Quantum Correlations, Lecture 10

Notation: I'm sometimes using \((A, \mathcal{R}_A)\) for a state space, and sometimes \((A, A^+, \mathcal{U}_A)\). This is completely equivalent.

7.5. Measurements (continued)

**Definition**: Let \((A, A^+, \mathcal{U}_A)\) be any state space. \(w_1, \ldots, w_n \in \mathcal{R}_A\) are called perfectly distinguishable if there is an \(n\)-outcome measurement \(e_1, \ldots, e_n \in \mathcal{E}_A\), \(\sum_{i=1}^n e_i = \mathcal{U}_A\) such that \(e_i(w_j) = \delta_{ij}\).

The maximal size of any set of perfectly distinguishable states is called \(N_A\).

Furthermore, use the notation \(K_A := \dim(A)\).

**Lemma**: \(N_A\) is also the maximal size of any set of perfectly distinguishable pure states.

**Proof**: Suppose \(w_1, \ldots, w_n\) perfectly dist. with \(n = N_A\). Corresponding effects: \(e_1, \ldots, e_n\); \(e_i(w_j) = \delta_{ij}\).
Decompose every $\psi_i$ into pure states:

$$\psi_i = \sum_{k=1}^{i-1} \alpha_{i, k} w^{(k)}_i, \quad \sum_{k=1}^{i-1} \alpha_{i, k} = 1.$$  

For $j \neq i$:

$$0 = e_j(\psi_i) = \sum_{k=1}^{i-1} \alpha_{i, k} e_j(w^{(k)}_i) \quad \forall \alpha_{i, k} > 0 > 0$$

$$\Rightarrow e_j(\psi^{(k)}_i) = e_j(w^{(k)}_i) = 0.$$  

For $j = i$:

$$1 = e_j(\psi_i) = \sum_{k=1}^{i-1} \alpha_{i, k} e_j(w^{(k)}_i) \quad \forall \alpha_{i, k} > 0 \leq 1$$

$$\Rightarrow e_j(\psi^{(k)}_i) = e_j(w^{(k)}_i) = 1.$$  

Thus, $e_1, \ldots, e_n$ also distinguishes the pure states $\psi^{(1)}_1, \ldots, \psi^{(n)}_n$. \hfill $\square$

**Example:** Quantum Theory.

$$A = \{ S \in C^{n \times n} \mid \forall S = 1, S \geq 0 \},$$  

$$A = \{ M \in C^{n \times n} \mid M = M^+ \}.$$  

Suppose $\psi_1, \ldots, \psi_N$ are a maximal set of pure perfectly distinguishable states; $e_1, \ldots, e_N$ correspond to effects. We know:

$$e_i(S) = \text{tr}(E_i S), \quad 0 \leq E_i \leq 1, \quad \sum_{i=1}^{N} E_i = 1.$$  

$w_i = \frac{\alpha_i}{\sqrt{\lambda_i}} > 0$.

$$\Rightarrow \delta_{ij} = e_i(w_j) = \text{tr}(E_j w_j) = \langle \psi_j | E_j | \psi_j \rangle$$

$$\Rightarrow E_i | \psi_j \rangle = \delta_{ij} | \psi_j \rangle.$$  

Suppose $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ are such that $\alpha_i | \psi_i \rangle + \ldots + \alpha_N | \psi_N \rangle = 0$
\[ 0 = E_i \cdot 0 = E_i (\chi_n | 4_n \rangle + \cdots + \chi_N | 4_n \rangle) = \chi_i | 4_i \rangle \]

\[ \Rightarrow \chi_i = 0 \]

\[ \Rightarrow 14_1, \ldots, 14_n \text{ linearly independent} \]

\[ \Rightarrow N \leq n \]

Conversely, if \( 14_1, \ldots, 14_n \rangle \) is any ONB, then those states are perfectly dist. via \( E_i = \chi_i \times | 4_i \rangle \).

\[ \Rightarrow N = n \text{, the dimension of the underlying Hilbert space.} \]

\[ \dim A = \# \text{ of real params in a n+n unnormalized density matrix:} \]

\[ \text{real} \quad n + 2 \cdot (1+2+\ldots+(n-1)) = n^2 + \frac{n(n-1)}{2} = n^2 \]

\[ \Rightarrow K_A = n^2 \]

\[ \text{i total} \quad K_A = N \cdot n^2 \]

**Lemma:** For every state space \((A, A_+, A_+)\), we have \( K_A \geq N_A \) with equality if and only if the state space is classical, i.e. equivalent to the classical \( K_A \)-level state space.
Proof: Exercise.

Example: qubit: $B = \mathbb{R}^3 \Rightarrow K_B = 3$

Show last time: Can perfectly distinguish e.g. $w_1$ and $w_2$

\[
\begin{align*}
3 = & \sum_{\mu} e_{\mu} = 1 + 0 \\
\Rightarrow & \quad N_B \geq 2.
\end{align*}
\]

But $3 = K_B \leq N_B$, and equality only if classical, i.e.,

\[
= \sum_{\mu} \mu \Rightarrow \quad N_B = 2 \quad (\text{thus } "qubit")
\]

7.6. Composite state spaces

\[
\begin{array}{c|c}
A & AB = \?
\end{array}
\]

How can we combine state spaces $(A, A_+, U_A)$ and $(B, B_+, U_B)$ into a joint state space $AB$?

Requirements on a meaningful composite $AB$:

\[
\begin{align*}
\text{No signaling: For every } w_{AB} \in AB, \text{ local measurement statistics are described by local states } y_A \in A, y_B \in B.
\end{align*}
\]
which do not depend on any actions of the other party.

In particular, if \( e^A_i, \ldots, e^A_m \) and \( e^B_i, \ldots, e^B_n \) are local measurements, then there is a measurement with effects (outcomes) \( e^A_i e^B_i, \ldots, e^A_i e^B_j \) on \( AB \) that give probabilities \( p_{ij} \) in \( \text{CO}_i \text{CO}_j \) on all states.

* Independent preparations: \[ \begin{pmatrix} W_A \\ W_B \end{pmatrix} = \begin{pmatrix} W_A \otimes W_B \end{pmatrix} \]

For every \( W_A \in A, W_B \in B, \) there is a state \( W_A \otimes W_B \) describing the independent local preparations of \( W_A \) and \( W_B \). They satisfy
\[
(e^A_i, e^B_j)(W_A \otimes W_B) = e^A_i(W_A) \cdot e^B_j(W_B).
\]

\( \Rightarrow \) It follows that we can represent
\[
W_A \otimes W_B = W_A \otimes W_B, \\
\otimes e^B = e^A \otimes e^B, \quad \otimes W_B = W_A \otimes W_B.
\]

For the vector space \( AB \) carrying \( \otimes AB \),
\( \otimes A \otimes B = AB \).

The following is sometimes assumed, but not always:

**Local Tomography:** States on \( AB \) are uniquely determined by the statistics and correlations of local measurements.
Formally, local tomography ("tomographic locality") means the following.

Suppose that \( \rho_{AB}, \rho_{A} \in \mathcal{S}_{AB} \) are states on \( AB \).

Then if \( e^{A} \otimes f^{B}(\rho_{AB}) = e^{A} \otimes f^{B}(\rho_{A}) \) for all \( e^{A} \in \mathcal{E}_{A}, f^{B} \in \mathcal{E}_{B} \), then \( \rho_{AB} = \rho_{A} \).

This is equivalent to \( AB = A \otimes B \) (as vector spaces) and to \( K_{AB} = K_{A} \cdot K_{B} \), because it implies that \( A^{*} \otimes B^{*} \) separates points in \( AB \).

It turns out that classical probability theory satisfies this principle, and so does standard complex quantum theory.

**Example:** Quantum Theory (over \( \mathbb{C} \)).

If \( A \) is a quantum \( n \)-level state space,

\[
A = \{ M \in \mathbb{C}^{n \times n} \mid M = M^{+} \}
\]

\[
\Omega_{A} = \{ S \in \mathbb{C}^{n \times n} \mid S^{*} = 1, \ S > 0 \}
\]

and \( B \) is a quantum \( n \)-level state space (analogous def.), then the standard textbook rules of QM say that \( AB \) is constructed by taking the tensor product of the underlying complex Hilbert spaces,

\[
\mathcal{H}_{AB} = \mathcal{H}_{A} \otimes \mathcal{H}_{B},
\]

where \( \mathcal{H}_{A} = \mathbb{C}^{m}, \mathcal{H}_{B} = \mathbb{C}^{n} \), such that \( \mathcal{H}_{AB} = \mathbb{C}^{m} \otimes \mathbb{C}^{n} \cong \mathbb{C}^{mn} \).
Thus
\[ AB = \{ M \in C^{(mn) \times (mn)} \mid M = M^+ \} , \]
\[ S_{AB} = \{ S \in C^{(mn) \times (mn)} \mid \text{tr} S = 1, S \geq 0 \} . \]

To prove local tomography, we check that \( K_{AB} = K_A K_B \).

The textbook rule above can be formulated in our language as

"The composite \( AB \) of quantum state spaces \( A \) and \( B \) is by definition the quantum state space with
\[ N_{AB} = \frac{N_A \cdot N_B}{m \cdot n} \] levels."

Now we use our earlier result

\[ K_A = N_A^2 \]

for standard complex quantum theory.

\[ \Rightarrow \]

\[ K_{AB} = (N_A N_B)^2 = N_A^2 N_B^2 = K_A K_B . \]

Local tomography holds!

So we can really determine (entangled) quantum states by doing local measurements, and looking at the statistics (including the correlations).

Example: Quantum theory over IR or IQH.

We can build state spaces out of density matrices over the real numbers IR or quaternions IQH. Those have been
studied very early on in physics by Jordan and others, and they appear in mathematics as examples of so-called Jordan algebras.

\[ A_R^\mathbb{R} := \{ M \in \mathbb{R}^{mxm} \mid M = M^T \} \]

\[ N^\mathbb{R}_R := \{ S \in A_R \mid \text{trace}(S) = 1, \text{S positive-semidefinite} \} \]

Similar definitions hold for the quaternionic case (more online). A quaternion can be written

\[ a = a_0 + a_1 i + a_2 j + a_3 k \]

with \( i^2 = j^2 = k^2 = ijk = -1 \).

Setting \( a = a_0 - a_1 i - a_2 j - a_3 k \) allows to define \( M^+ \) for quaternionic matrices in the obvious way.

Parameter counting shows that

\[ K_A_R = \frac{N_A_R(N_A_R + 1)}{2}, \quad K_{A_{\mathbb{H}}} = N_{A_{\mathbb{H}}}(2N_{A_{\mathbb{H}}} - 1) \]

Repeating the calculation above shows that

\[ K_{AB} = K_A \cdot K_B \]

**Local tomography does not hold!**

This gives a first hint as to "why" the complex numbers appear in quantum theory and physics.

More on this later.