8. A generalized no-cloning theorem (continued)

Last lecture: composite systems \((A \otimes B, A \otimes B, A \otimes \overline{B})\)
and the following is true in any theory with composition:

**Theorem:** States \(w_1, \ldots, w_n \in A^*\) are cloneable if and only if they are perfectly distinguishable.

**Proof Sketch:**
- If perfectly distinguishable, then cloneable.
  
If \(w_1, \ldots, w_n \in A^*\) are perfectly distinguishable then there is a measurement with effects \(e_1, \ldots, e_n \in A^*\) such that \(\sum_i e_i = 1\) and \(e_i(w_j) = \delta_{ij}\).

Define the map \(T : A \rightarrow A \otimes A\) as \(T(w) = \sum_{i=1}^n e_i(w) \cdot w_i \otimes w_i\).

\(T\) is linear, \(T(w_j) = w_j \otimes w_j\) (produces 2 independent copies).

\(T\) is normalization-preserving: \(\langle w_{\otimes} | T(w) \rangle = \sum_{i=1}^n e_i(w) \langle w_i | w_i \rangle = 1\).
and it is positive (i.e., maps states to states): Consider \( w_i \otimes w_i \). This is an element of \( \mathbb{S}_{\min A_B} \).

If \( w \in A \), then \( e_i(w) > 0 \) and \( \sum e_i(w) = U_A(w) = 1 \) \( \Rightarrow T(w) \) is a convex combination of the \( w_i \otimes w_i \) \( \Rightarrow T(w) \in \mathbb{S}_{\min A_B} \leq \mathbb{S}_{A_B} \) is a valid state.

In fact, \( T \) describes a "measure-and-prepare" transformation.

If cloneable then perfectly distinguishable:

Claim: if \( w \sim w \) cloneable then \( \forall \varepsilon > 0 \exists \sum e_i^{(\varepsilon)} = U_A \) such that \( |e_i^{(\varepsilon)}(w_j) - \delta_{ij}| < \varepsilon \).

Topological closure of \( A^+ \) proves that we can also find exactly perfectly distinguishing effects.

Sketch: build a device that reliably distinguishes \( w \sim w \) via

Do state tomography on \( w_i \otimes (2^n) \) \( \Rightarrow \) can find reliably what "i" is.
Details: Law of large numbers etc., see last lecture's notes.

9. A derivation of QT from simple operational postulates

Situation so far:

\[ \text{landscape of all general probabilistic theories} \]

state spaces: arbitrary compact convex sets of arbitrary dimension.

many different ways to build composite state spaces

Different physical predictions:

supershong nonlocality, higher-order interference, other possible dynamics and uncertainty relations etc.

"Why" QT and not another prob. theory?

Theorem (Masanes + Pusey 2010):

\[ (A_N, (A+)N, U_N, G_N) \]

Suppose we have a collection of dynamical state spaces, with the union of perfectly distinguishable states, such that the following 3 postulates hold:

1) Local tomography: Every pair of state spaces \( A, B \) has a locally-tomographic composite \( AB \) in the collection.
2) Reversibility: For any pair of pure states \( w, \psi \in \Omega_N \) there is a reversible transformation \( T \in G_N \) such that \( Tw = \psi \).

3) Subspace Action:
If \( (e_1, \ldots, e_N) \) is any measurement on \( A_N \) that perfectly distinguishes \( N \) states, then the set of those states \( w \in \Omega_N \) with \( e_N(w) = 0 \) is equivalent to \( \Omega_{N-1} \).

4-level system in the lab.
System guaranteed NOT
System guaranteed to be found in 4th level
\( \Rightarrow \) we get a 3-level system.

Then one of the two following statements must be true:

I. We have classical probability theory:
\( \Omega_N \) is the set of \( N \)-level probability distributions,
and \( G_N \) is the group of all permutations.
II. (Standard complex) quantum theory:

$\mathbb{C}^N$ is the set of $N \times N$ density matrices, and $\mathbb{G}_N$ is the group of unitary conjugations,

$S \mapsto U S U^\dagger$, with $U \in \mathbb{SU}(N)$

Consequences:

- Schrödinger equation: the only possible continuous reversible time evolution satisfies

$\dot{S} = -i [S, H]$ with $H \in C^{N \times N}$, $H = H^\dagger$

- Usual tensor product rule of combining $n$ qubit systems,

- Measurements are POVM or

- Open system evolutions are given by completely positive, trace-preserving maps,

- Tsirelson bound for CHSH inequality,

If Postulate 2 is replaced by

2') Continuous reversibility

the only solution II/07 survives

Discuss similarity with "theories of geometry" and
Einstein's Postulates.

History:

- Long prehistory

- 2004: Lucien Hardy: "At first 5 reasonable axioms". But: so many solutions; need "Simplicity Axion"?

- 2010/11: 3 groups give solution almost at the same time.

See links on course website!
9.1. Why bits are balls

From now on, assume Postulates 1), 2) and 3)

Lemma 9.1.1: The bit state space \( \Omega_2 \) is strictly convex, i.e., does not contain lines in its boundary.

Proof: Convex geometry. Let \( \Omega_2 \) be a hyperplane \( H \) that touches \( \Omega_2 \) in only one point \( w_0 \):

\[ H = \{ x \in \mathbb{R}^2 \mid u_2(x) = 1 \} \]

Let \( H' \) be a parallel hyperplane such that \( \Omega_2 \) is between \( H \) and \( H' \), and \( H' \cap \Omega_2 = \emptyset \).

Thus, there exists an affine function \( e \) such that \( H = \{ x \mid e(x) = 0 \} \) and \( H' = \{ x \mid e(x) = 1 \} \).

\[ 0 \leq e(w) \leq 1 \quad \forall w \in \Omega_2 \quad \Rightarrow \quad e \in E \]

\[ (u_2, e, e) \text{ is a perfect dist. 2 states.} \]

Subspace Pachl. \( H \cap \Omega_2 = \{ w_0 \} \) is equivalent to \( \Omega_1 \).

\[ \Omega_1 \text{ contains only one single state.} \]

Now suppose that \( \Omega_2 \) was not strictly convex.

Analogous argumentation:
Subspace postulate: \( \{ w \in \mathcal{L}_2 \mid e'(w) = 0 \} \) is equivalent to \( \mathcal{L}_1 \) state.

**Lemma 9.1.2**: There is a unique ("maximally mixed") state \( \mu \in \mathcal{S}_N \) such that
\[
Gw = \mu \text{ for all } G \in G_N.
\]

**Proof**: We use the reversibility postulate:

\[
G'w' = \mu
\]

Let \( \omega \in \mathcal{S}_N \) be any pure state, and define
\[
\mu_N = \int_{G \in G_N} G\omega \, dG \quad \text{(integration with respect to Haar measure on group)}.
\]

\[
\Rightarrow \int G\mu_N = \int (HG)\omega \, dG = \int G'\omega \, dG' = \mu_N.
\]

Similar argumentation: \( \mu_N \) does not depend on choice of \( \omega \).

Uniqueness: Suppose \( Gw = w \forall G \in G_N \)
\[
w = \sum_i \lambda_i \omega_i \Rightarrow w = Gu = \sum_i \lambda_i G\omega_i \Rightarrow \int Gw \, dG = \sum_i \lambda_i \int G\omega_i \, dG = \lambda_i \mu = \mu.
\]

For \( w \in \mathcal{S}_N \), define its "block vector"
\[ \hat{w} = w - \mu \Rightarrow \hat{w} \in \hat{A}_N = \{ x \in A_N | u_N(x) = 0 \} \]

and \[ G\hat{w} = Gw - G\mu = (Gw)^\perp, \quad G\mu = 0 \]
so \( Gw \) acts linearly on \( \hat{A}_N \). **Done!**

**Lemma 9.43:** \( A_2 \) is equivalent to a (Euclidean) unit ball (of some dimension \( d \in \mathbb{N} \)).

- **Strict convexity \( \Rightarrow \) all \( w \in A_2 \) are pure states.
- **Reversibility postulate \( \Rightarrow \) all \( w \in A_2 \) connected by a symmetry.

**Proof:** \((\cdot, \cdot)\): standard inner product on \( \hat{A}_2 \).

For \( x, y \in \hat{A}_2 \), set

\[
\langle x, y \rangle : = c \int_{G \in G_2} (Gx, Gy) dG \quad \forall G \in G_2
\]

Easy to check:

This is an inner product.

Augmentation as before:

\[
\Rightarrow \langle Gx, Gy \rangle = \langle x, y \rangle \quad \forall G \in G_2,
\]

so \( G \) is orthogonal w.r.t. \((\cdot, \cdot)\).

If \( w, w' \) are pure then there exists \( G \in G_2 \) such that

\[
w' = Gw \quad \text{(reversibility)} \Rightarrow \langle w', w' \rangle = \langle Gw, Gw \rangle = \langle w, w \rangle.
\]
Choose $c > 0$ such that

$$\langle \hat{\psi}, \hat{\psi} \rangle = 1 \text{ for all pure states } \hat{\psi} \in \Omega_2.$$ 

$$\Rightarrow \Omega_2 := \{ \hat{\psi} \mid \hat{\psi} \in \Omega_2 \} \text{ is a compact convex subset of }$$

- Euclidean unit ball $\hat{\mathbb{R}}^n = (\hat{\mathbb{R}}, \langle \cdot, \cdot \rangle)$.

All boundary points (pure states) have norm 1 $\Rightarrow$ $\Omega_2$ equals the unit ball. $\square$

$$d := \dim (\Omega_2) = \dim (\hat{\mathbb{R}}^2) - 1 \text{ is still unknown.}$$

$d = 1$ classical bit

$d = 2$

$d = 3$ quantum bit

$d > 4$

$d = 2, 5, 9$: bits in quantum theory over

- real numbers $\mathbb{R}$ / quaternions $\mathbb{H}$ /
- octonions $\mathbb{O}$.

For all $d$, we have $N=2$ perf. dist. states!

(Recall: 3-level QT state space much more complicated.)
Further steps (next time):

- show that $N_{AB} = N_A N_B$, and thus
- $\dim (\hat{\mathcal{H}}_2) = d \in \{1, 3, 7, 15, 31, \ldots\}$
- show that $d=1$ leads to CPT
- show that $d > 7$ leads to inconsistencies on 2 bits
- show that $d = 3$ leads to QT.