Quantum operations: some more details

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(Dated: November 13, 2018)

Definition 1 (Quantum operation – cf. Nielsen, Chuang, Quantum Computation and Quantum Information). Given two finite-dimensional Hilbert spaces \(A\) and \(B\), a quantum operation is a map \(T : \mathcal{B}(A) \to \mathcal{B}(B)\) such that

(i) \(T\) is trace-preserving: \(\text{tr}(T(\rho)) = \text{tr}(\rho)\) for all \(\rho \in \mathcal{B}(A)\);

(ii) \(T\) is linear;

(iii) \(T\) is completely positive. That is, not only does \(T\) map positive operators to positive operators, but also, if \(E\) is an additional finite-dimensional Hilbert space, then the map \(T \otimes 1\) from \(\mathcal{B}(A \otimes E)\) to \(\mathcal{B}(B \otimes E)\) has this property, too.

(Sometimes, one considers the more general class of trace non-increasing CP maps, i.e. linear completely positive maps \(T\) such that \(\text{tr}(T(\rho)) \leq \text{tr}(\rho)\) for all \(\rho\). This is important, for example, if we have filters like in the lecture on higher-order interference.)

As a consequence of Definition 1, \(T\) maps density operators on \(A\) to density operators on \(B\). However, not every linear map with this property is a quantum operation. For example, consider the transposition

\[ T(\rho) := \rho^\top. \]

Exercise 2. The matrix components of any operator \(X\) are \(X_{ij} := \langle i | X | j \rangle\), where \(i, j = 0, 1\) labels some orthonormal basis of \(\mathbb{C}^2\). Show that \(T\) is linear and trace-preserving, that \(X = \sum_{ij} X_{ij} |i\rangle \langle j|\), that \(X^\top = \sum_{ij} X_{ji} |j\rangle \langle i|\), and that \(T(|i\rangle \langle j|) = |j\rangle \langle i|\equiv |i\rangle (j)\top\).

Let \(\rho = |\psi\rangle \langle \psi|\) with \(|\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)\) (a maximally entangled state), then

\[ T \otimes 1(\rho) = \frac{1}{2} T \otimes 1 \left( |00\rangle \langle 00| + |00\rangle \langle 11| + |11\rangle \langle 00| + |11\rangle \langle 11| \right) \]

\[ = \frac{1}{2} \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right). \]

(Note that \(|ij\rangle = |i\rangle \otimes |j\rangle\) and \(|ij\rangle \langle kl|\) is a density matrix because it has a negative eigenvalue. Thus \(T\) is not a quantum operation – it is positive, but not completely positive.

Theorem 3. The set of quantum operations from \(\mathcal{B}(A)\) to \(\mathcal{B}(B)\) is exactly the set of maps

\[ T[\rho] = \sum_k A_k \rho A_k^\dagger, \text{ with } \sum_k A_k^\dagger A_k = 1. \]

This is called the "Kraus representation", and the \(A_k\) are linear maps from \(A\) to \(B\).

The unitaries are a special case: \(\rho \mapsto U \rho U^\dagger\), with \(U^\dagger U = 1\).

A natural way to describe the dynamics of an open quantum system is to regard it as arising from an interaction between the system of interest (\(A\)) and an environment (\(E\)).

Theorem 4 ("Stinespring dilation" / environment representation). Every quantum operation \(T : \mathcal{B}(A) \to \mathcal{B}(A)\) can be written in the form

\[ T[\rho] = \text{Tr}_E \left( U_{AE} \rho_A \otimes |0\rangle \langle 0| E U_{AE}^\dagger \right), \]
where $U_{AE}$ is some unitary on $AE = A \otimes E$, and $|0\rangle$ is an arbitrary pure state on $E$. In fact, it is always possible to have $\dim E = (\dim A)^2$. More generally, any quantum operation $T : \mathcal{B}(A) \to \mathcal{B}(B)$ can be written

$$T[\rho] = \text{Tr}_{E'} \left( U_{AE} \rho_A \otimes |0\rangle \langle 0|_E U_{AE}^\dagger \right),$$

where $AE = BE'$ are two bipartitions of the same composite Hilbert space.

It is nice that three different physically motivated definitions turn out to be equivalent, yielding the same definition of a quantum operation.

For our purpose, the most important quantum operations are the reversible transformations, i.e. those operations that can be physically inverted. It turns out that these are exactly the unitary maps.

**Theorem 5.** Let $T : \mathcal{B}(A) \to \mathcal{B}(A)$ be a quantum operation that is invertible, and for which $T^{-1}$ is a quantum operation too. Then there is a unitary matrix $U$ on $A$ such that $T[\rho] = U \rho U^\dagger$. (And thus, for arbitrary $t > 0$, there is a self-adjoint matrix $H$ on $A$ such that $T[\rho] = e^{itH} \rho e^{-itH}$).