Quantum operations: some more details

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(Dated: November 13, 2018)

Definition 1 (Quantum operation – cf. Nielsen, Chuang, Quantum Computation and Quantum Information)**.** *Given two finite-dimensional Hilbert spaces A and B, a* quantum operation *is a map $T:\mathcal{B}(A)\to\mathcal{B}(B)$ such that*

(i) T is trace-preserving: $\mathrm{tr}(T(\rho))=\mathrm{tr}(\rho)$ for all $\rho\in\mathcal{B}(A)$;

(ii) T is linear;

(iii) T is completely positive*. That is, not only does T map positive operators to positive operators, but also, if E is an additional finite-dimensional Hilbert space, then the map $T\otimes\mathbf{1}$ from $\mathcal{B}(A\otimes E)$ to $\mathcal{B}(B\otimes E)$ has this property, too.*

(Sometimes, one considers the more general class of *trace non-increasing* CP maps, i.e. linear completely positive maps T such that $\mathrm{tr}(T(\rho))\leq\mathrm{tr}(\rho)$ for all ρ. This is important, for example, if we have *filters* like in the lecture on higher-order interference.)

As a consequence of Definition 1, T maps density operators on A to density operators on B. However, not every linear map with this property is a quantum operation. For example, consider the *transposition*

$$T(\rho):=\rho^\top.$$

Exercise 2. *The matrix components of any operator X are $X_{ij}:=\langle i|X|j\rangle$, where $i,j=0,1$ labels some orthonormal basis of $\mathbb{C}^2$. Show that T is linear and trace-preserving, that $X=\sum_{ij}X_{ij}|i\rangle\langle j|$, that $X^\top=\sum_{ij}X_{ji}|i\rangle\langle j|$, and that $T(|i\rangle\langle j|)=|j\rangle\langle i|\equiv|i\rangle\langle j|^\top$.*

Let $\rho=|\psi\rangle\langle\psi|$ with $|\psi\rangle=\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right)$ (a maximally entangled state), then

$$\begin{aligned}T\otimes\mathbf{1}(\rho) &= \frac{1}{2}T\otimes\mathbf{1}\left(|00\rangle\langle 00|+|00\rangle\langle 11|+|11\rangle\langle 00|+|11\rangle\langle 11|\right)\\ &= \frac{1}{2}\left(|00\rangle\langle 00|+|10\rangle\langle 01|+|01\rangle\langle 10|+|11\rangle\langle 11|\right)\\ &= \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}.\end{aligned}$$

(Note that $|ij\rangle=|i\rangle\otimes|j\rangle$ and $|ij\rangle\langle kl|=|i\rangle\langle k|\otimes|j\rangle\langle l|$.) This is not a density matrix because it has a negative eigenvalue. Thus T is not a quantum operation – it is positive, but not completely positive.

Theorem 3. *The set of quantum operations from $\mathcal{B}(A)$ to $\mathcal{B}(B)$ is exactly the set of maps*

$$T[\rho]=\sum_k A_k\rho A_k^\dagger,\ \textit{with}\ \sum_k A_k^\dagger A_k=\mathbf{1}.$$

This is called the "Kraus representation", and the A_k are linear maps from A to B.

The unitaries are a special case: $\rho\mapsto U\rho U^\dagger$, with $U^\dagger U=\mathbf{1}$.

A natural way to describe the dynamics of an open quantum system is to regard it as arising from an interaction between the system of interest (A) and an *environment* (E).

Theorem 4 ("Stinespring dilation" / environment representation)**.** *Every quantum operation $T:\mathcal{B}(A)\to\mathcal{B}(A)$ can be written in the form*

$$T[\rho]=\mathrm{Tr}_E\left(U_{AE}\rho_A\otimes|0\rangle\langle 0|_E U_{AE}^\dagger\right),$$

where U_{AE} is some unitary on $AE = A \otimes E$, and $|0\rangle$ is an arbitrary pure state on E. In fact, it is always possible to have $\dim E = (\dim A)^2$. More generally, any quantum operation $T : \mathcal{B}(A) \to \mathcal{B}(B)$ can be written

$$T[\rho] = \mathrm{Tr}_{E'}\left(U_{AE}\rho_A \otimes |0\rangle\langle 0|_E U_{AE}^\dagger\right),$$

where $AE = BE'$ are two bipartitions of the same composite Hilbert space.

It is nice that three different physically motivated definitions turn out to be equivalent, yielding the same definition of a quantum operation.

For our purpose, the most important quantum operations are the *reversible transformations*, i.e. those operations that can be physically inverted. It turns out that these are exactly the unitary maps.

Theorem 5. *Let $T : \mathcal{B}(A) \to \mathcal{B}(A)$ be a quantum operation that is invertible, and for which T^{-1} is a quantum operation too. Then there is a unitary matrix U on A such that $T[\rho] = U\rho U^\dagger$. (And thus, for arbitrary $t > 0$, there is a self-adjoint matrix H on A such that $T[\rho] = e^{itH}\rho e^{-itH}$).*