Testing quantum theory with generalized noncontextuality

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Abstract

It is a fundamental prediction of quantum theory that states of physical systems are described by complex vectors or density operators on a Hilbert space. However, many experiments admit effective descriptions in terms of other state spaces, such as classical probability distributions or quantum systems with superselection rules. Here, we ask which probabilistic theories could reasonably be found as effective descriptions of physical systems if nature is fundamentally quantum. To this end, we employ a **generalized version of noncontextuality: processes that are statistically indistinguishable in an effective theory should not require explanation by multiple distinguishable processes in a more fundamental theory.** We formulate this principle in terms of embeddings and simulations of one probabilistic theory by another, show how this concept subsumes standard notions of contextuality, and prove a multitude of fundamental results on the exact and approximate embedding of theories (in particular into quantum theory). We show how results on Bell inequalities can be used for the robust certification of generalized contextuality. From this, **we propose an experimental test of quantum theory by probing single physical systems without assuming access to a tomographically complete set of procedures,** arguably avoiding a significant loophole of earlier approaches.

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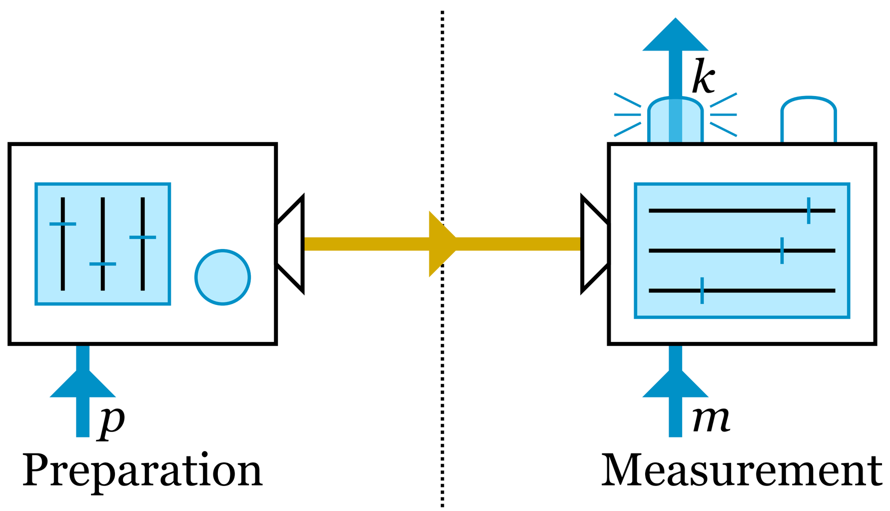
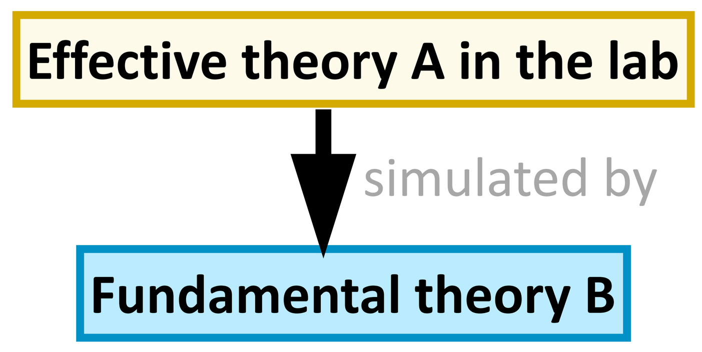
When could theory-agnostic tomography falsify QM?

Suppose we prepare and measure a system in many different ways, and fit a generalized probabilistic theory (GPT) to it [1]. If QM is correct, then we see a "shadow" of an (in general infinite-dimensional) quantum system. Which "shadows" are possible/plausible?



Which results would tell us that QM is an **implausible** explanation, i.e. challenge QM?



To address this, we introduce a notion of **simulation** of an (effective) GPT A by another (more fundamental) GPT B. Simulations can be **contextual or non-contextual** (see below).

- **A=Quantum Theory, B=Classical Prob. Theory**: this reduces to Spekkens' contextuality. Addresses the question of whether CPT (hidden variables) can plausibly explain QT.
- **A=experimental GPT result**, **B=Quantum Theory**: the case we are most interested in. Addresses the question of whether QT is a plausible explanation of our laboratory data.

[1] M. D. Mazurek, M. F. Pusey, K. J. Resch, and R. W. Spekkens, *Experimentally bounding deviations from quantum theory in the landscape of generalized probabilistic theories*, PRX Quantum **2**, 020302 (2021).

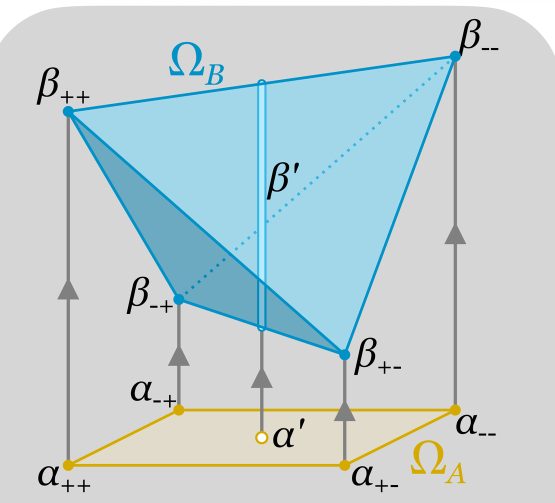
[2] R. W. Spekkens, *Contextuality for preparations, transformations, and unsharp measurements*, Phys. Rev. A **71**, 052108 (2005).

Definition: Contextual and noncontextual simulations

Definition 1 (Simulation). *Consider $\mathcal{A}=(A,\Omega_A,E_A)$ (the "effective GPT") and $\mathcal{B}=(B,\Omega_B,E_B)$ (the "fundamental GPT"), and let $\varepsilon\geq 0$. An* ***ε-simulation*** *of $\mathcal{A}$ by $\mathcal{B}$ assigns to each $\omega_A\in\Omega_A$ a nonempty set of states $\Omega_B(\omega_A)\subset\Omega_B$ ("the states that simulate ω_A"), and to every normalized effect $e_A\in E_A$ a nonempty set of effects $E_B(e_A)\subset E_B$ ("the effects that simulate e_A"), such that the following conditions hold:*

- *all outcome probabilities are reproduced up to ε: for all $\omega_A\in\Omega_A, e_A\in E_A$, we have*
$$|(\omega_A,e_A)-(\omega_B,e_B)|\leq\varepsilon\quad\forall\omega_B\in\Omega_B(\omega_A),e_B\in E_B(e_A);\tag{4}$$
- *mixtures of simulating states (effects) are valid simulations of mixtures of states (effects):*
$$\lambda\Omega_B(\omega_A)+(1-\lambda)\Omega_B(\varphi_A)\subseteq\Omega_B(\lambda\omega_A+(1-\lambda)\varphi_A)\tag{5}$$
for all $0\leq\lambda\leq 1$ and $\omega_A,\varphi_A\in\Omega_A$ (and the analogous inclusion for E_B on mixtures of effects);
- *the fundamentally impossible effect is a valid simulation of the effectively impossible effect:*
$$0\in E_B(0).\tag{6}$$

An ($\varepsilon=0$)-simulation is called an ***exact simulation****. The simulation is called* ***preparation–noncontextual*** *if $|\Omega_B(\omega_A)|=1$ for all $\omega_A\in\Omega_A$,* ***measurement–noncontextual*** *if $|E_B(e_A)|=1$ for all $e_A\in E_A$, and* ***noncontextual*** *if it is both preparation– and measurement–noncontextual.*



Ex.: Contextual simulation of a gbit (square bit) by CPT ("Holevo projection").

Noncontextual Ɛ-simulations are (linear) Ɛ-**embeddings**:

Definition 2 (Embedding). *Let $\mathcal{A}=(A,\Omega_A,E_A)$ and $\mathcal{B}=(B,\Omega_B,E_B)$ be GPTs, and let $\varepsilon\geq 0$. A pair of linear maps $\Phi:A\to B$ and $\Psi:A^*\to B^*$ is said to be an ε-***embedding*** *of $\mathcal{A}$ into $\mathcal{B}$ if*

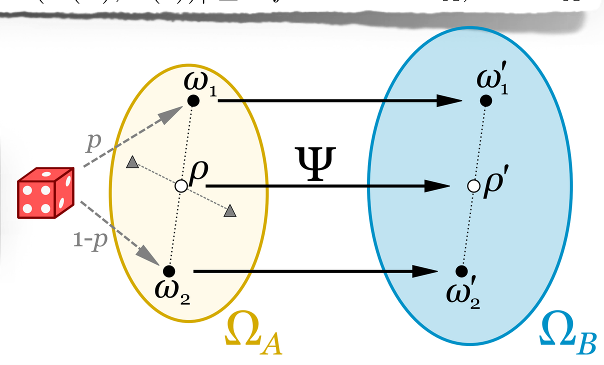
(i) Φ and Ψ are positive *and Ψ is normalization-preserving, i.e. $\Phi(E_A)\subseteq E_B$ and $\Psi(\Omega_A)\subseteq\Omega_B$;*

(ii) Φ and Ψ preserve outcome probabilities up to ε; *i.e. $|(\omega,e)-(\Psi(\omega),\Phi(e))|\leq\varepsilon$ for all $e\in E_A$, $\omega\in\Omega_A$.*

But there are no **non**contextual simulations:

Example 2. *Let $\varepsilon\leq 0.1014$. Then the gbit cannot be ε-embedded into any $\mathcal{Q}_n$ or $\mathcal{Q}_\infty$.*

(Here, $\mathcal{Q}_n$= *n*-dimensional quantum theory).



Theorem: All the GPTs with non-contextual quantum simulation

Among the "unrestricted" GPTs (i.e. states and effects are full duals of each other), we classify those that have an exact non-contextual quantum simulation:

Theorem 2. *An unrestricted GPT can be exactly embedded into finite-dimensional quantum theory if and only if it corresponds to a* special Euclidean Jordan algebra.

Essentially, these are QM over the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$, "Bloch balls" of some dimension *d*, and direct sums of those (including CPT).

Theorem: Certifying non-embeddability via Bell nonlocality

We introduce a method to certify that an *(in general, restricted!)* GPT A does not have any Ɛ-approximate noncontextual quantum (or classical) simulation.

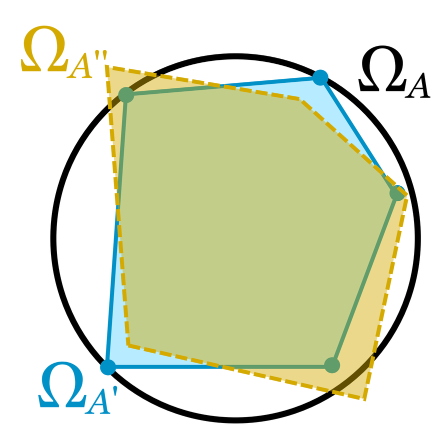
Theorem 3. *Suppose that $\mathcal{A}$ is a GPT that "admits of post-quantum self-correlations" in the following sense: there is some Bell functional B such that $B_{\mathcal{A}\mathcal{A}}>B_{\mathcal{Q}}$, i.e. some state on two copies of $\mathcal{A}$ violates the corresponding Bell inequality by more than any bipartite quantum state. Then, for every*
$$\varepsilon<\frac{B_{\mathcal{A}\mathcal{A}}-B_{\mathcal{Q}}}{4|B|(1+2\mathcal{R}(\mathcal{A}))},\tag{46}$$
the GPT $\mathcal{A}$ cannot be ε-embedded into any $\mathcal{Q}_n$ or $\mathcal{Q}_\infty$.

Known results on Bell inequalities can thus be "lifted" to prove non-embeddability of (single-system) GPTs.

A rigorous experimental test of QT

Arguably, **contextual simulations are implausible** for the same reasons as Spekkens' contextuality (Reichenbach's principle), but we prove further results that support this:

- Noncontextuality extends **transitively** across several different levels / layers of theories;
- noise and decoherence **cannot create** contextuality.



- Ω_A: actual effective state space of a laboratory system
- $\Omega_{A'}$: experimentally actually implemented states
- $\Omega_{A''}$: fitted GPT from data. Should have Ɛ-approx. embedding in QT.

This suggests a **novel experimental test of QT:**
Determine a GPT experimentally on a laboratory system. Certify that it cannot be Ɛ-embedded into QT (of any dimension), where Ɛ quantifies the experimental errors/ uncertainty. If this works, then QT is (arguably) falsified.