. How would we describe them?)*

Exercise 8 (Classical measurements)**.** *Consider the classical N-level state space (C, Ω_C) with $C = \mathbb{R}^N$ and $\Omega_C = \{(p_1, \ldots, p_N)^\top \mid p_i \geq 0,\ \sum_i p_i = 1\}$. Determine the order unit u_C, the set of effects $\mathcal{E}_C$, and the dual cone C_+^*. Interpret the result! What sorts of measurements can one do in classical probability theory?*

Hint: We can identify linear functionals $e : C \to \mathbb{R}$ with vectors $\vec{e} \in C$ via the standard (dot) inner product, such that $e(x) = \vec{e} \cdot x$. Then u_C becomes a vector in $\mathbb{R}^N$, and both $\mathcal{E}_C$ and C_+^* are subsets of $\mathbb{R}^N$.

Exercise 9 (Perfectly distinguishable quantum states)**.** *Consider the quantum n-level state space (A, Ω_A), where $A = \mathbb{C}_{\rm sa}^{n\times n}$ and $\Omega_A = \{\rho \in A \mid \mathrm{tr}(\rho) = 1,\ \ \rho \geq 0\}$.*

- *Show that n pure quantum states $\rho_1 = |\psi_1\rangle, \langle\psi_1|, \ldots, \rho_n = |\psi_n\rangle\langle\psi_n|$ are perfectly distinguishable if and only if $\langle\psi_i|\psi_j\rangle = \delta_{ij}$.*
- *When are n (not necessarily pure) quantum state $\rho_1, \ldots, \rho_n$ perfectly distinguishable?*
- *Show that if n quantum states $\rho_1, \ldots, \rho_n$ are* pairwise *perfectly distinguishable, then they are all (jointly) perfectly distinguishable. Show that the analogous statement does not hold, however, for the gbit, even if we restrict ourselves to pure states.*

Exercise 10 (Conditional states, reduced states, and purity of products of pure states)**.** *We now discuss some notions of GPTs that generalize quantum theory's conditional states and partial trace (i.e. reduced states). To this end, recall that the cone of unnormalized states on a state space (A, Ω_A) is $A_+ = \{\lambda\omega \mid \lambda \geq 0,\ \ \omega \in \Omega_A\}$. Furthermore, the dual cone is*

$$A_+^* = \{e \in A^* \mid e(\omega) \geq 0 \text{ for all } \omega \in A_+\} = \{\lambda e \mid \lambda \geq 0, e \in \mathcal{E}_A\}$$

(the second equality says that it is also obtained as the set of non-negative multiples of all effects). It turns out (convex geometry) that the dual of the dual cone is exactly the original cone of unnormalized states, $(A_+^*)^* = A_+$; that is,

$$A_+ = \{\omega \in A \mid e(\omega) \geq 0 \text{ for all } e \in A_+^*\}.$$

This will be very useful for the following, when we prove properties of locally tomographic composites $(A \otimes B, \Omega_{AB})$ of two state spaces (A, Ω_A) and (B, Ω_B).

- *Suppose that Bob performs a local measurement on $\omega_{AB} \in \Omega_{AB}$ and obtains outcome $e_B \in \mathcal{E}_B$. We will now show that there exists a* conditional state *$\omega_{A|e_B} \in \Omega_A$ that Alice will assign to her half of the physical system, after Bob has communicated his outcome to Alice.*

  To this end, argue via probability theory that this state, if it exists, must satisfy

  $$e_A(\omega_{A|e_B}) = \frac{e_A \otimes e_B(\omega_{AB})}{u_A \otimes e_B(\omega_{AB})} \qquad \text{for all } e_A \in \mathcal{E}_A.$$

  Now argue that the right-hand side can be seen as a linear functional φ that takes the covector $e_A \in A^*$ to a real number, i.e. $\varphi \in (A^*)^* = A$. Moreover, argue that $\varphi \in (A_+^*)^*$. Finally, show that $\varphi \in \Omega_A$, and so we can choose $\omega_{A|e_B} := \varphi$. Clearly, we can exchange the roles of Alice and Bob, obtaining conditional states $\omega_{B|e_A}$.

- *We can now define the reduced state of Alice as* $\omega_A := \omega_{A|u_B}$*, and of Bob as* $\omega_B := \omega_{B|u_A}$*. Argue that* $\omega_A = (\mathrm{Id}_A \otimes u_B)(\omega_{AB})$*, and that the reduced states are exactly those states that Alice and Bob assign to their halves of the joint state if there is no communication between them whatsoever. This generalizes the partial trace from quantum theory.*

- *Prove the following: if* ω_{AB} *is any bipartite state such that* ω_A *is pure, then* $\omega_{AB} = \omega_A \otimes \omega_B$*, i.e.* A *and* B *are uncorrelated.*

  Hint: Choose any $e_B \in \mathcal{E}_B$ *such that* $\lambda := e_B(\omega_B) \in (0,1)$*, let* $\bar{e}_B := u_B - e_B$*, and set* $\varphi_A := \lambda\omega_{A|e_B} + (1-\lambda)\omega_{A|\bar{e}_B}$*. Prove that* $e_A(\varphi_A) = e_A(\omega_A)$ *for all* $e_A \in \mathcal{E}_A$*, and argue that it follows that* $\varphi_A = \omega_A = \omega_{A|e_B}$*. From this it follows that* $(e_A \otimes e_B)(\omega_A \otimes \omega_B) = (e_A \otimes e_B)(\omega_{AB})$ *for all* $e_A \in \mathcal{E}_A, e_B \in \mathcal{E}_B$ *(why?), and thus, by local tomography,* $\omega_{AB} = \omega_A \otimes \omega_B$*.*

- Challenging bonus exercise, only for the interested freaks: *show that if* ω_A, ω_B *are both pure then* $\omega_A \otimes \omega_B$ *is pure, too.*

Exercise 11 (The subspace axiom in quantum theory)**.** *Let* Ω_N *be the quantum* N*-level state space (where* $N \in \mathbb{N}$*), and suppose that* $E_1, \ldots, E_N$ *is a POVM (positive operator-valued measure, i.e. a general measurement) that perfectly distinguishes* N *quantum states. Consider*

$$S_{N-1} := \{\rho \mid \mathrm{tr}(\rho E_N) = 0\}.$$

Show that this subset of states is equivalent to Ω_{N-1}*, i.e. the quantum* $(N-1)$*-level state space. You can do this via the following steps:*

- *Let* $\rho_1, \ldots, \rho_N$ *be a set of quantum states that is perfectly distinguished by the* $E_1, \ldots, E_N$*. Show that there is an orthonormal basis* $|\psi_1\rangle, \ldots, |\psi_N\rangle$ *such that* $\rho_i = |\psi_i\rangle\langle\psi_i|$*.*

- *You can represent every operator* E_j *as an* $N \times N$*-matrix in the orthonormal basis just constructed. Use the result of Exercise 2 [which applies to all positive semidefinite matrices, not just to density matrices] to say something about the matrix entries, and show that* $E_j = |\psi_j\rangle\langle\psi_j|$*.*

- *Now use again the result of Exercise 2. Which matrices* ρ *lie in* S_{N-1}*?*