ÖAW AUSTRIAN ACADEMY OF SCIENCES

IQOQI – INSTITUTE FOR QUANTUM OPTICS AND QUANTUM INFORMATION VIENNA

Testing quantum theory with generalized noncontextuality

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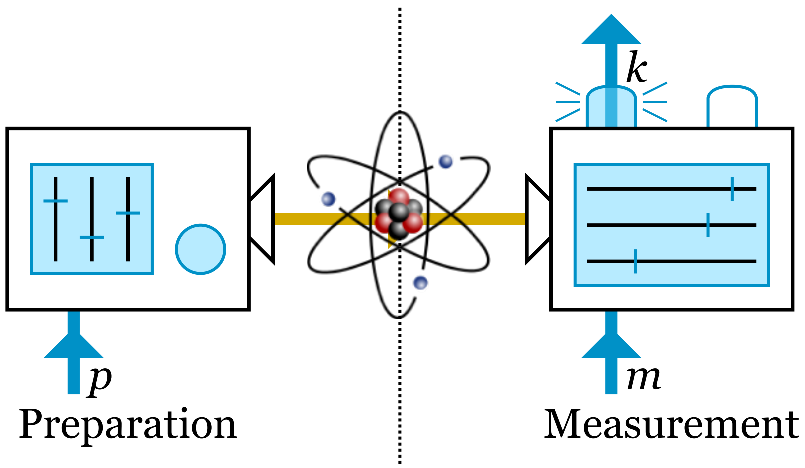
Two motivations

Suppose we **prepare** and **measure** a physical system in all ways accessible to us.



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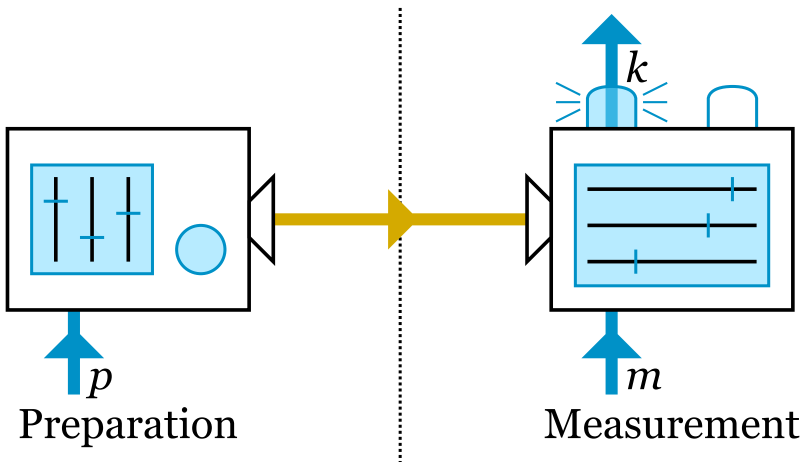

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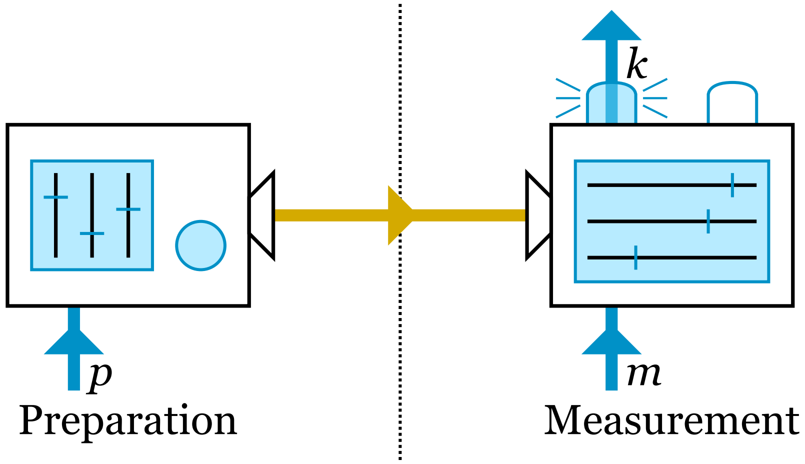

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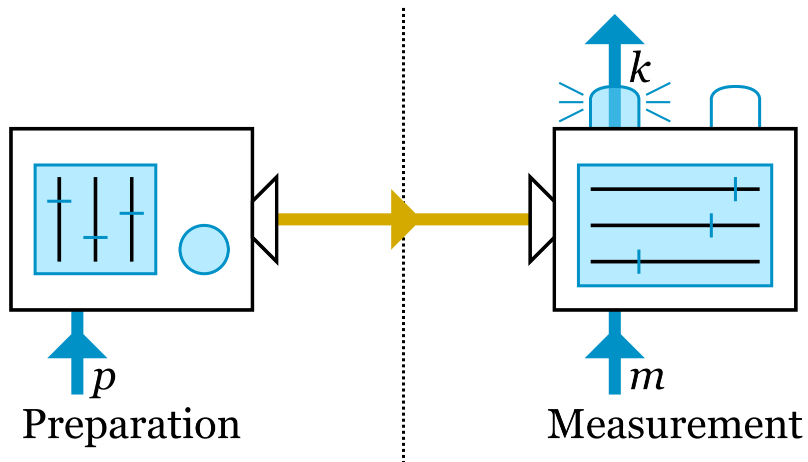

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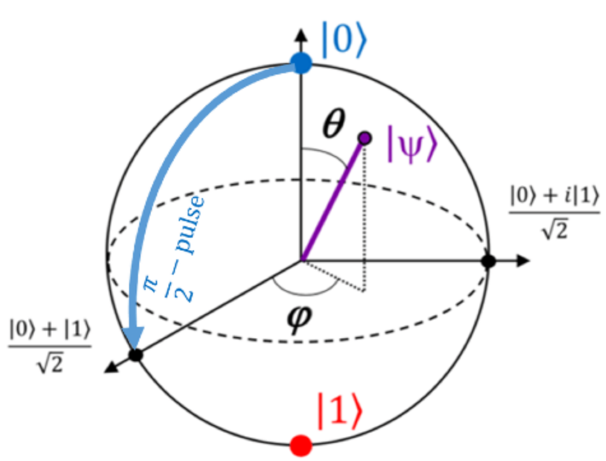

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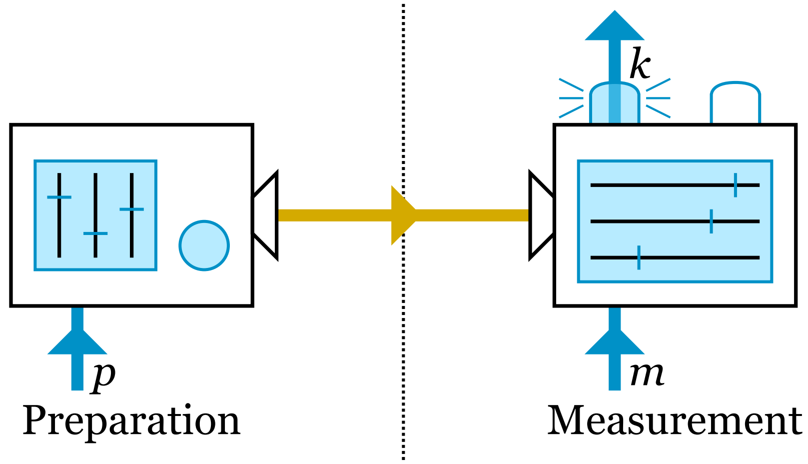

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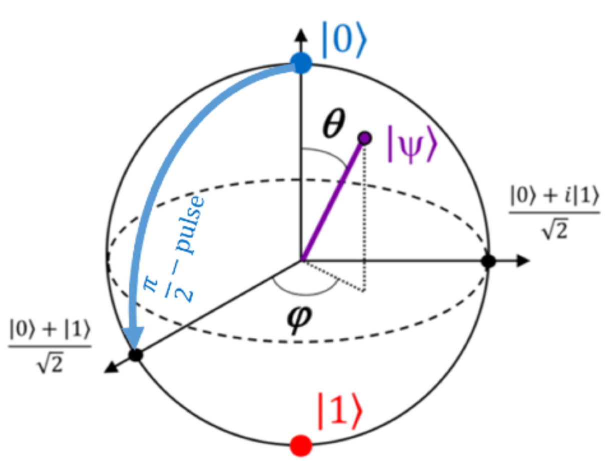

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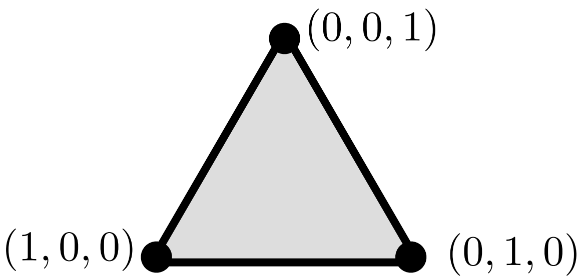
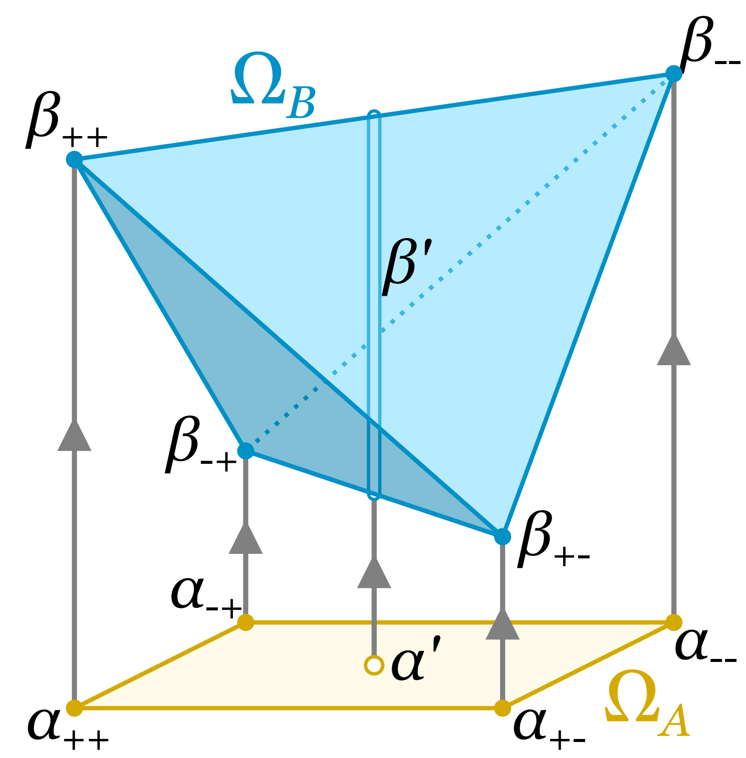


$$\Omega = \{p = (p_1, \ldots, p_n) \mid p_i \geq 0, \sum p_i = 1\}$$

- classical probability theory
- noisy qubits etc.
- QT *w/* superselection rules
- … **?**

Overview

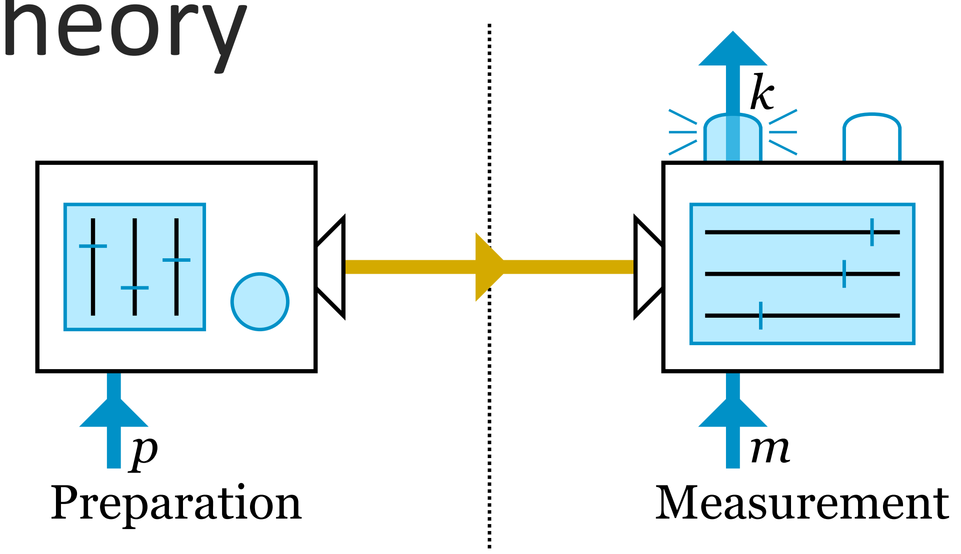


1. Testing QT via theory-agnostic tomography

2. Simulations, embeddings, and contextuality

3. Exact embeddings into quantum theory

4. Certifying non-embeddability



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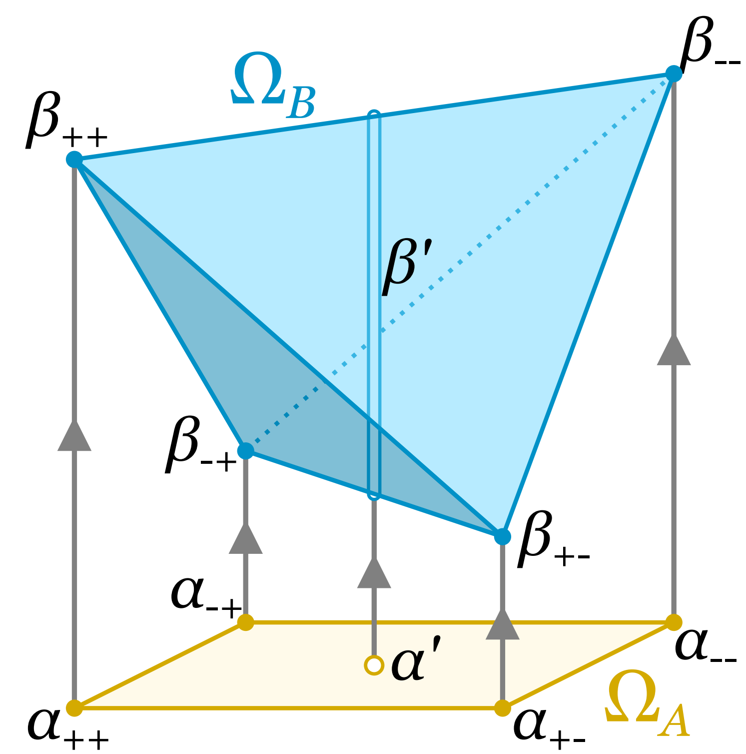

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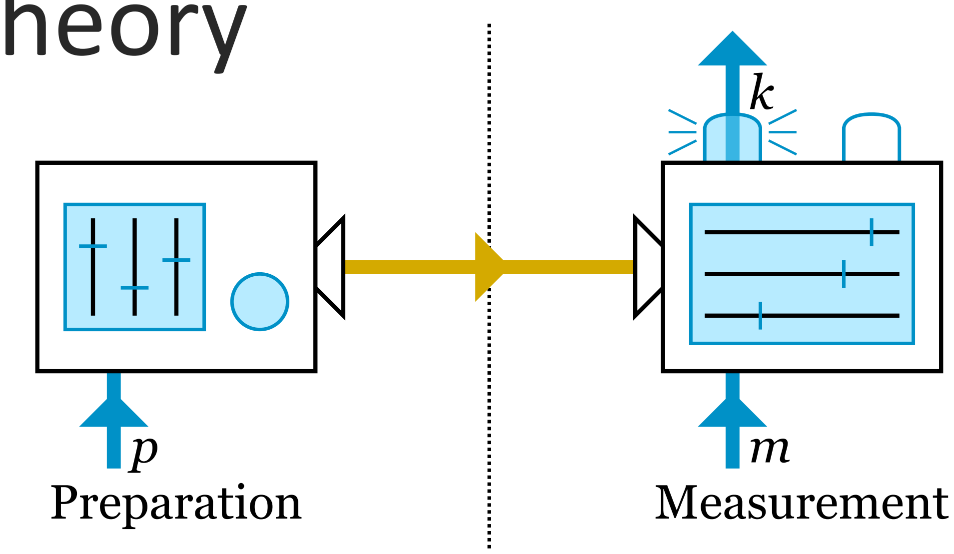

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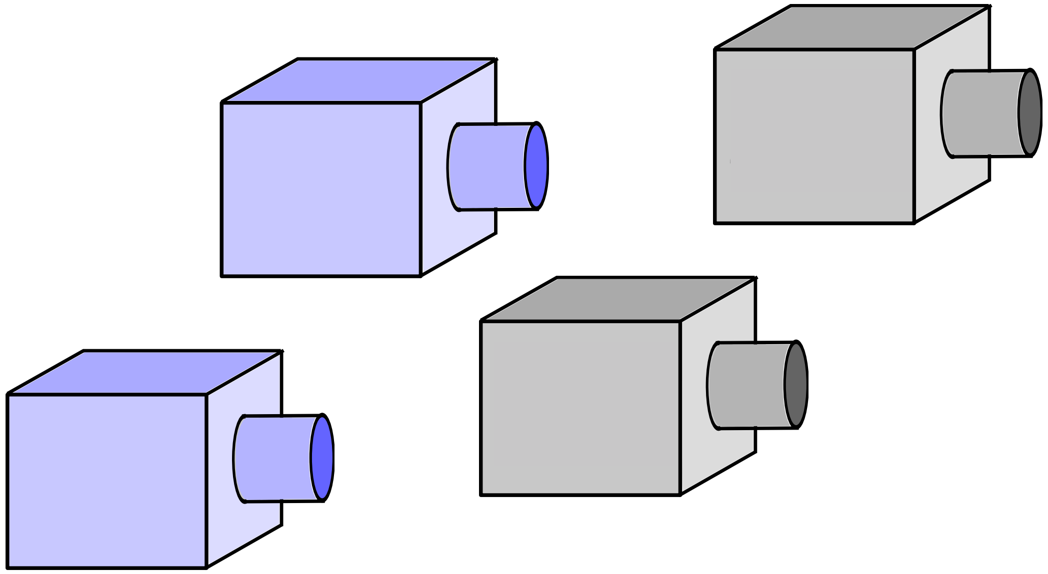

(all accessible preparation procedures)

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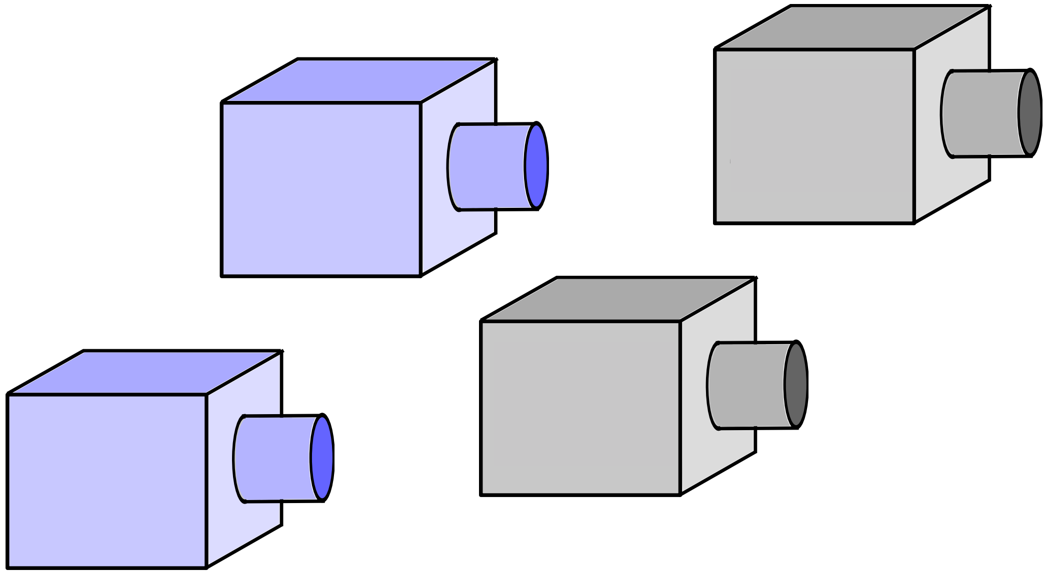

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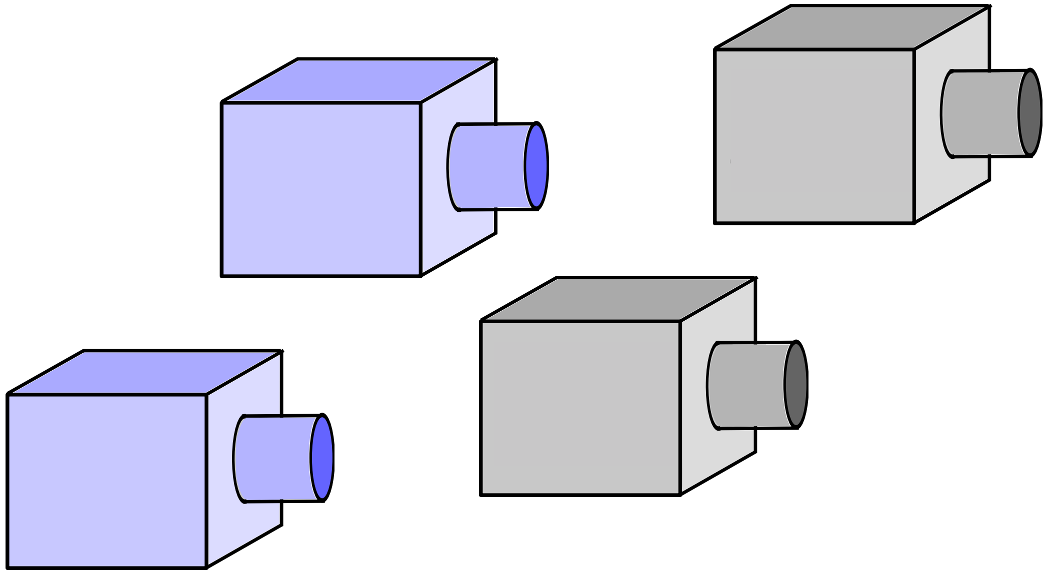

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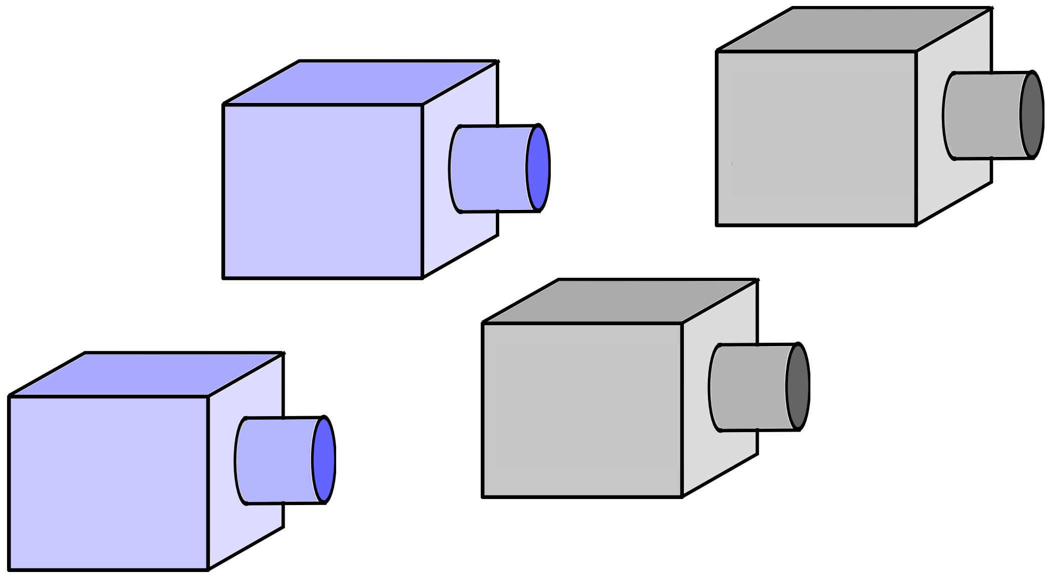

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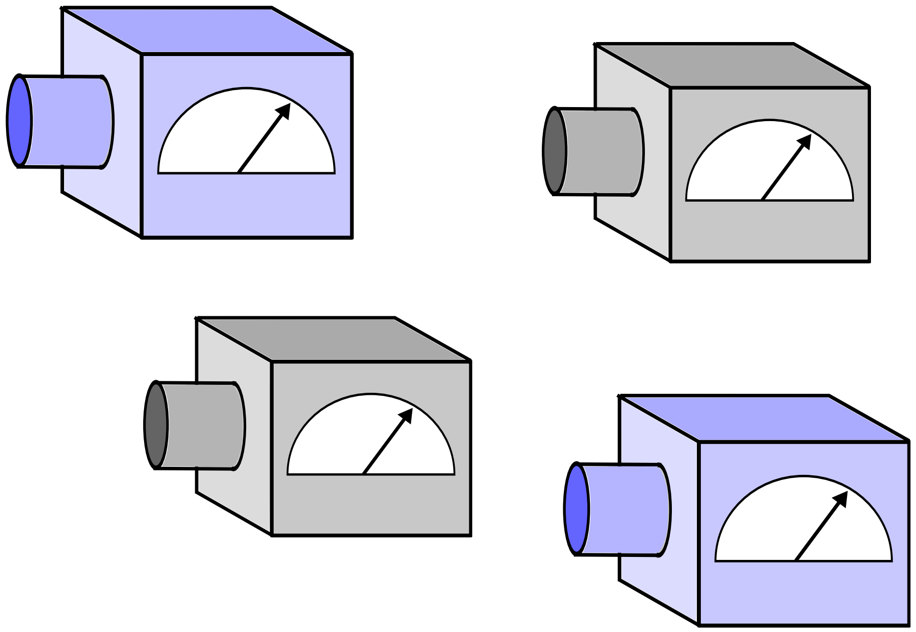

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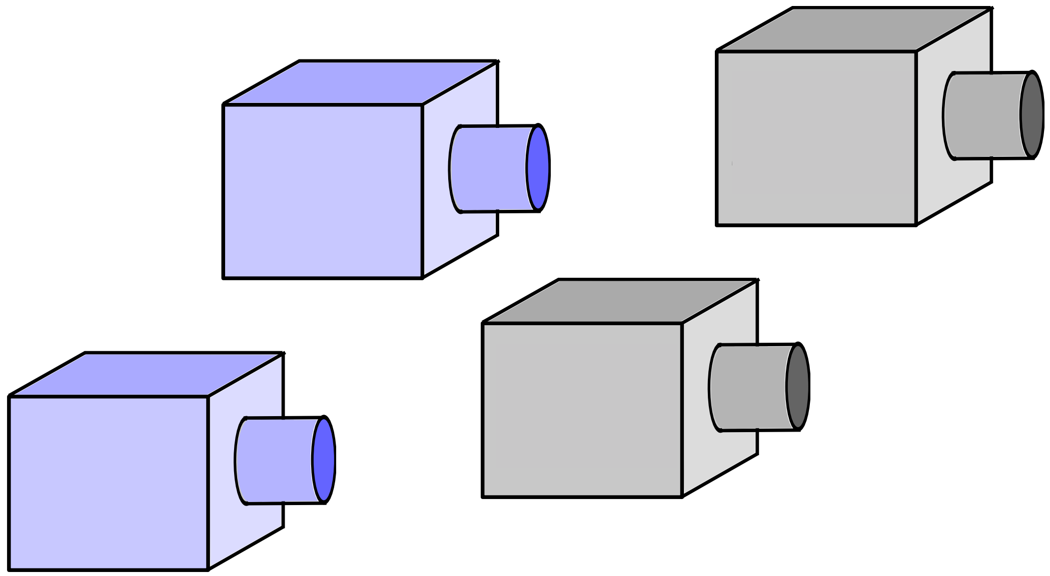

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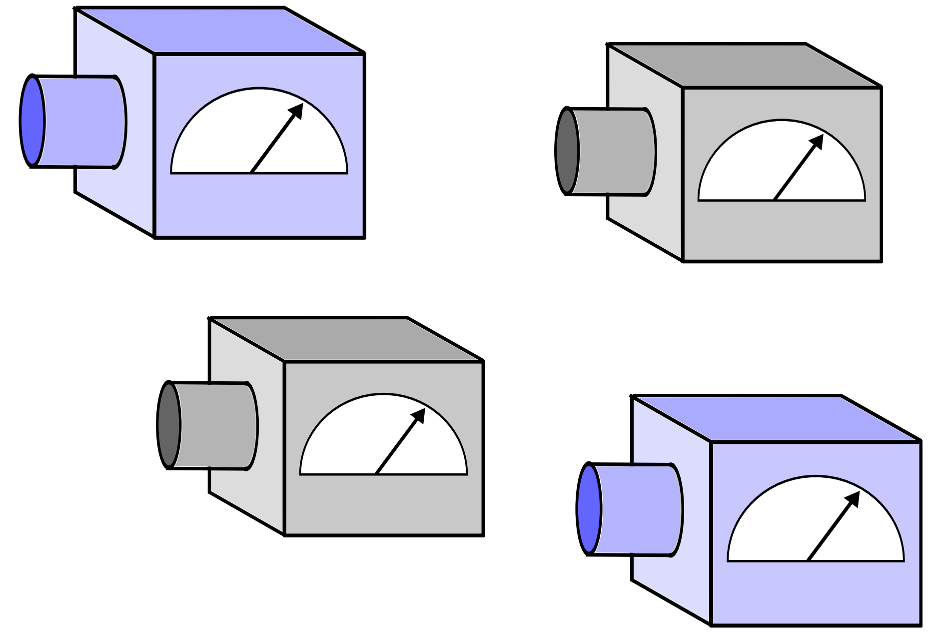

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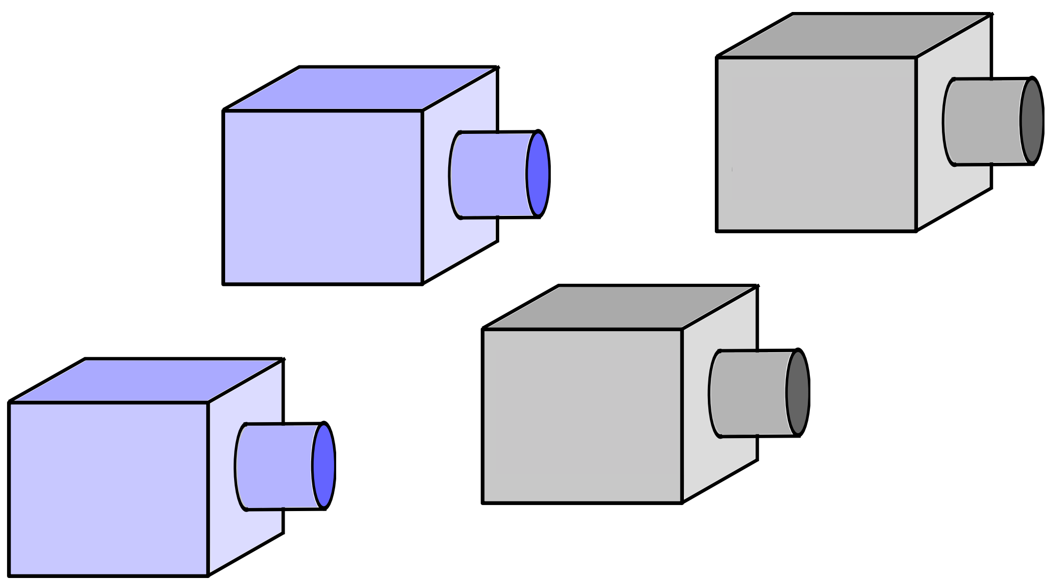

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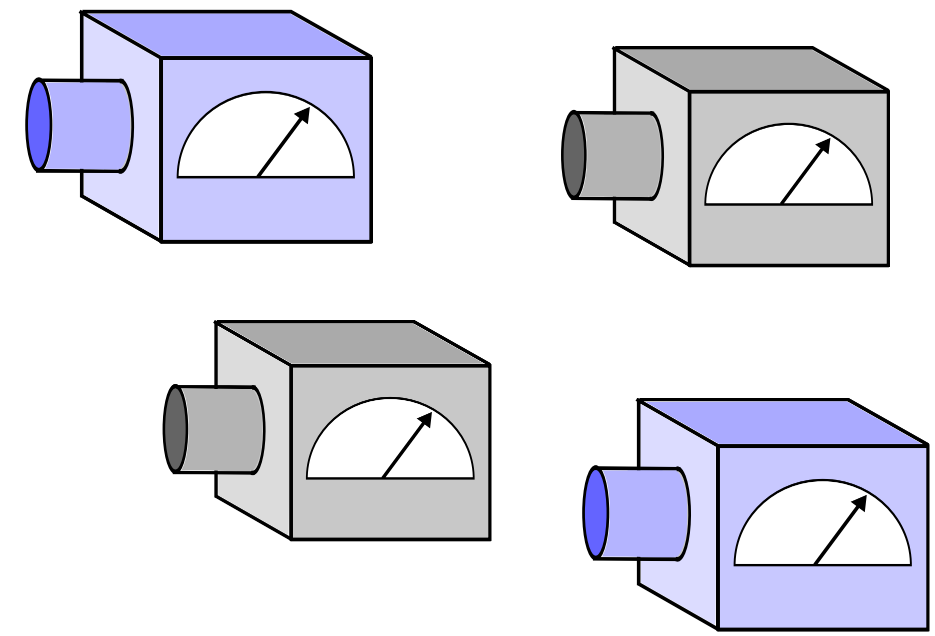

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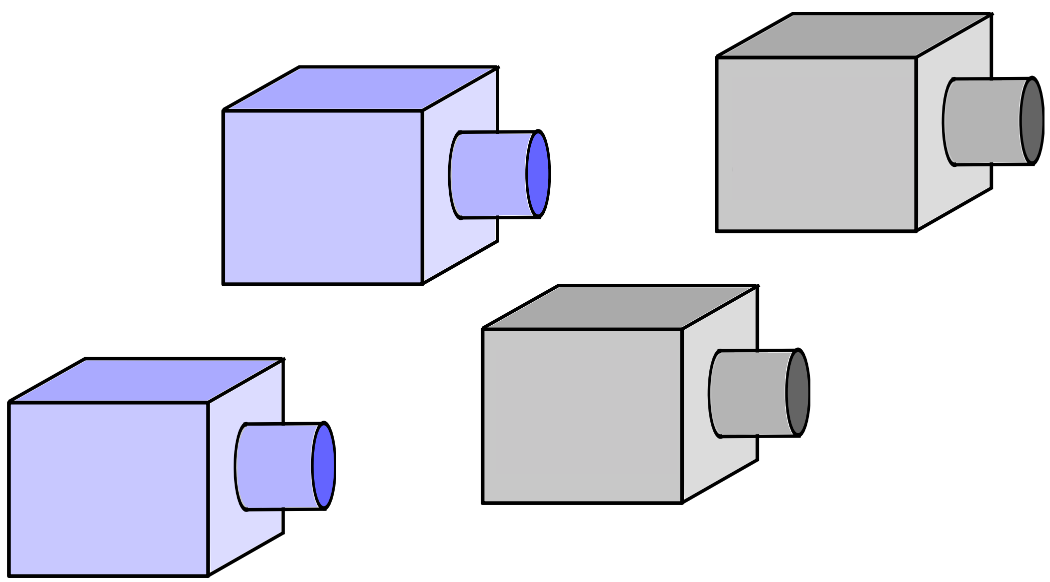

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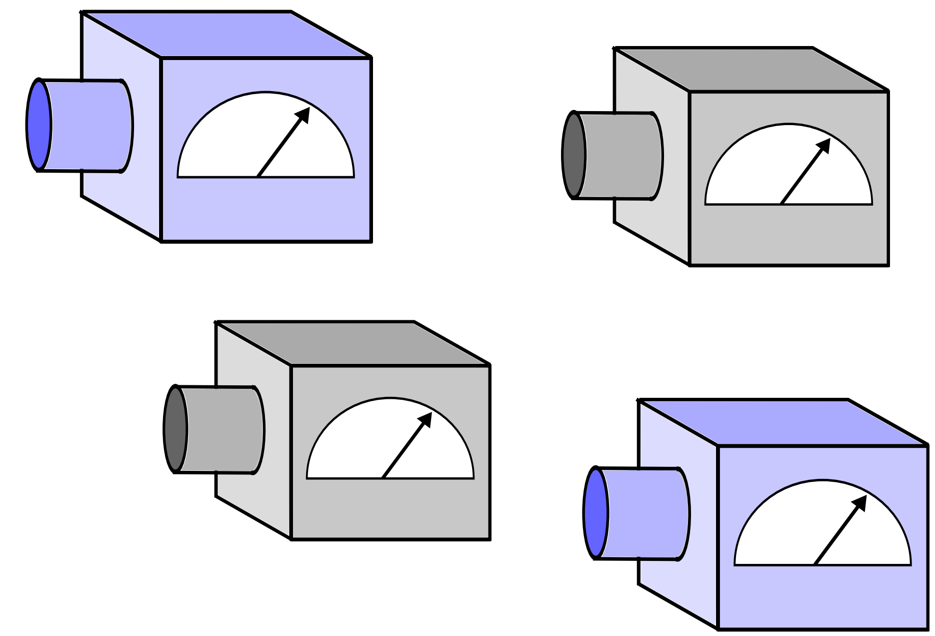

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$$\mathrm{Prob}(k|P, M) = \langle \omega_P, e_{k,M} \rangle \qquad (e_{k,M} \in A, \omega_P \in A^*).$$

General probabilistic theories

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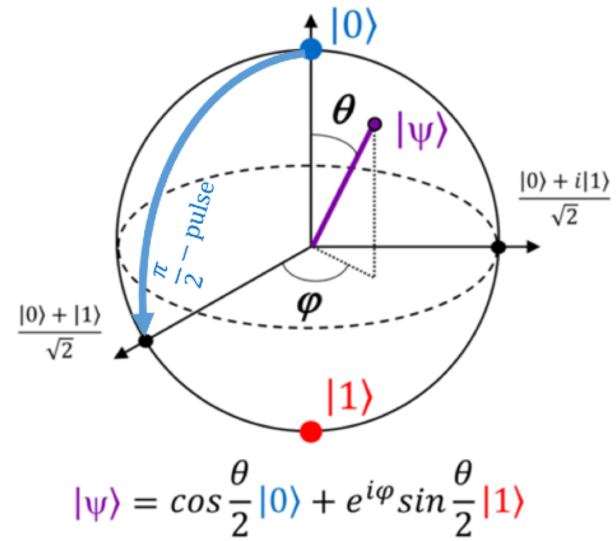

Quantum theory (QT): $\mathcal{Q}_n$

$A = \mathbb{H}_n(\mathbb{C})$ (complex Hermitian $n \times n$ matrices)

$E_A = \{E \mid 0 \leq E \leq \mathbf{1}\}$ (POVM elements)

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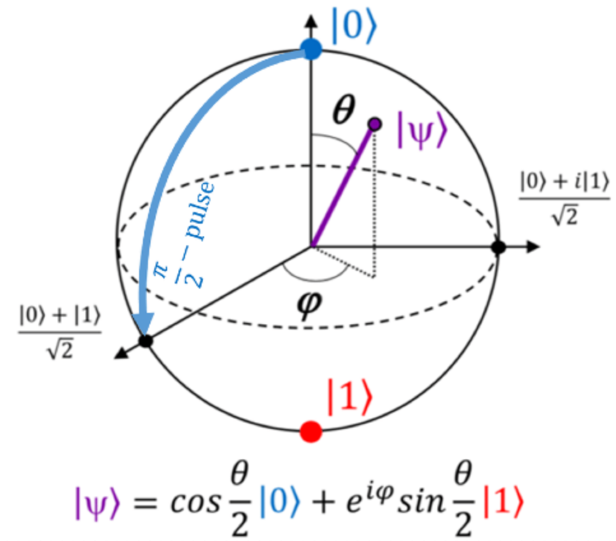

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Classical probability theory (QT): $\mathcal{C}_n$

$$A = \mathbb{R}^n \simeq A^*$$

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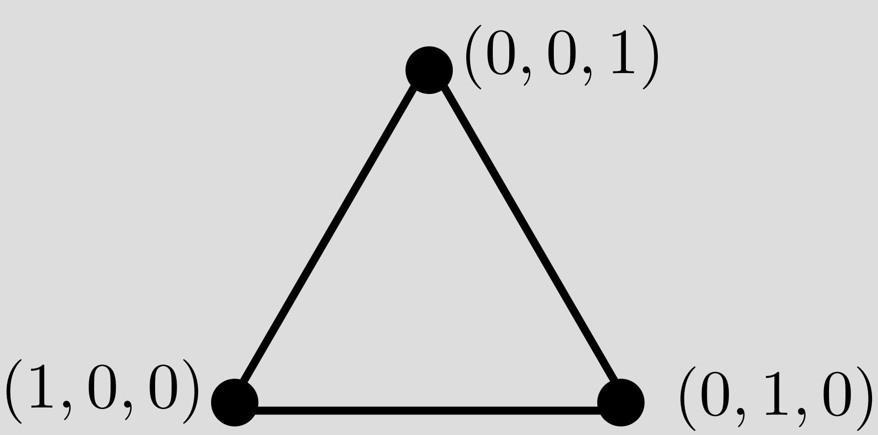

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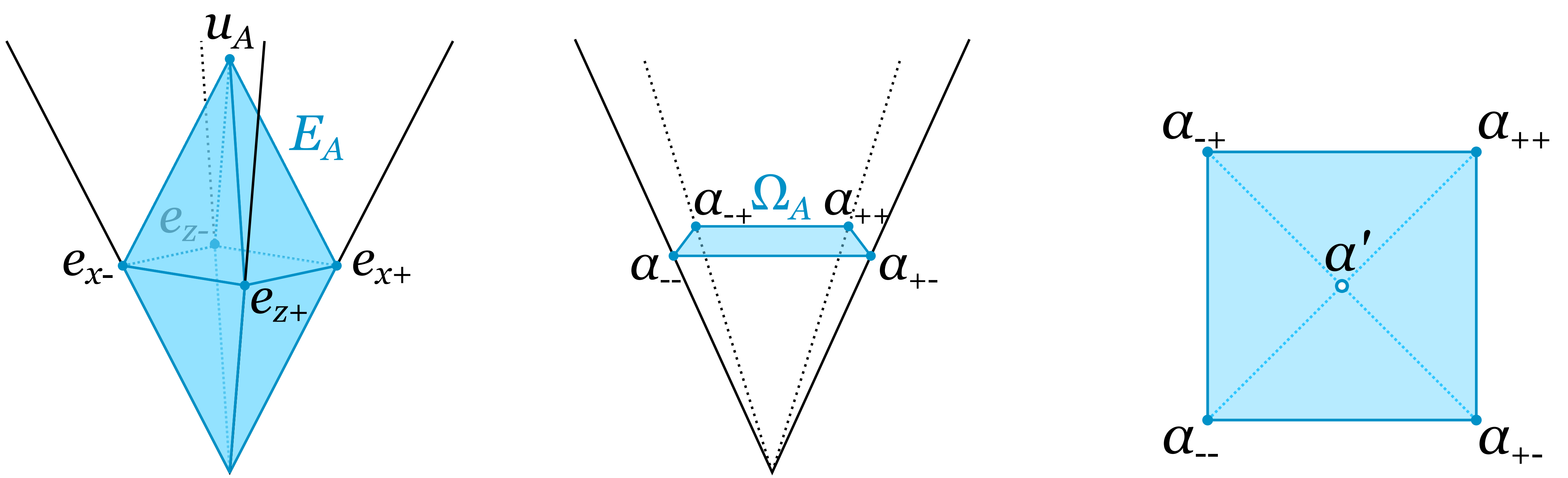

a) Cone of effects A_+ b) Cone of states A^*_+ c) Normalized states Ω_A

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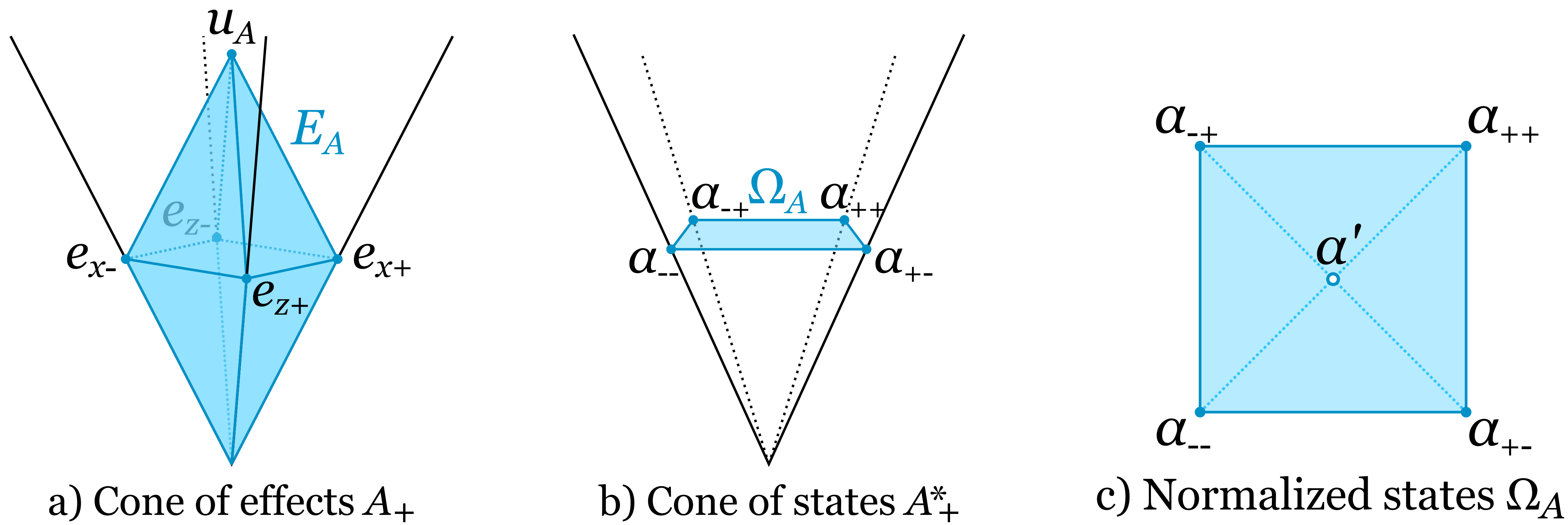

The four pure states $\alpha_{\pm,\pm}$ are **pairwise** perfectly distinguishable, but **not jointly** $\Longrightarrow$ this cannot be a classical or quantum system.

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Idea: Isolate a **physical system**. Perform as many preparations and measurements as possible; **fit a GPT to the data**; compare with $\mathcal{Q}_n$.

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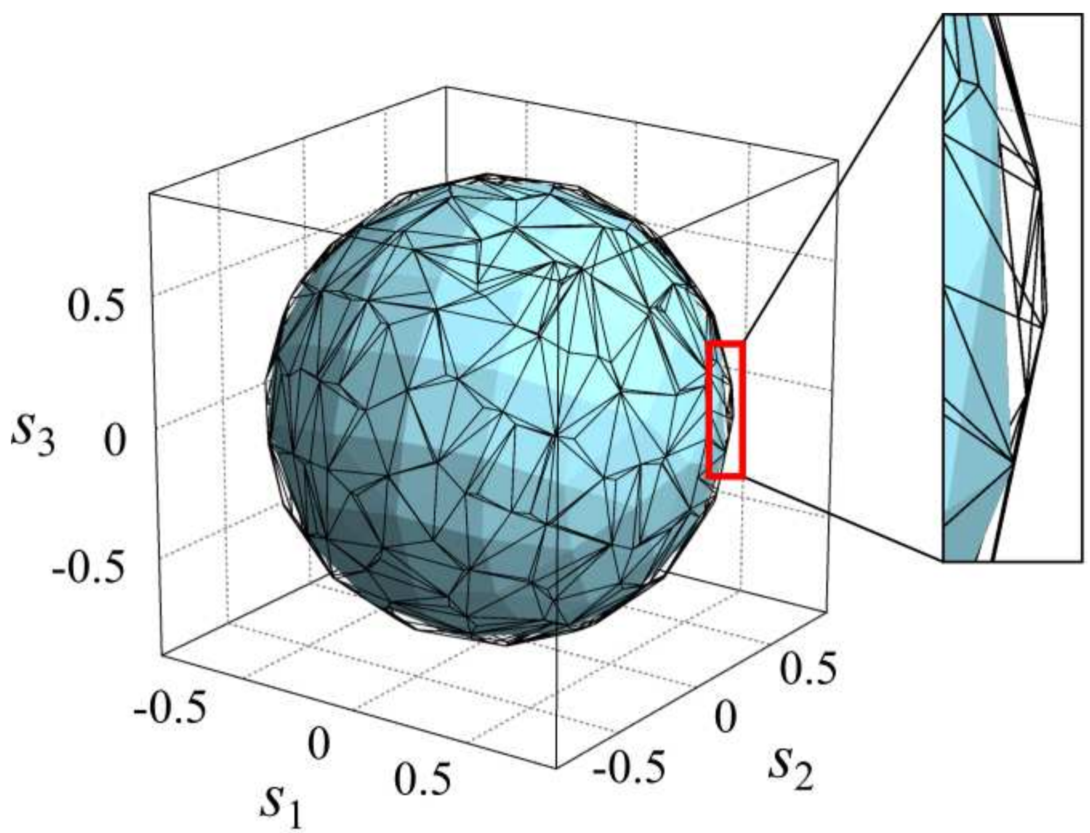

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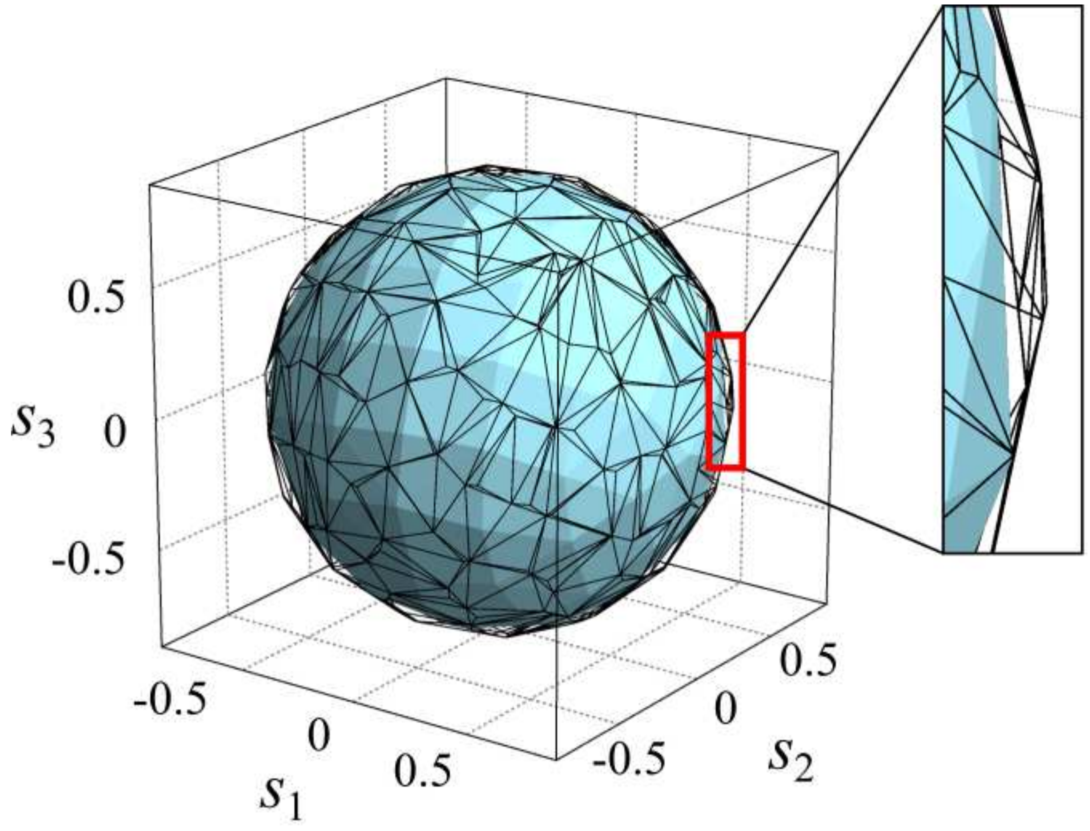

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Tomographic completeness loophole: can never be sure that we probed the system *completely*.

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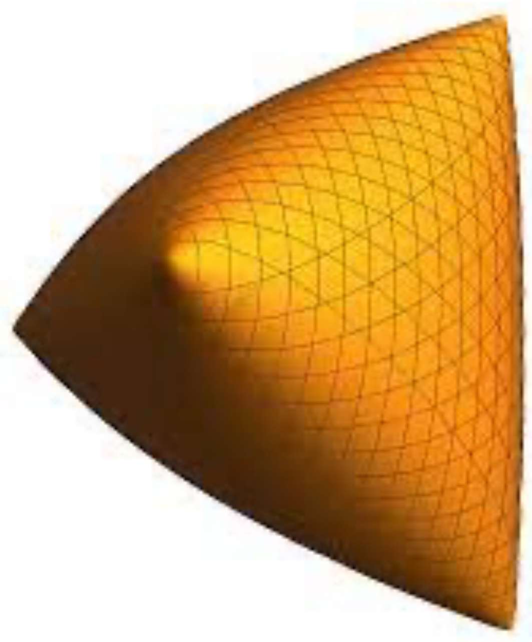

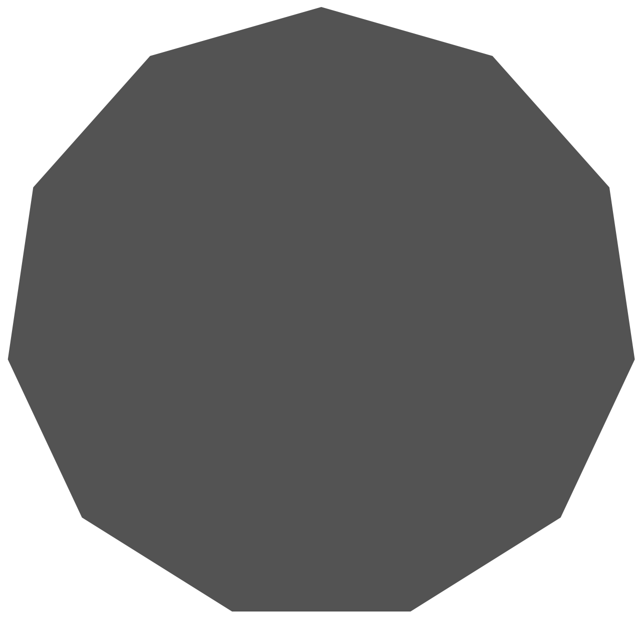

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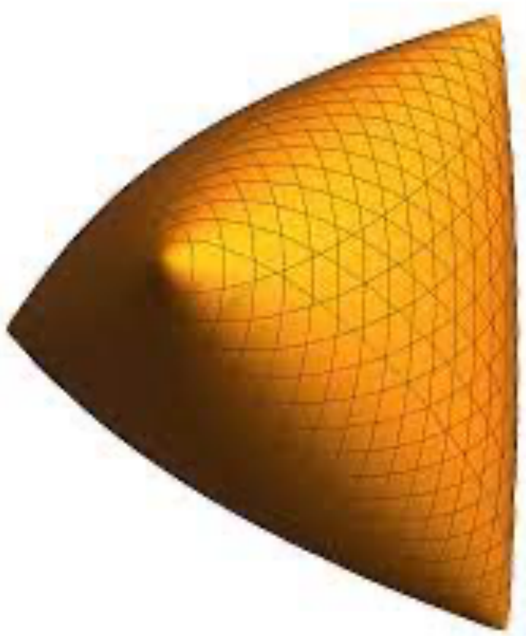

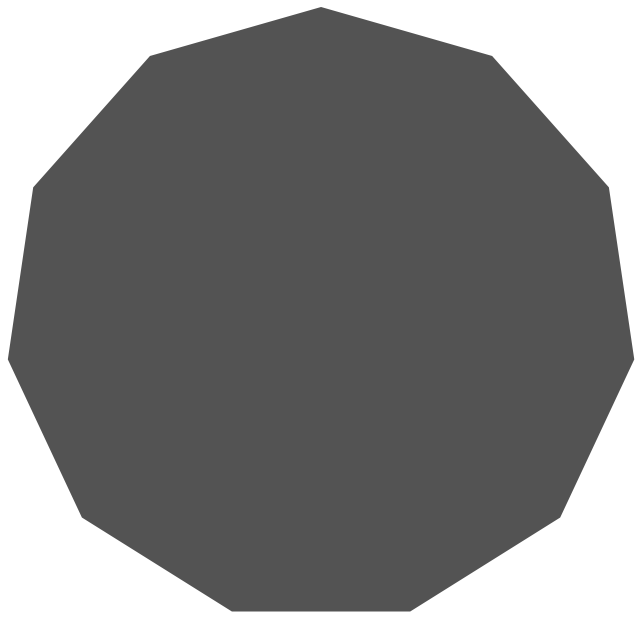

Is **fundamental QT** a plausible explanation of a given **effective GPT**?

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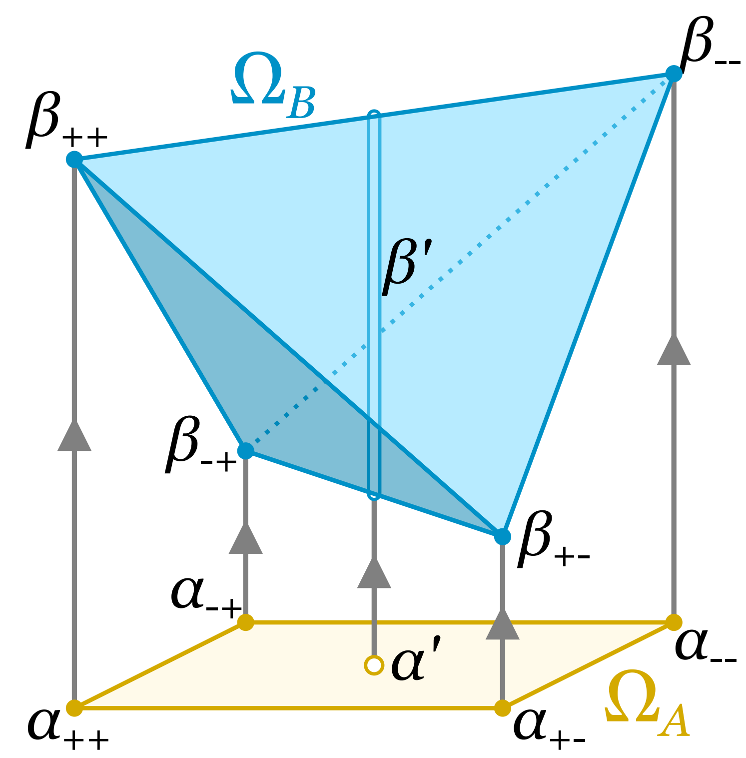

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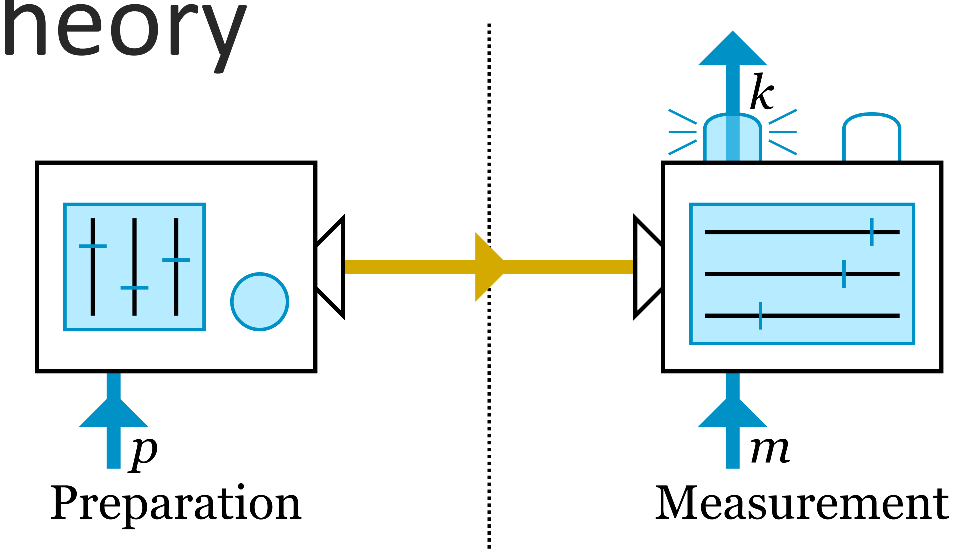

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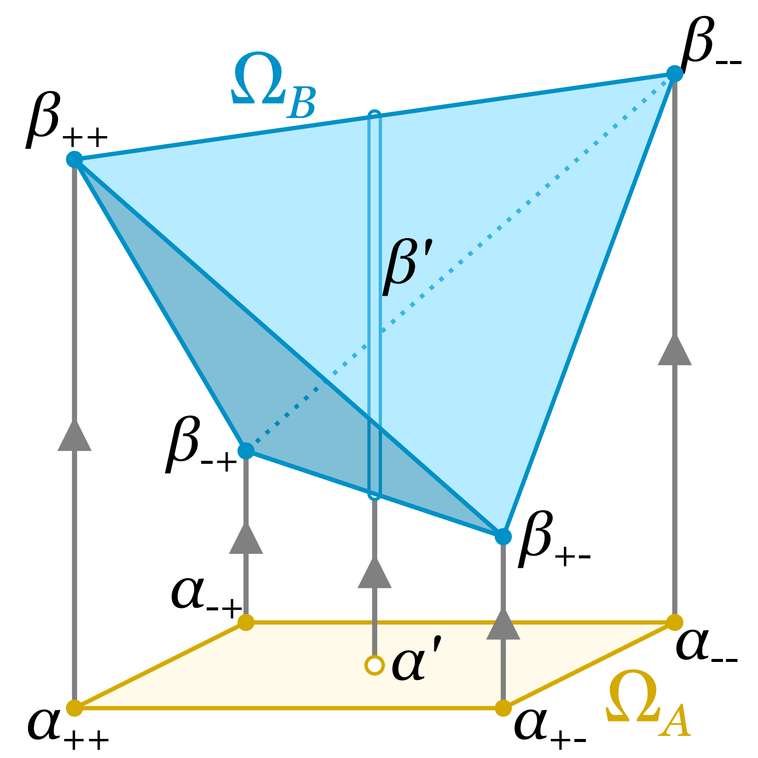

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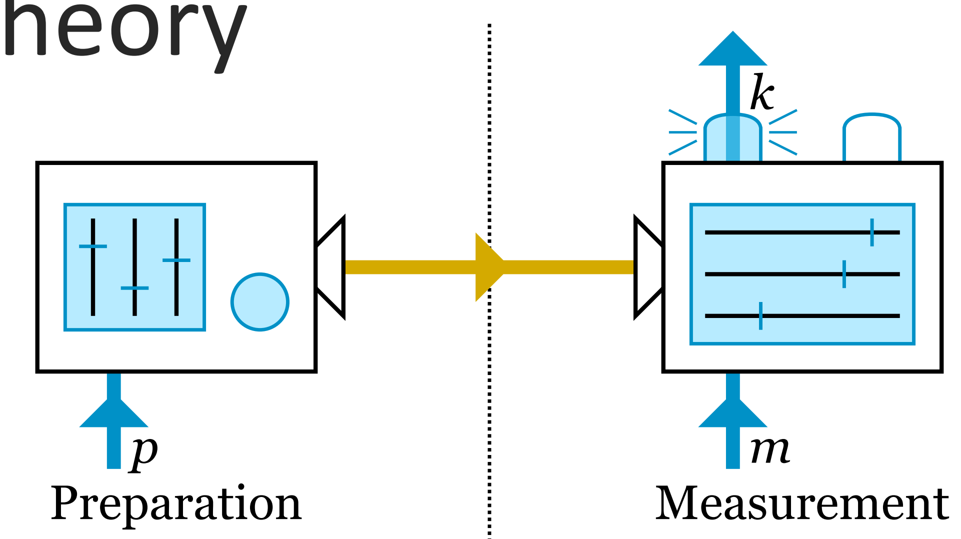

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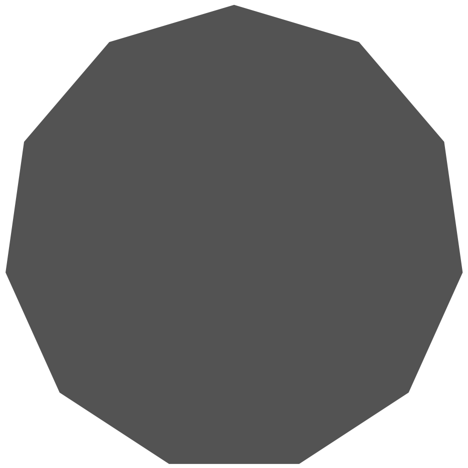

Effective GPT $\mathcal{A} = (A, \Omega_A, E_A)$ found in the lab

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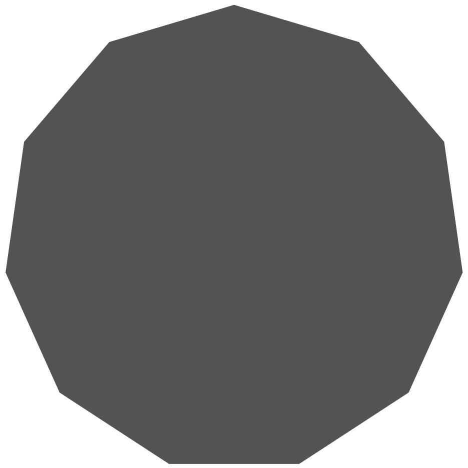

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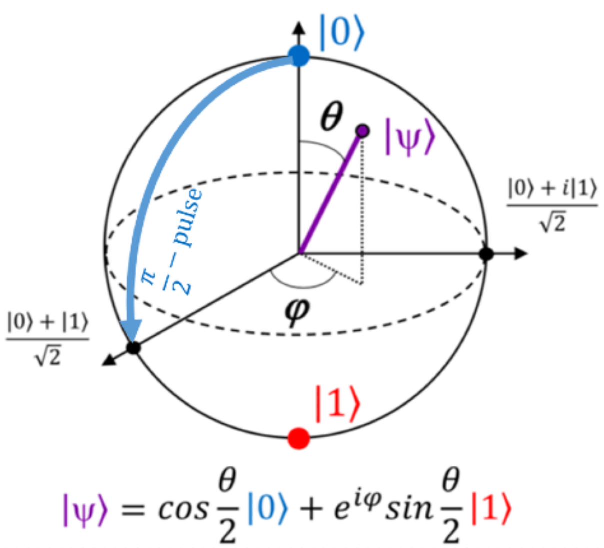

Fundamental GPT $\mathcal{B} = (B, \Omega_B, E_B)$

for example $\mathcal{Q}_n$ for very large n or $n = \infty$.

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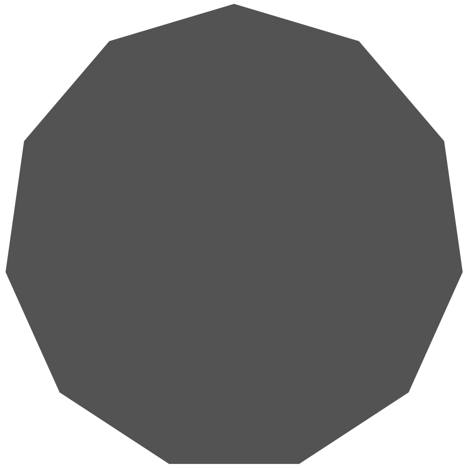

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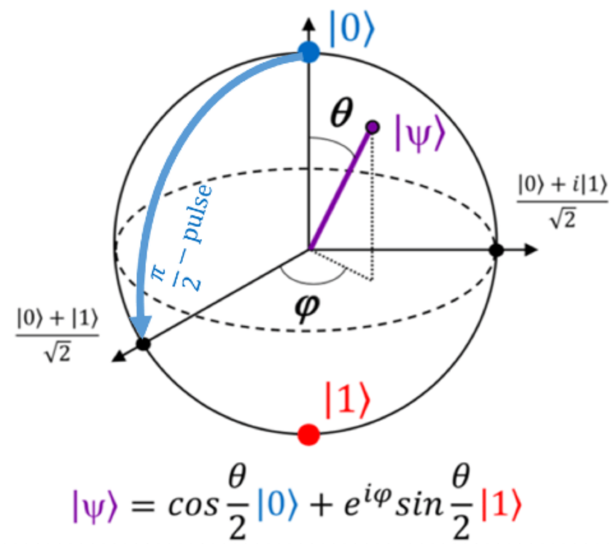

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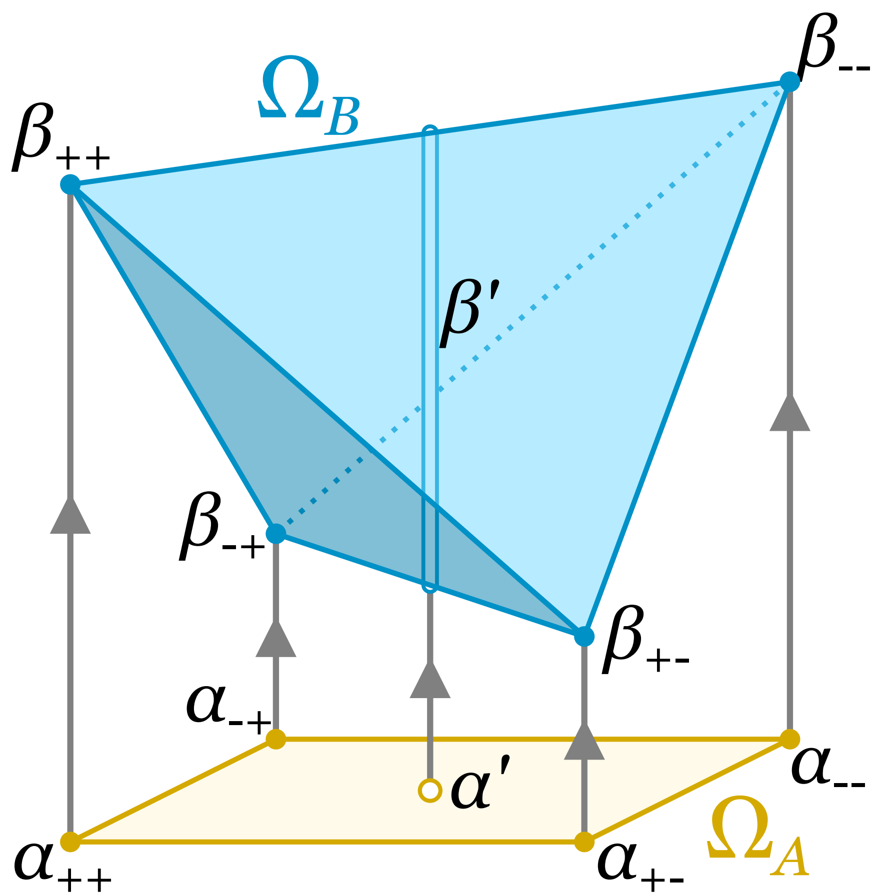
Effectively preparing state ω_A means **fundamentally** preparing some ω_B, but ω_B may depend on the preparation *procedure*, i.e. the *context*. Collect all those states into a set $\Omega_B(\omega_A) := \{\omega_B\}$.

Example ("Holevo projection"): simulating the gbit $\mathcal{A} = (\mathbb{R}^3, \Omega_A, E_A)$ with a classical 4-level system $\mathcal{C}_4$.



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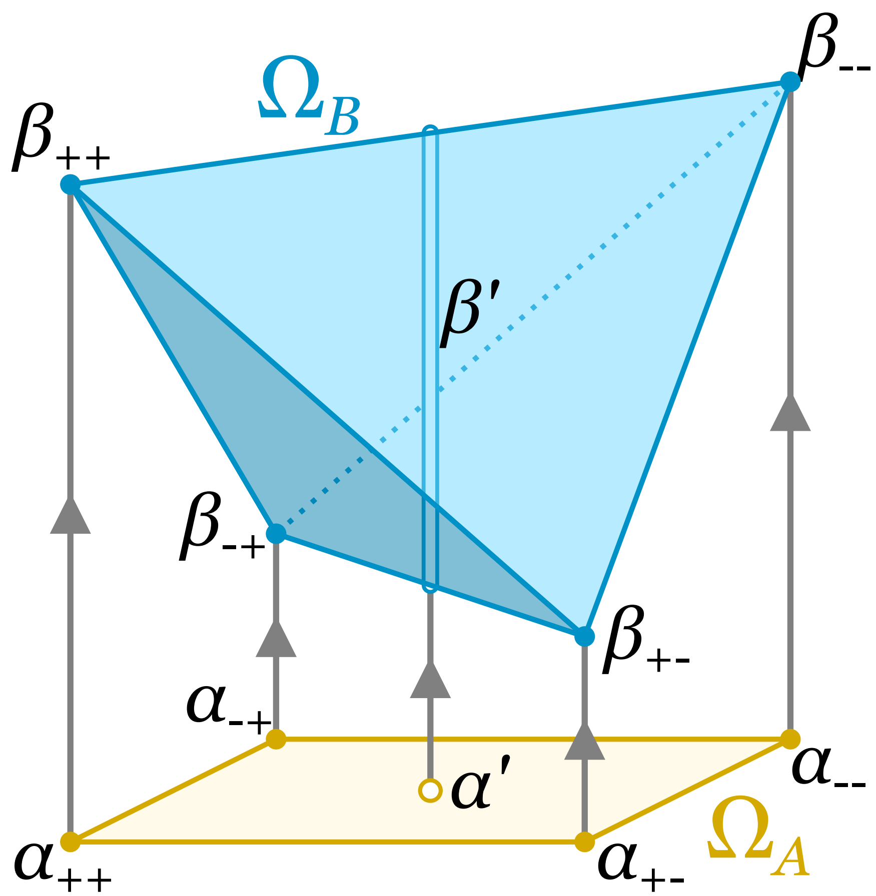

$$\Omega_B(\alpha_{\pm\pm}) = \{\beta_{\pm\pm}\},$$

but $\Omega_B(\alpha') =$ {states β' on blue line}.

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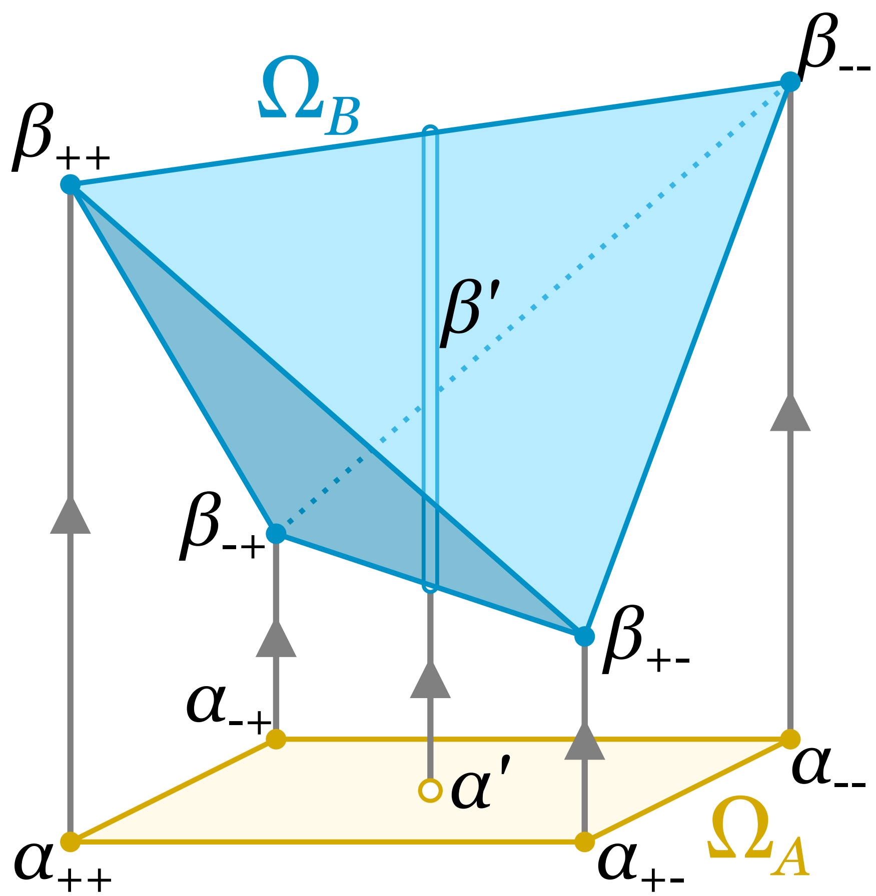

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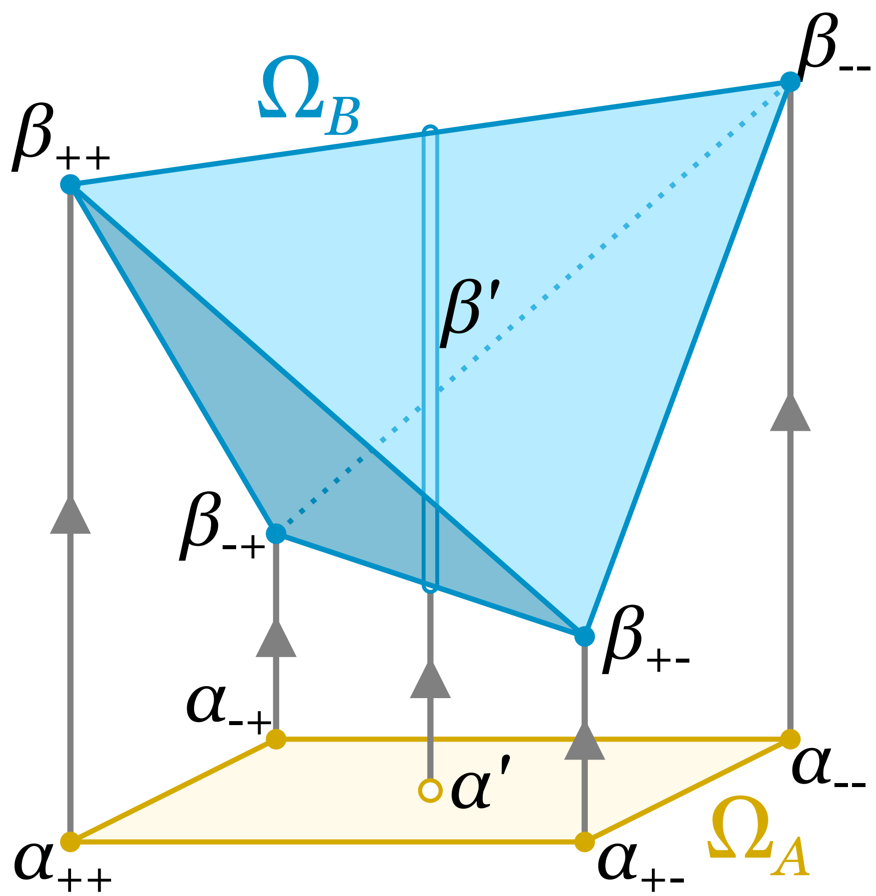

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This is an instance of implausible fine-tuning:
the statistical differences among the fundamental states are miraculously *exactly "washed out"* on the effective level.

2. Simulations, embeddings, and contextuality

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for all $0 \leq \lambda \leq 1$ *and* $\omega_A, \varphi_A \in \Omega_A$ *(and the analogous inclusion for* E_B *on mixtures of effects);*

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Classical probability theory can *contextually* simulate all GPTs:

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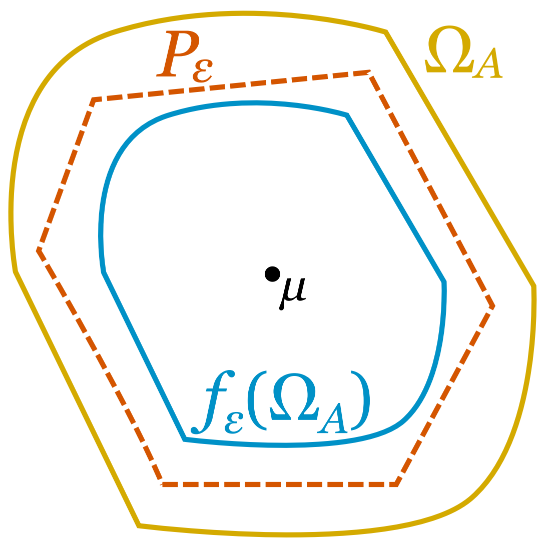
Lemma 1. *Let $\mathcal{A}$ be any GPT. Then, for every $\varepsilon > 0$, there is a measurement-noncontextual (but, in general, preparation-contextual) ε-simulation of $\mathcal{A}$ by $\mathcal{C}_n$ (and thus by $\mathcal{Q}_n$) for some $n \leq \left\lceil \left(\frac{c}{\varepsilon}\right)^{(\dim A - 2)/2} \right\rceil$, where $c > 0$ is a constant that only depends on Ω_A.*



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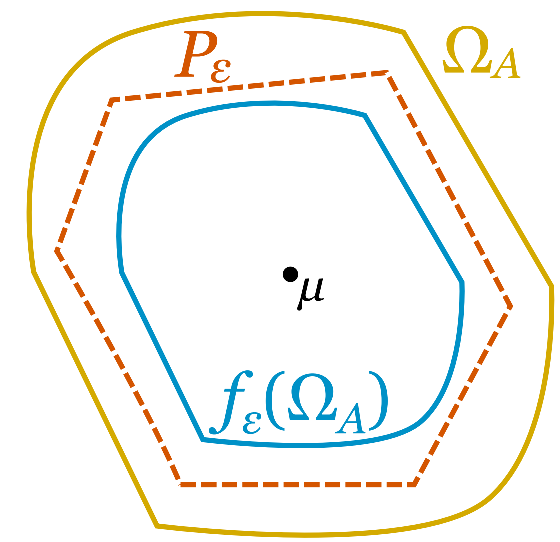

In special case $\mathcal{B} = \mathcal{C}_n$ (fundamental GPT is classical), this notion reduces exactly to **Spekkens'** notion [3] of contextuality.

$$P(k|p,m) = \sum_{\lambda \in \Lambda} \mu_p(\lambda) \chi_{k,m}(\lambda)$$

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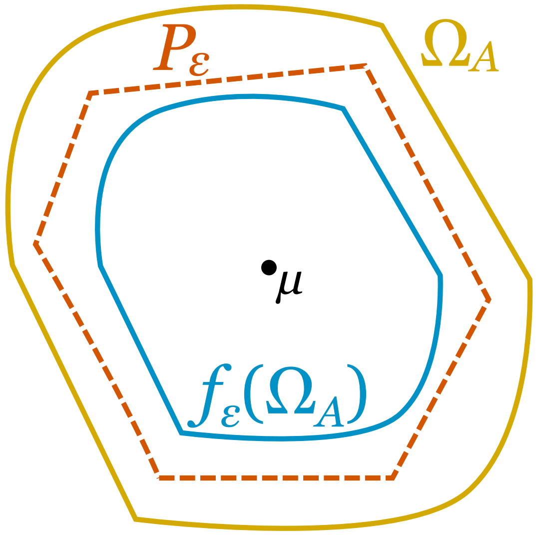

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Theorem 1. *Every discrete ontological model of an operational theory defines an exact simulation of the corresponding GPT by some $\mathcal{C}_n$, and vice versa. Moreover, the simulation is preparation–noncontextual / measurement–noncontextual / noncontextual if and only if the corresponding ontological model has this property.*

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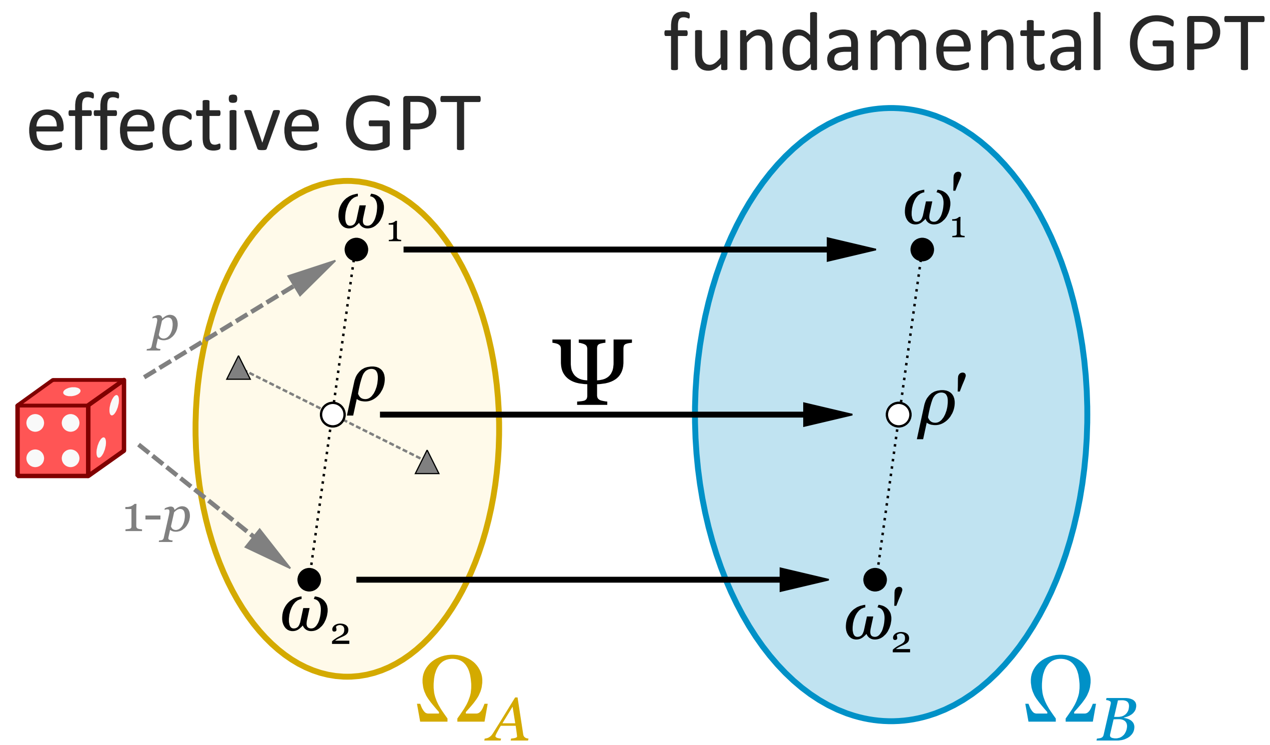

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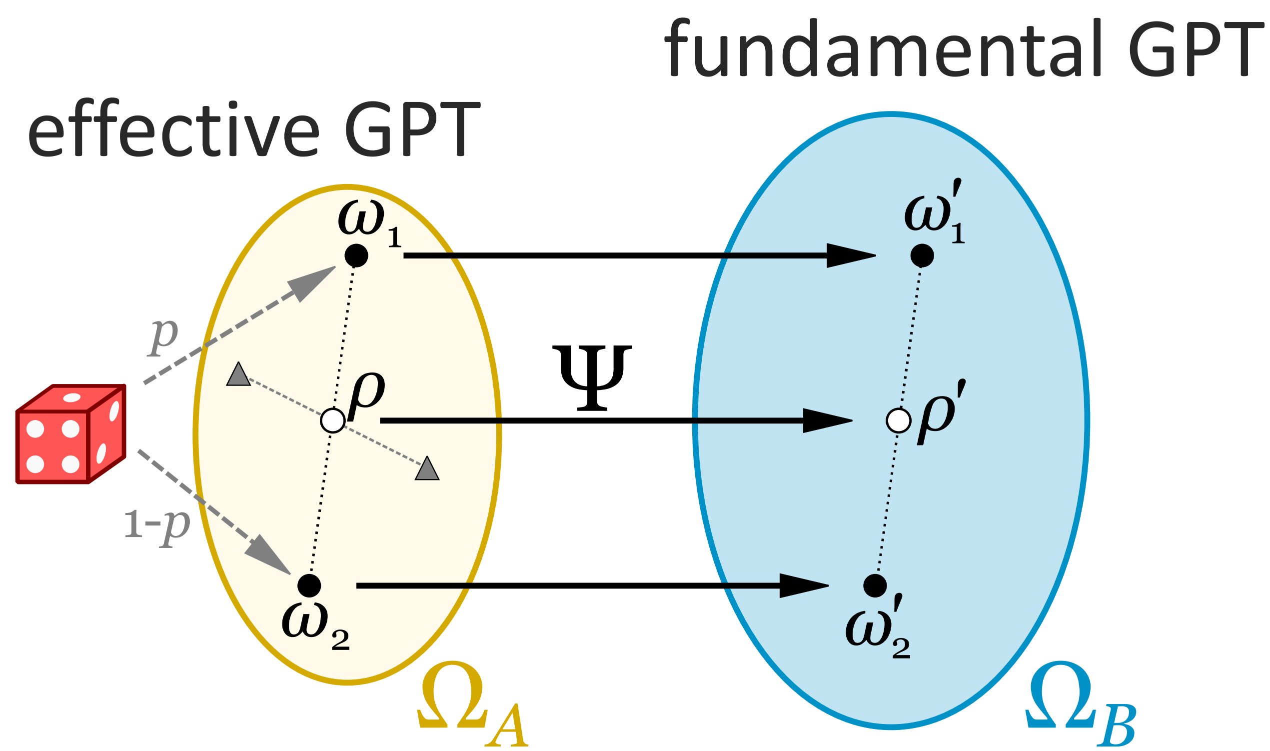

Definition 2 (Embedding). *Let* $\mathcal{A} = (A, \Omega_A, E_A)$ *and* $\mathcal{B} = (B, \Omega_B, E_B)$ *be GPTs, and let* $\varepsilon \geq 0$. *A pair of linear maps* $\Phi : A \to B$ *and* $\Psi : A^* \to B^*$ *is said to be an* ε**-embedding** *of* $\mathcal{A}$ *into* $\mathcal{B}$ *if*

(i) Φ *and* Ψ *are* positive *and* Ψ *is normalization-preserving, i.e.* $\Phi(E_A) \subseteq E_B$ *and* $\Psi(\Omega_A) \subseteq \Omega_B$*;*

(ii) Φ *and* Ψ preserve outcome probabilities up to ε*; i.e.* $|(\omega, e) - (\Psi(\omega), \Phi(e))| \leq \varepsilon$ *for all* $e \in E_A$, $\omega \in \Omega_A$.

Lemma 2. Every noncontextual ε-simulation of $\mathcal{A}$ by $\mathcal{B}$ defines an ε-embedding of $\mathcal{A}$ into $\mathcal{B}$, and vice versa.

Noncontextual inequalities and approximate embeddings

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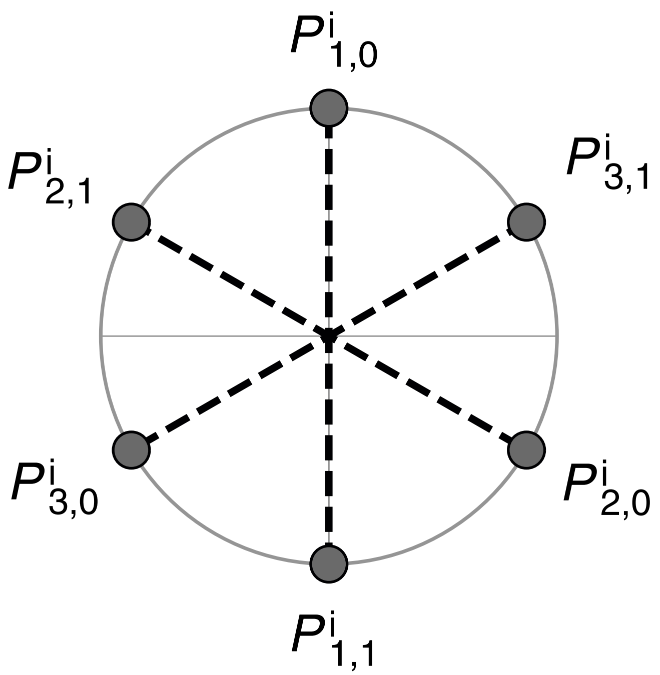

The qubit (actually, rebit) does not have a noncontextual ontological model.

Quantitative statement:

$$A := \frac{1}{6} \sum_{t\in\{1,2,3\}} \sum_{b\in\{0,1\}} P(b\,|\,p_{t,b}, m_t) \leq \frac{5}{6}.$$

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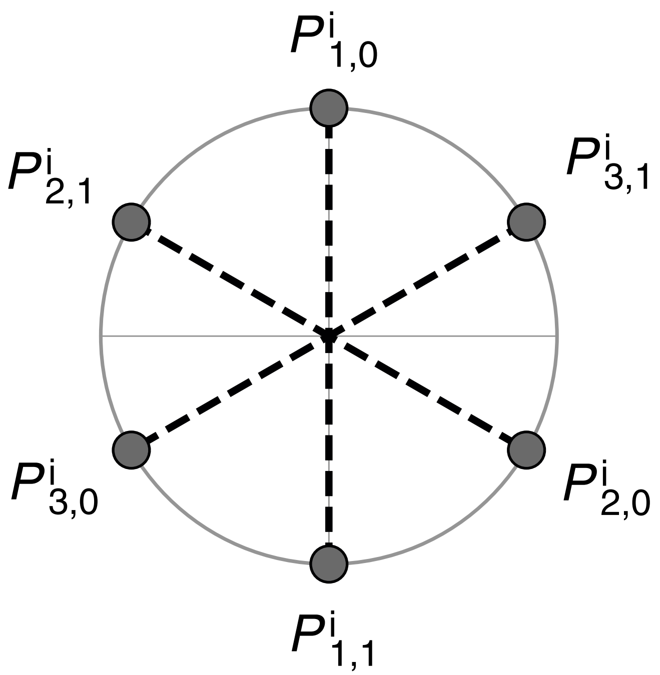

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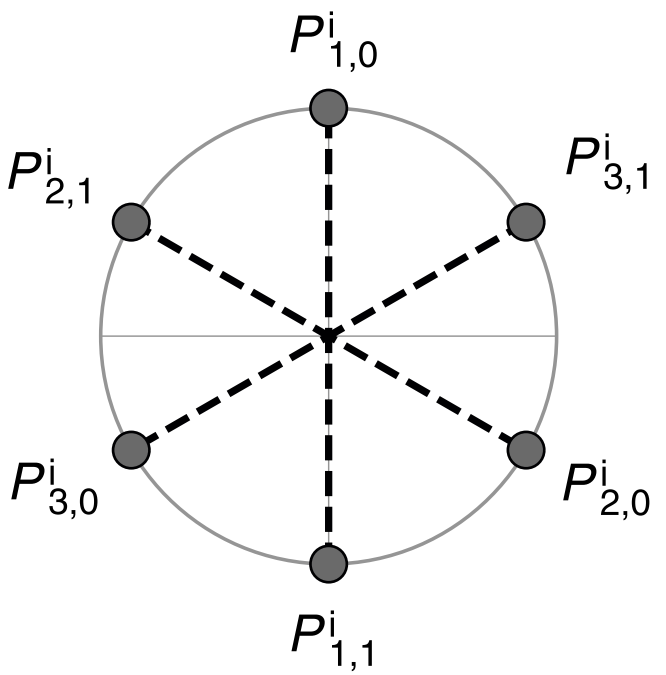

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Proof of ε-nonembeddability admits experimental falsification of noncontextuality.

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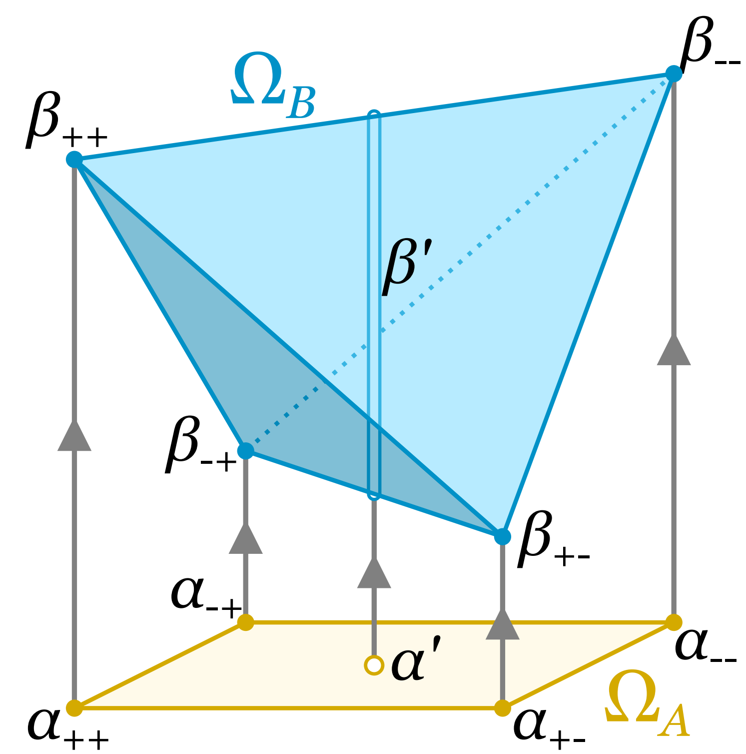

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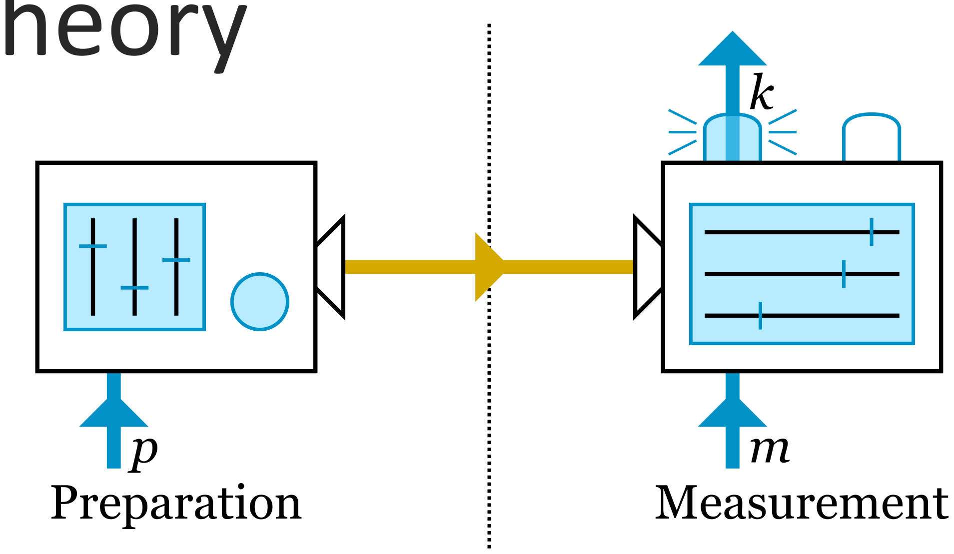

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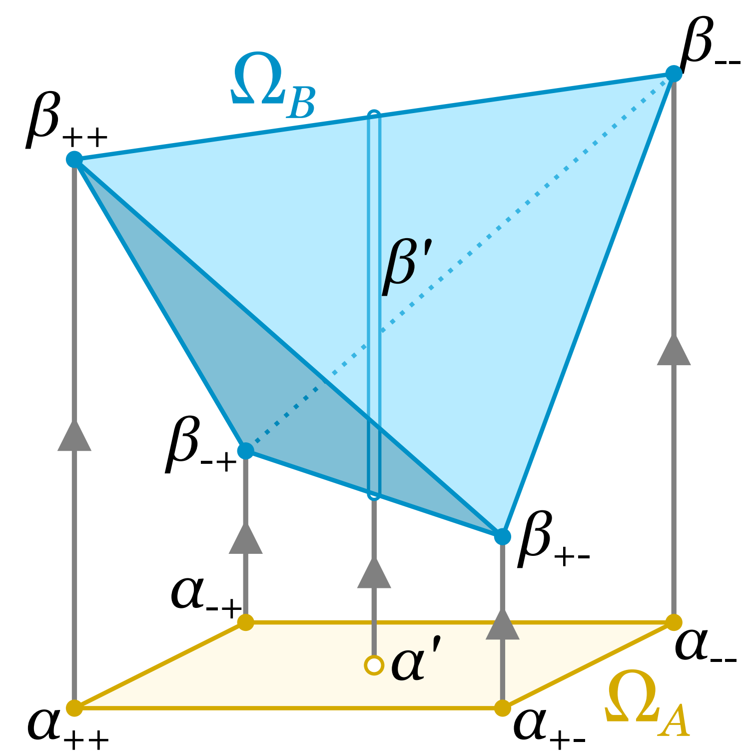

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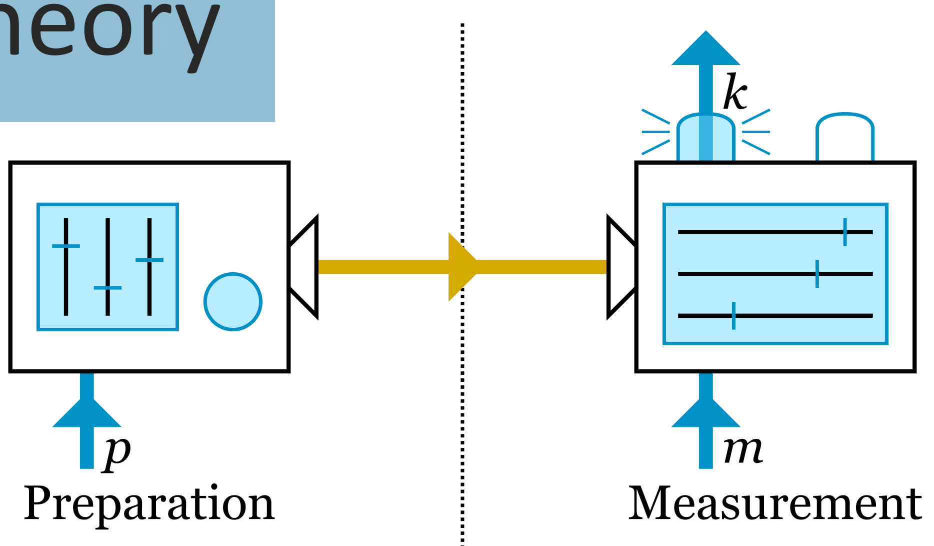

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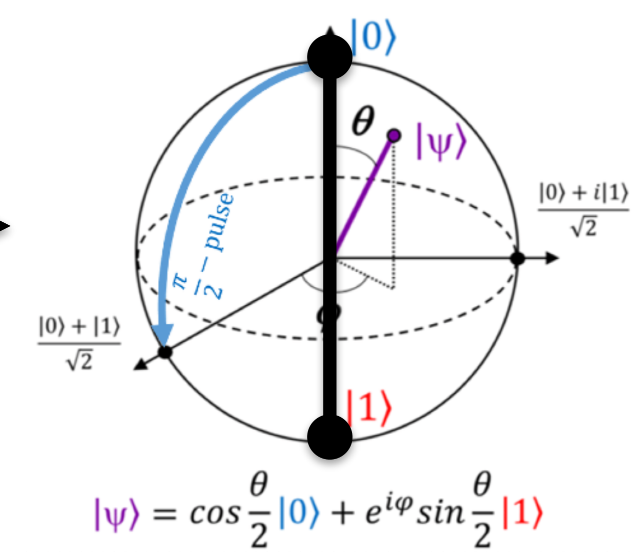

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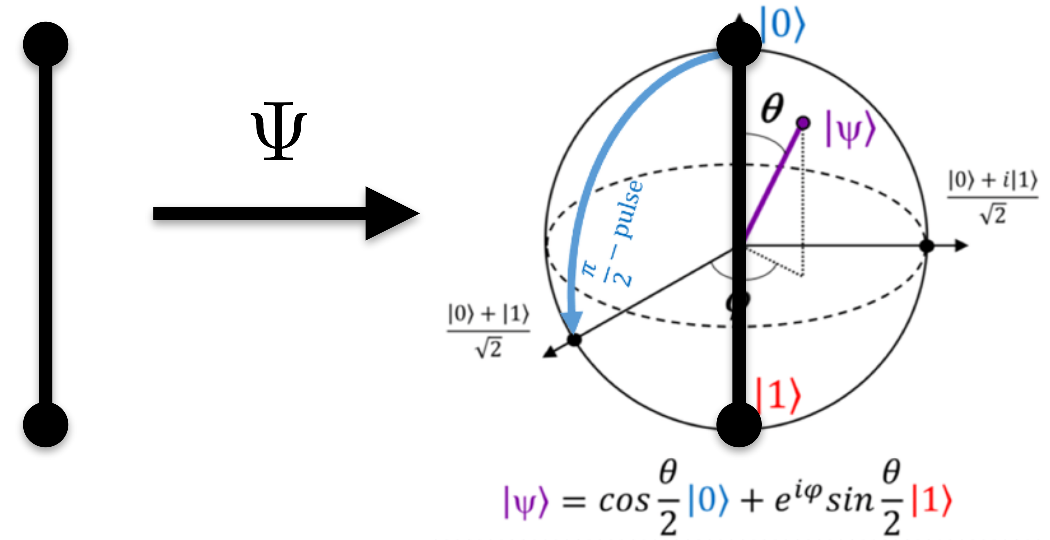
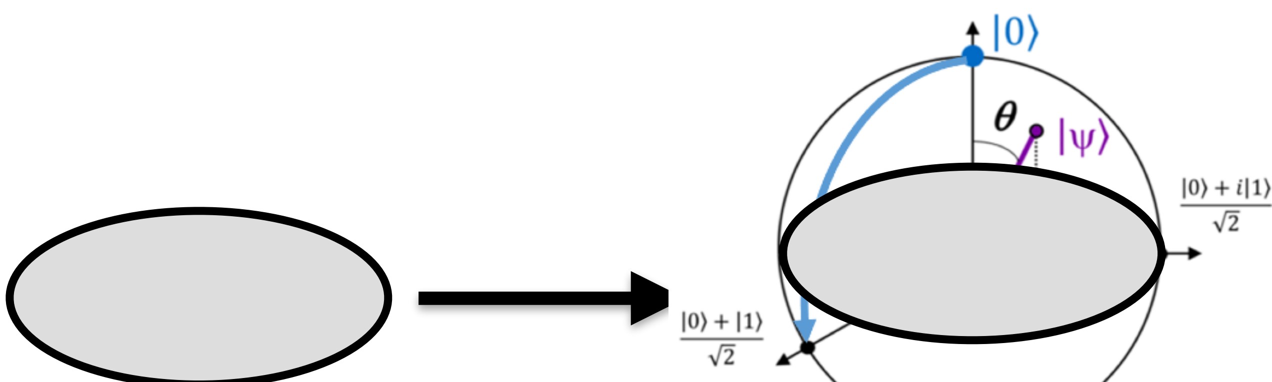


Similarly, **QT over the real numbers** can be embedded into QT.



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Focus on the "**unrestricted** GPTs" where all vectors yielding valid probabilities on all states are effects: $\mathcal{A} = (A, \Omega_A, E_A)$

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We should not be (and are not) surprised to find any of those in the lab.

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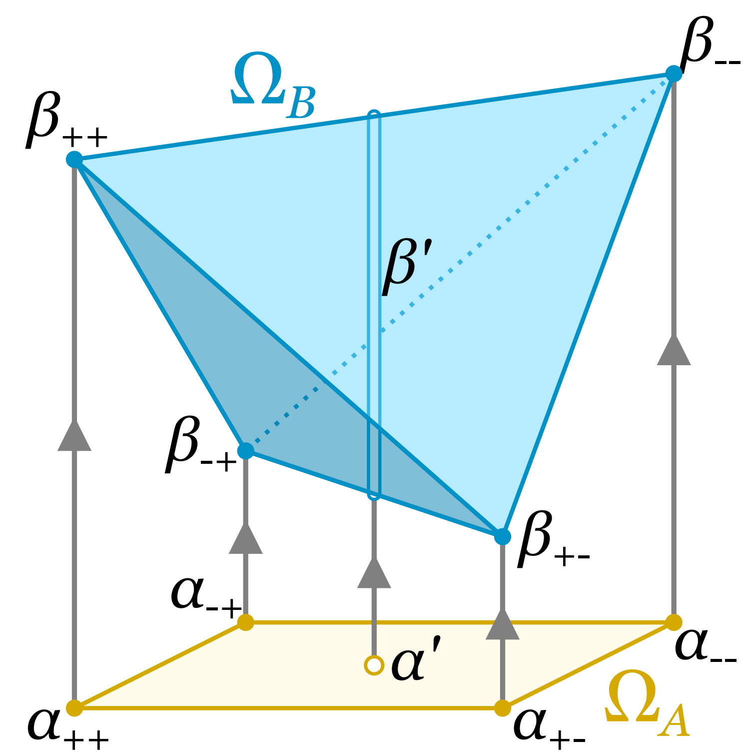

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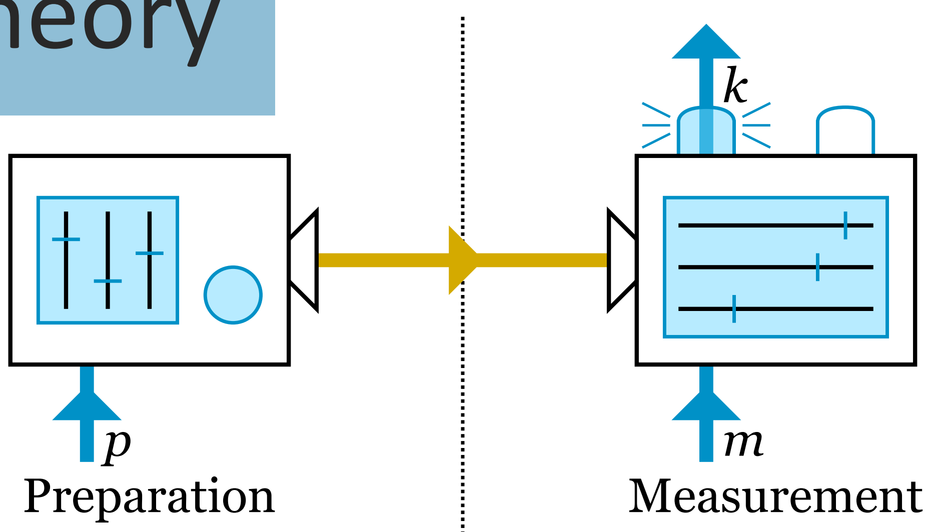

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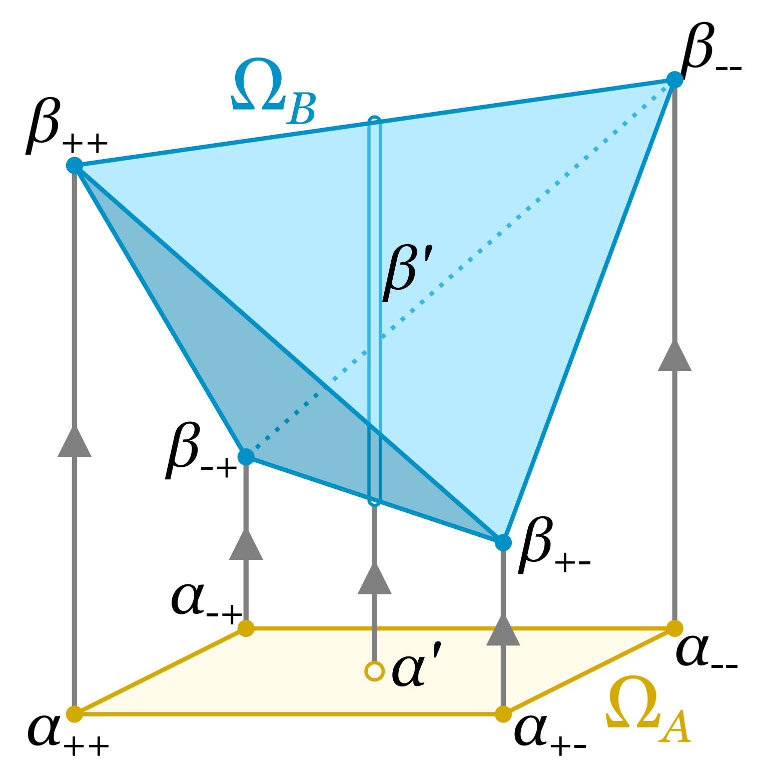

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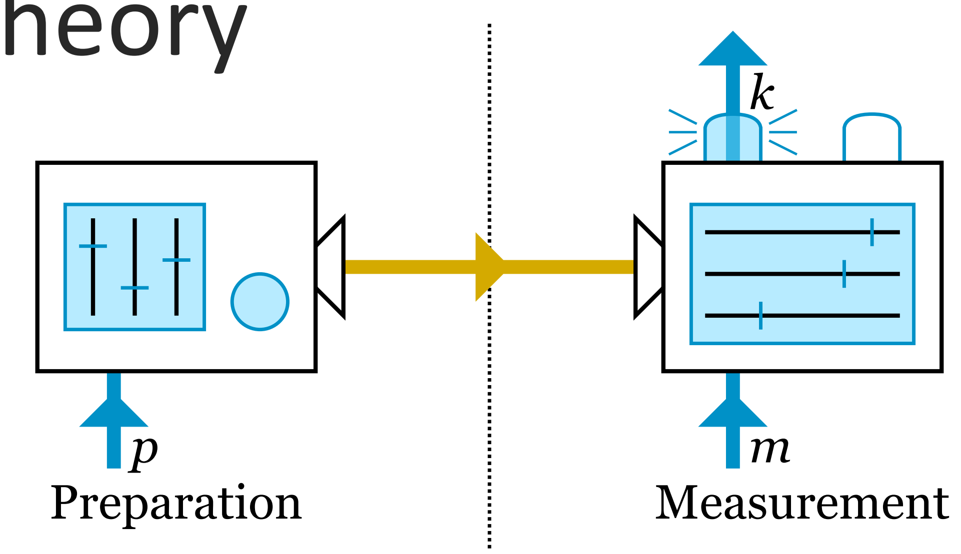

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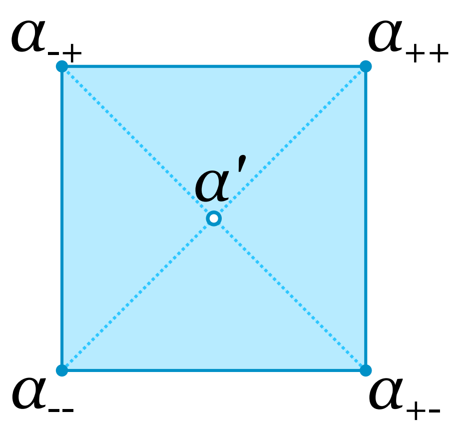

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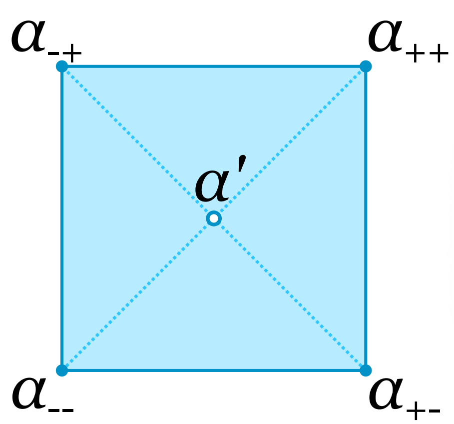

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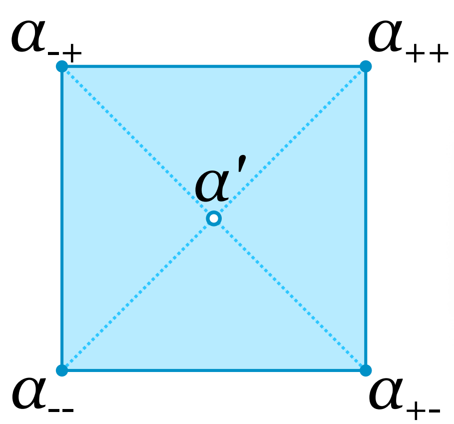

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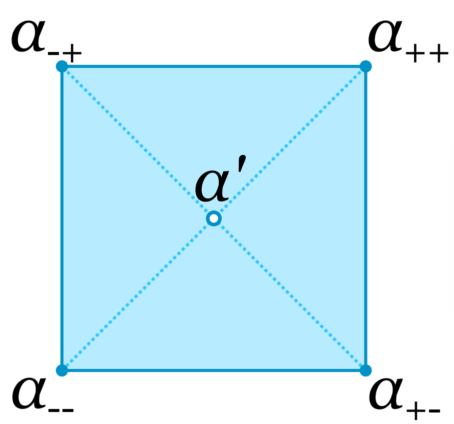

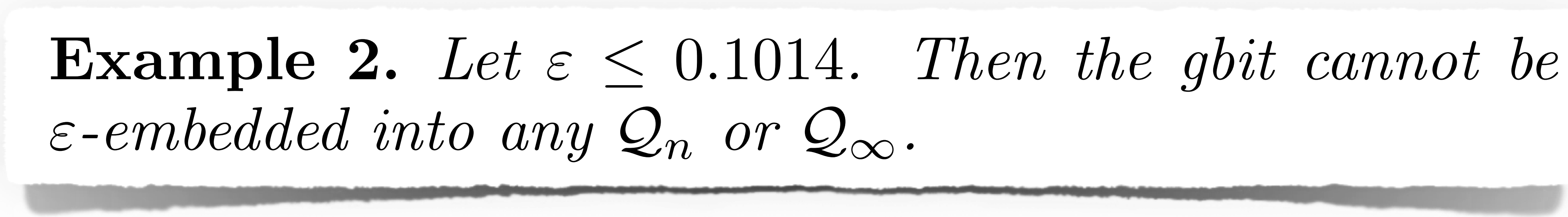

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Proof sketch: the four pure states $\alpha_{\pm\pm}$ are simulated by four quantum states $\rho_{\pm\pm}$ which are pairwise *almost* perfectly distinguishable. Imagine a device that approx. distinguishes all four successively. Contradicts $\frac{1}{2}\rho_{-+} + \frac{1}{2}\rho_{+-} = \frac{1}{2}\rho_{--} + \frac{1}{2}\rho_{++}$.

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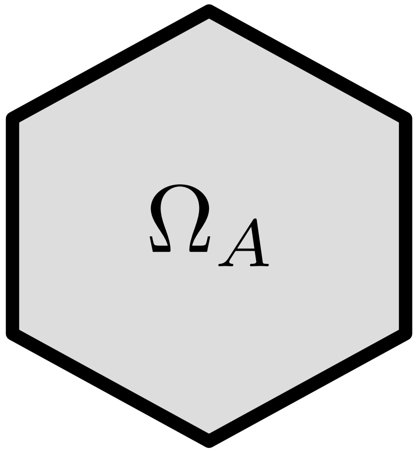
Example. If $\mathcal{A}$ is an even-sided polygon, then some states on $\mathcal{A}\mathcal{A}$ violate the Tsirelson bound of $2\sqrt{2}$ for the Bell-CHSH inequality. From this, we can compute some $\varepsilon > 0$ such that $\mathcal{A}$ cannot be ε-embedded.



Summary

- Have generalized Spekkens' notion of generalized noncontextuality: *"Processes that are statistically indistinguishable in an effective theory should not require explanation by multiple distinguishable processes in a more fundamental theory."*

- → approximate **simulations** and **embeddings** of one GPT by another.

- We have classified all unrestricted GPTs exactly embeddable into QT…

- … and we have given methods for certifying the impossibility of an approximate embedding. *Not optimal. Open: find a better method!*

- This admits a **novel experimental test of QT** that does not suffer from a "tomographic completeness loophole".

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Thank you!