ÖAW AUSTRIAN ACADEMY OF SCIENCES

IQOQI – INSTITUTE FOR QUANTUM OPTICS AND QUANTUM INFORMATION VIENNA

Computational irreducibility

and notions of simulation for Turing machines

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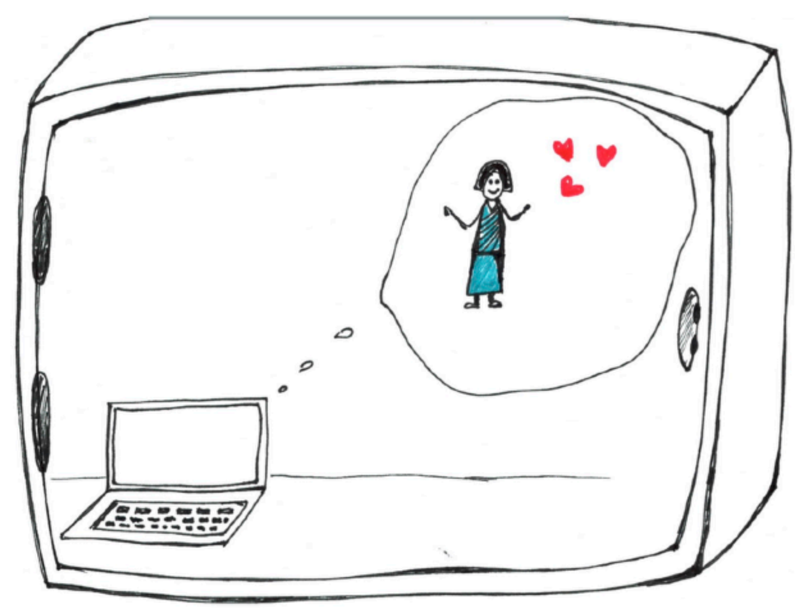


FQXi FOUNDATIONAL QUESTIONS INSTITUTE

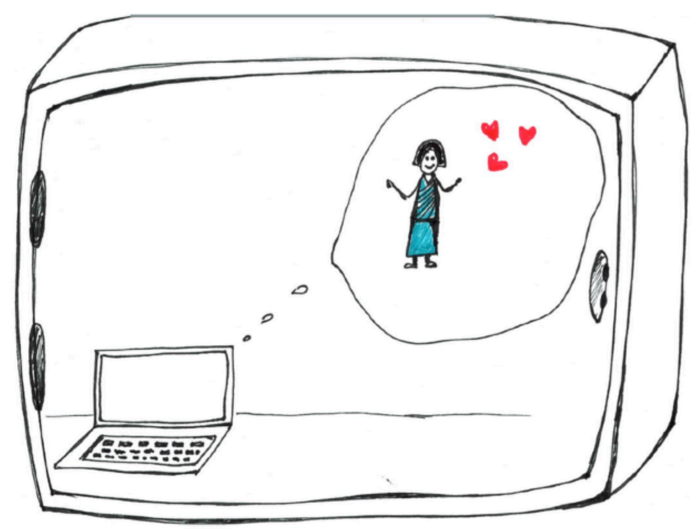
PERIMETER INSTITUTE

Overview

1. **Wolfram's** computational irreducibility and free will

inadequacy of that approach

2. John the cook and **computational sourcehood**



motivates the main technical question

3. Universality and a **simulation preorder for TMs**?

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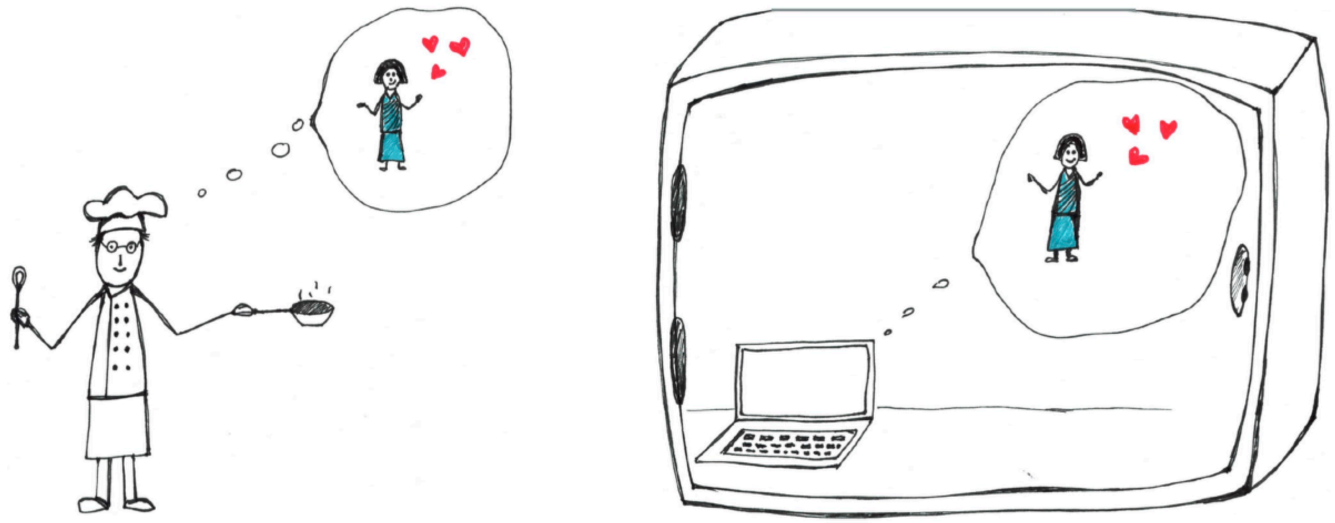

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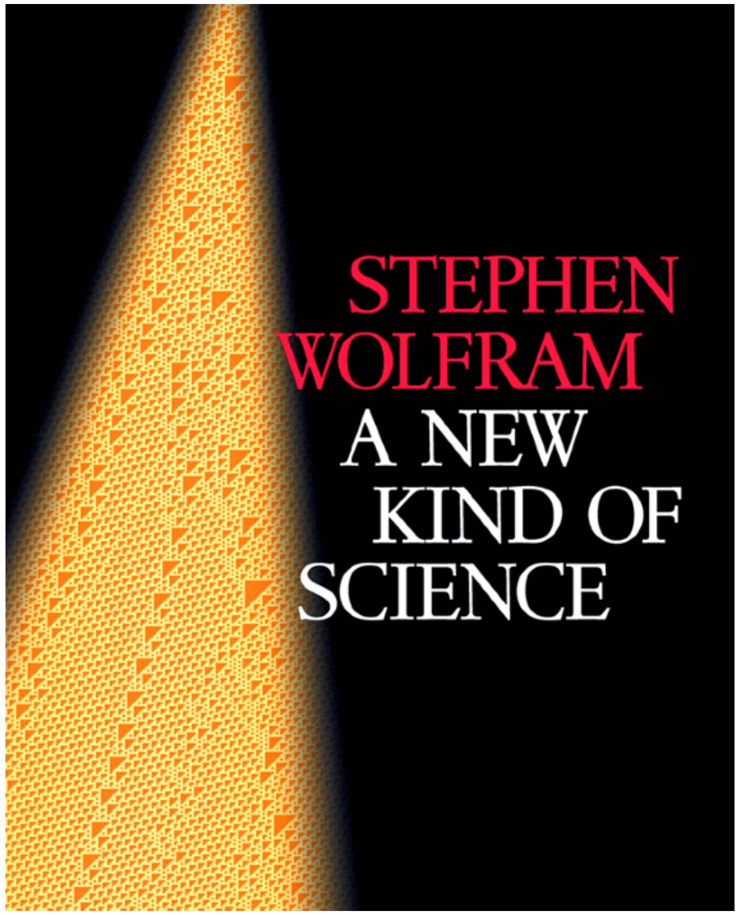
Wolfram’s computational irreducibility



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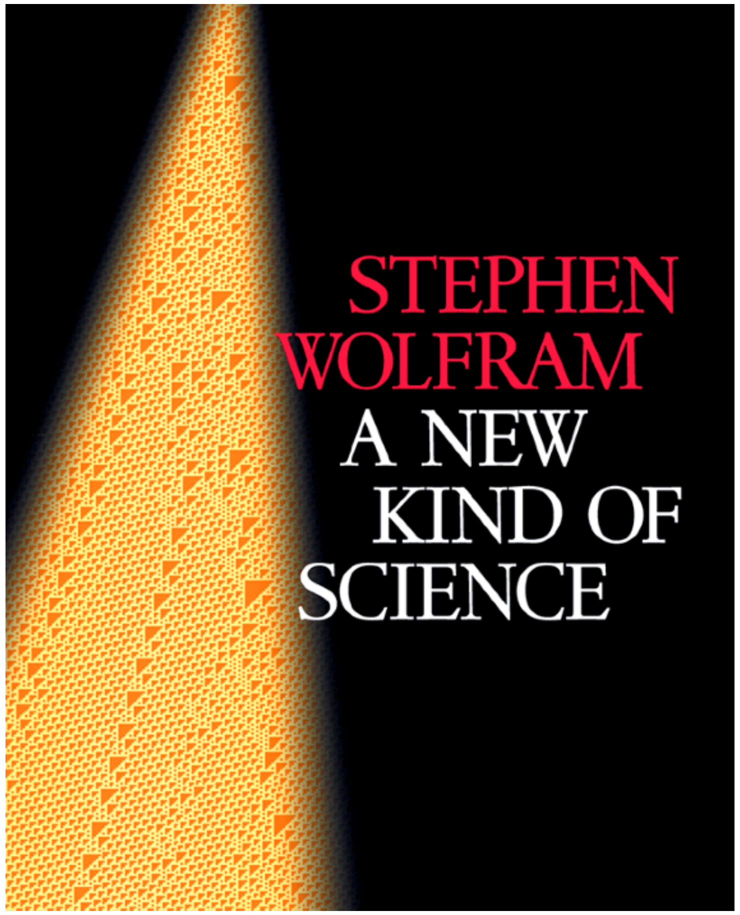

Some systems in nature are **comp. reducible:** we can predict their future behavior with simple equations / theories / algorithms.

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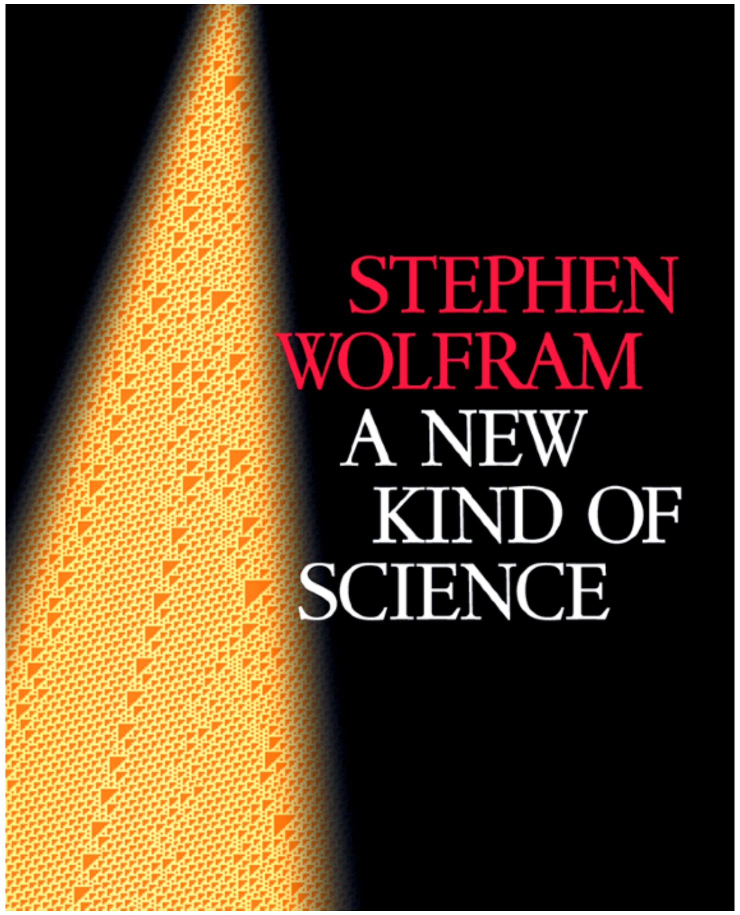

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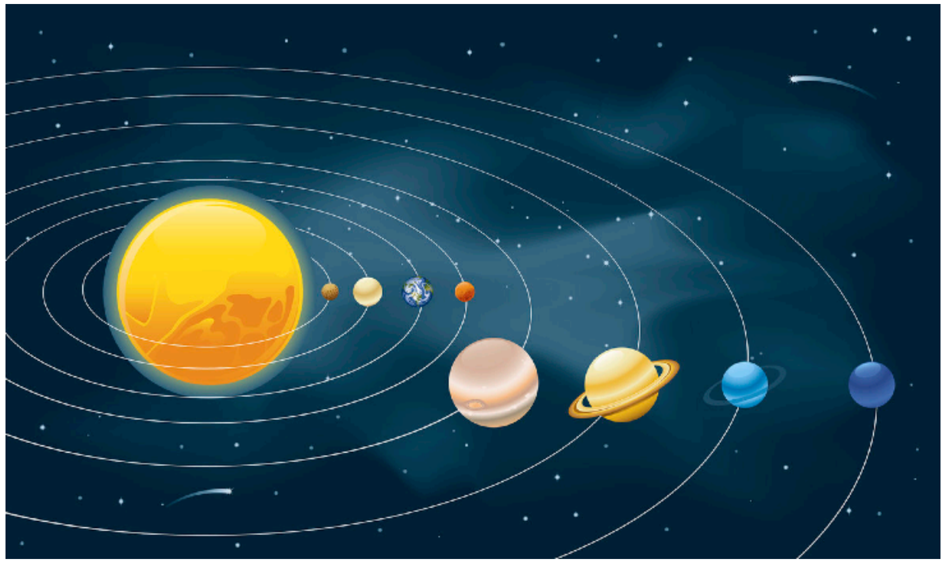
Examples:
Position of **Jupiter** on Jan. 31, 2520;



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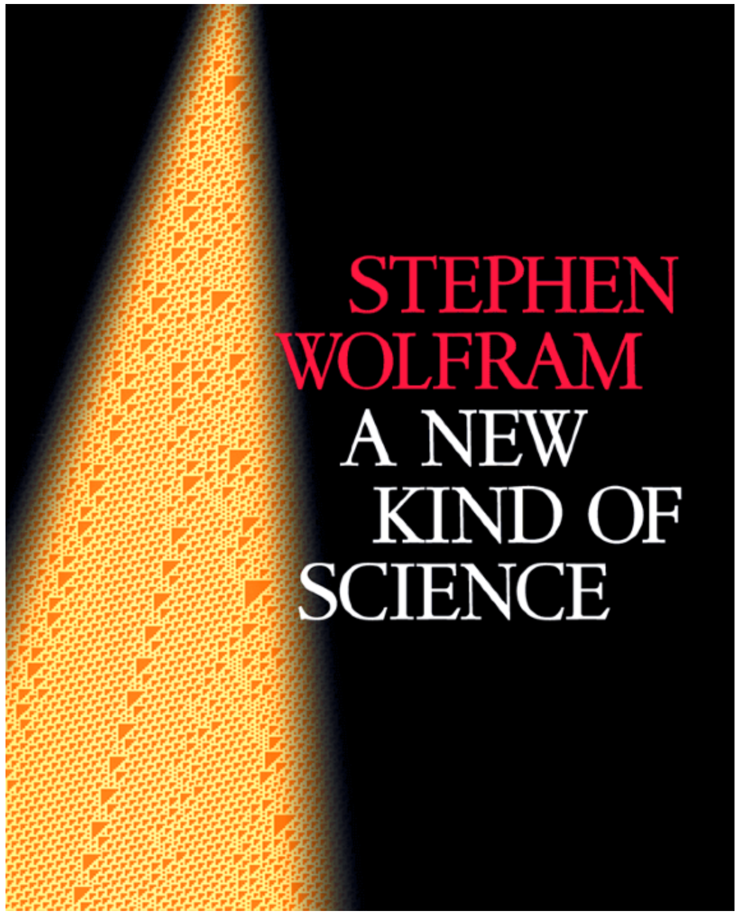

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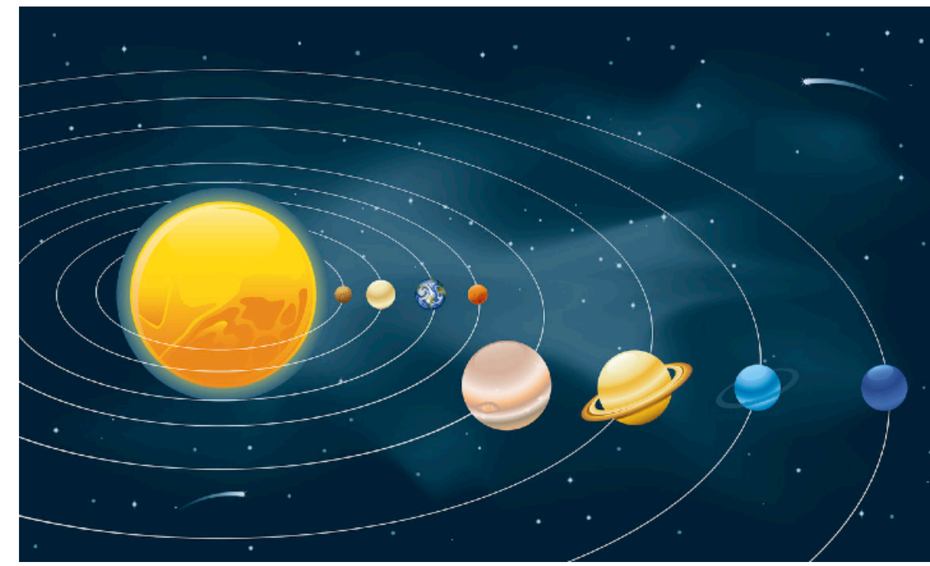
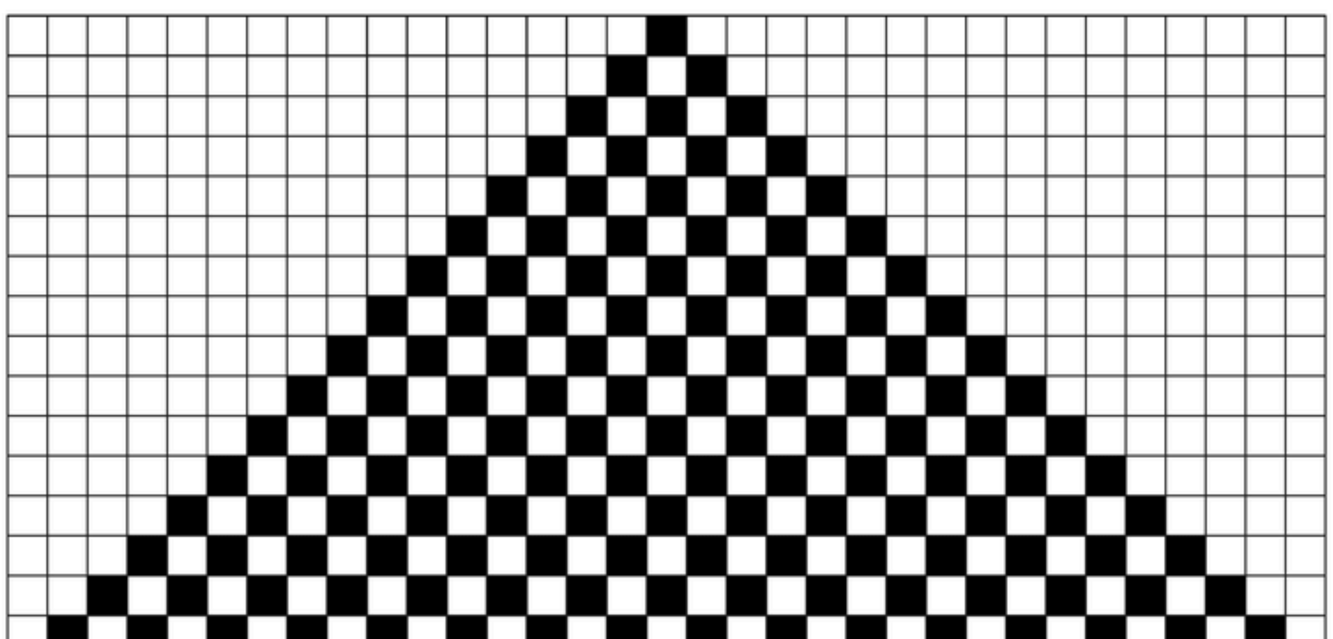
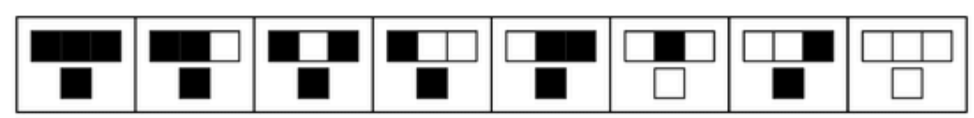
Examples:
Position of **Jupiter** on Jan. 31, 2520;
some **cellular automata**.



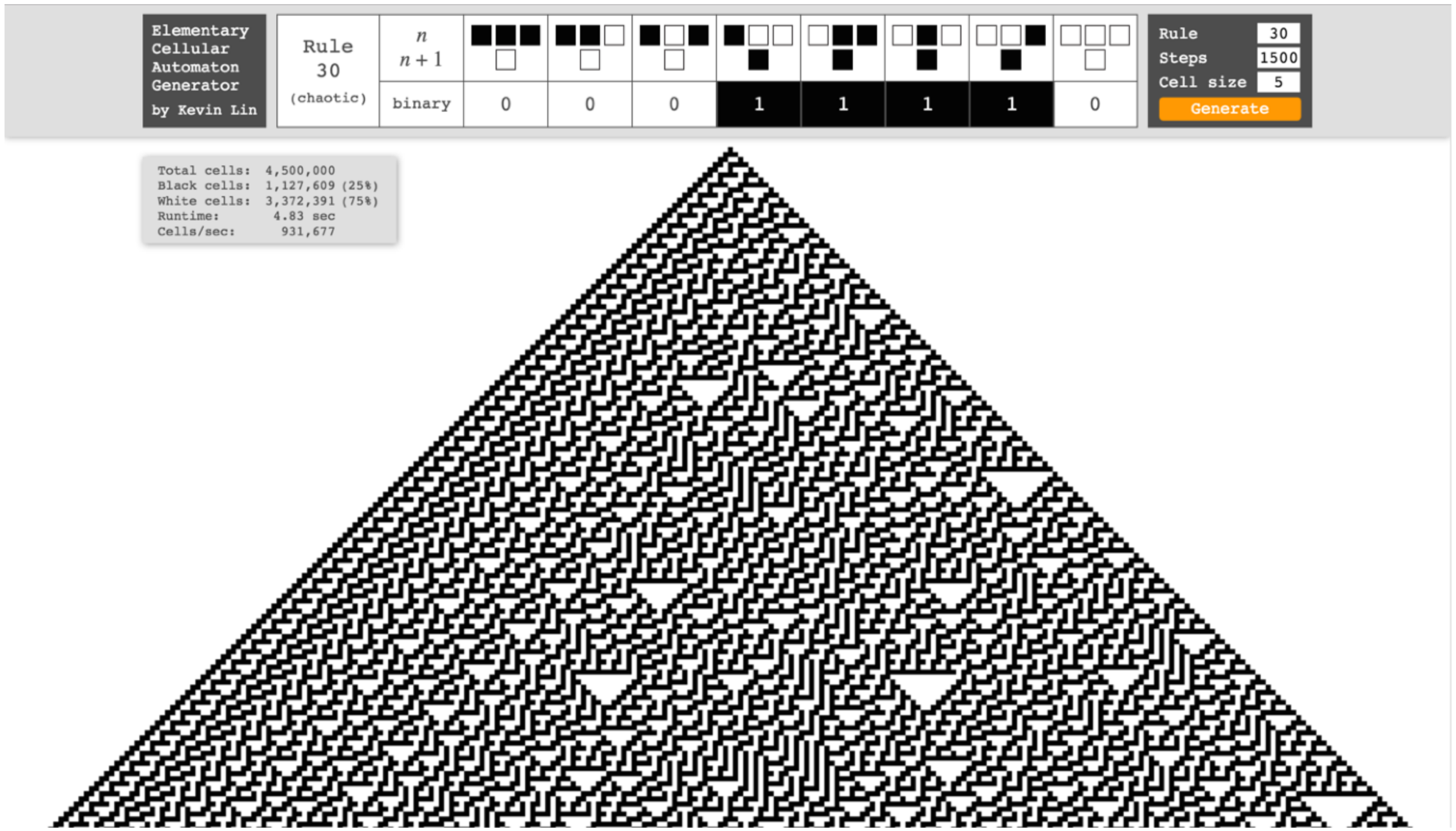




A cellular automaton with a slightly different rule. The rule makes a particular cell black if either of its neighbors was black on the step before, and makes the cell white if both its neighbors were white. Starting from a single black cell, this rule leads to a checkerboard pattern. In the numbering scheme of Chapter 3, this is cellular automaton rule 250.

Wolfram’s computational irreducibility

Some systems admit no such shortcuts: **computationally irreducible**.



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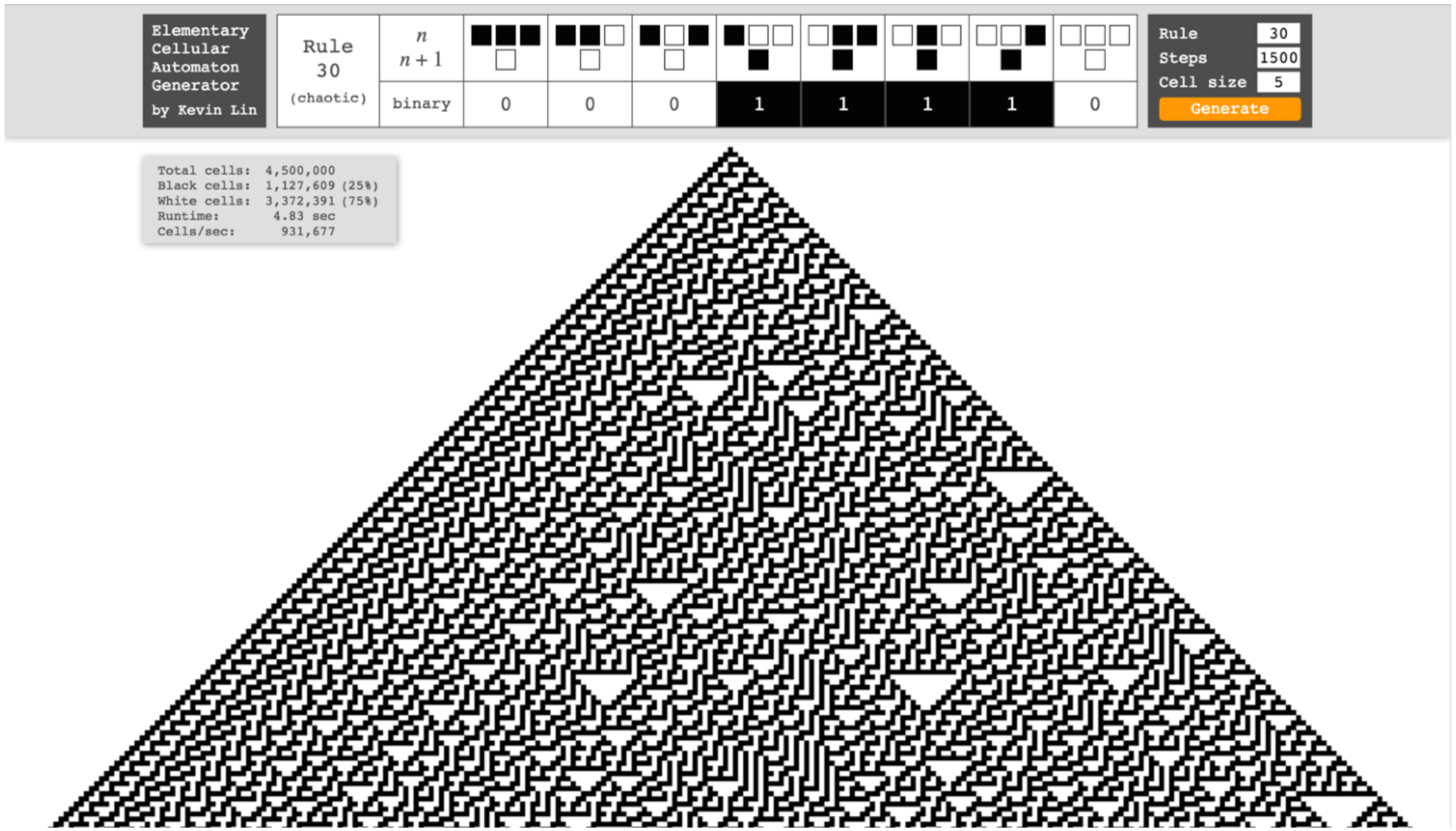

- To predict the behavior of a CI system, we have to **emulate it exactly.**
- Happens as soon as **computational universality** is reached.
- Wolfram's claim: except for the simplest systems, this is the **typical behavior**.

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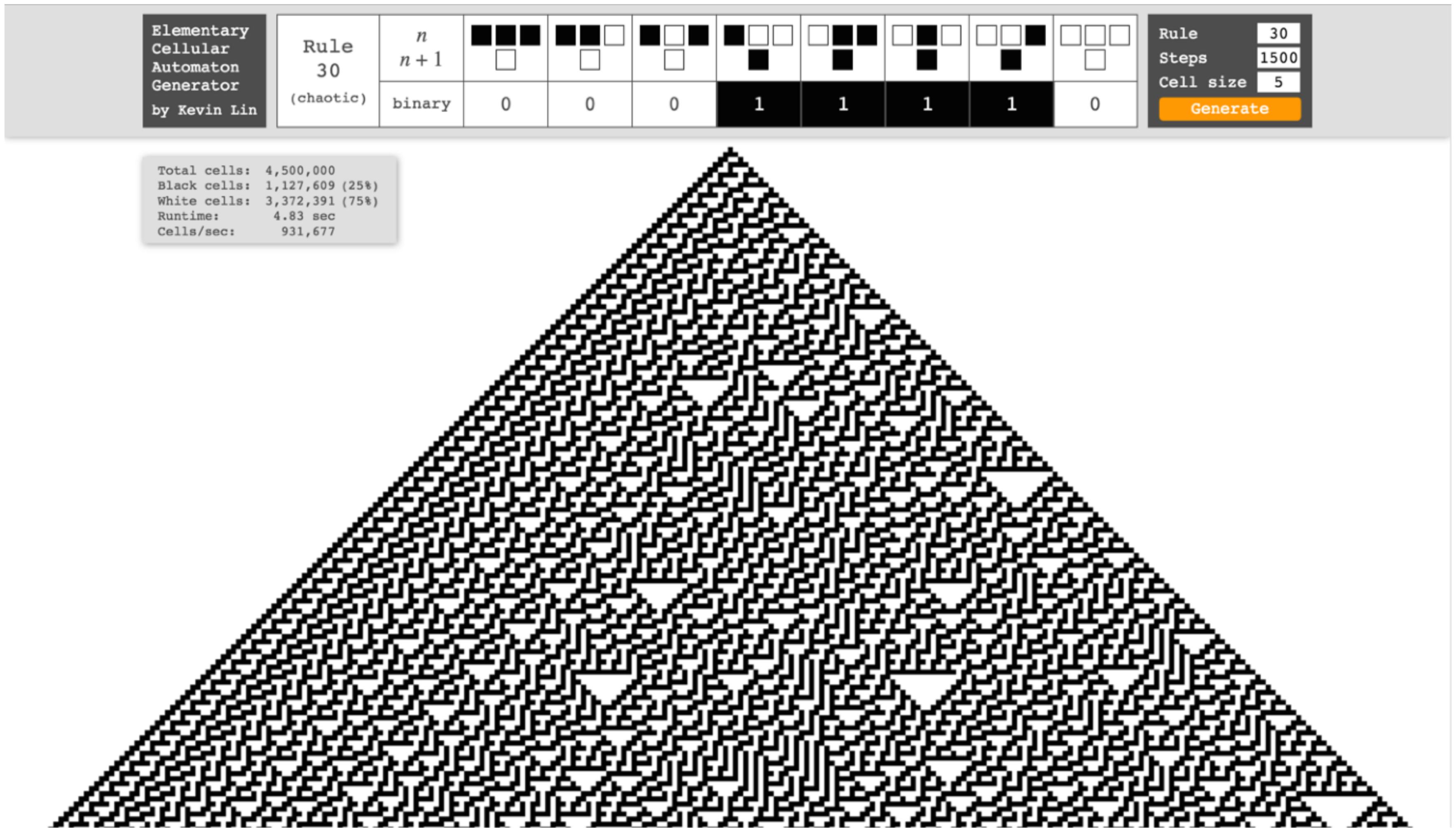

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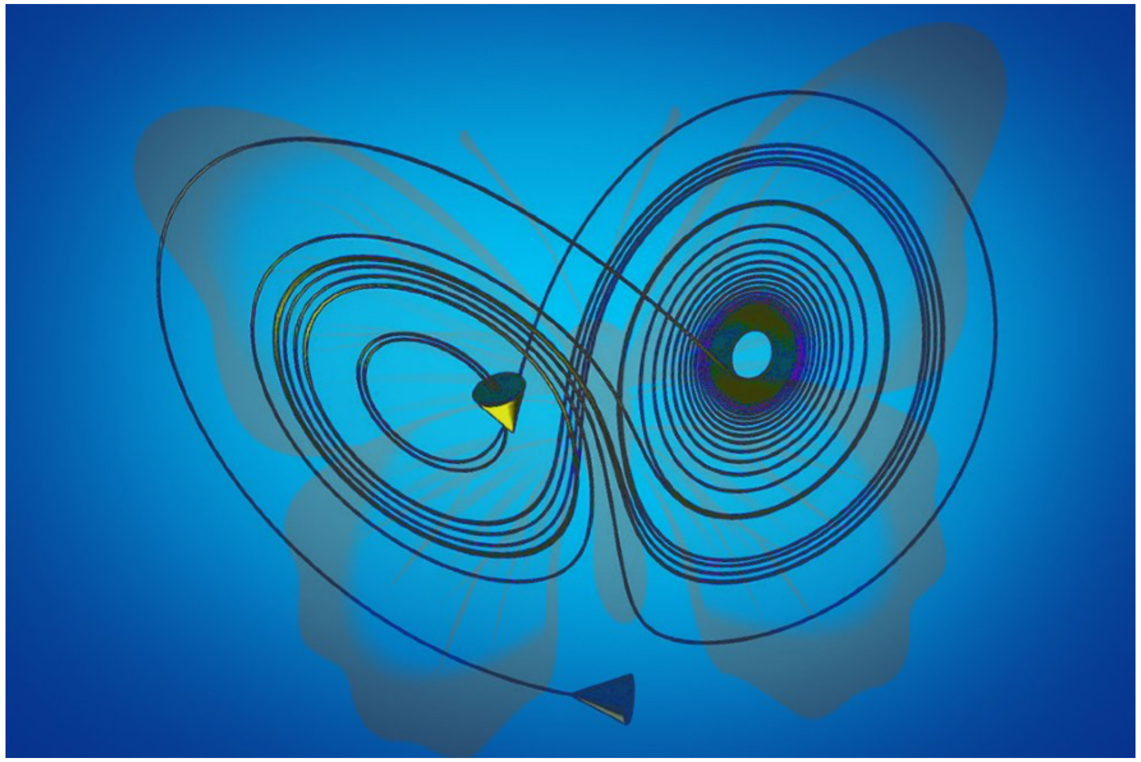
Problem: there is **no formal definition** of computational irreducibility!

Computational irreducibility ≠ chaos



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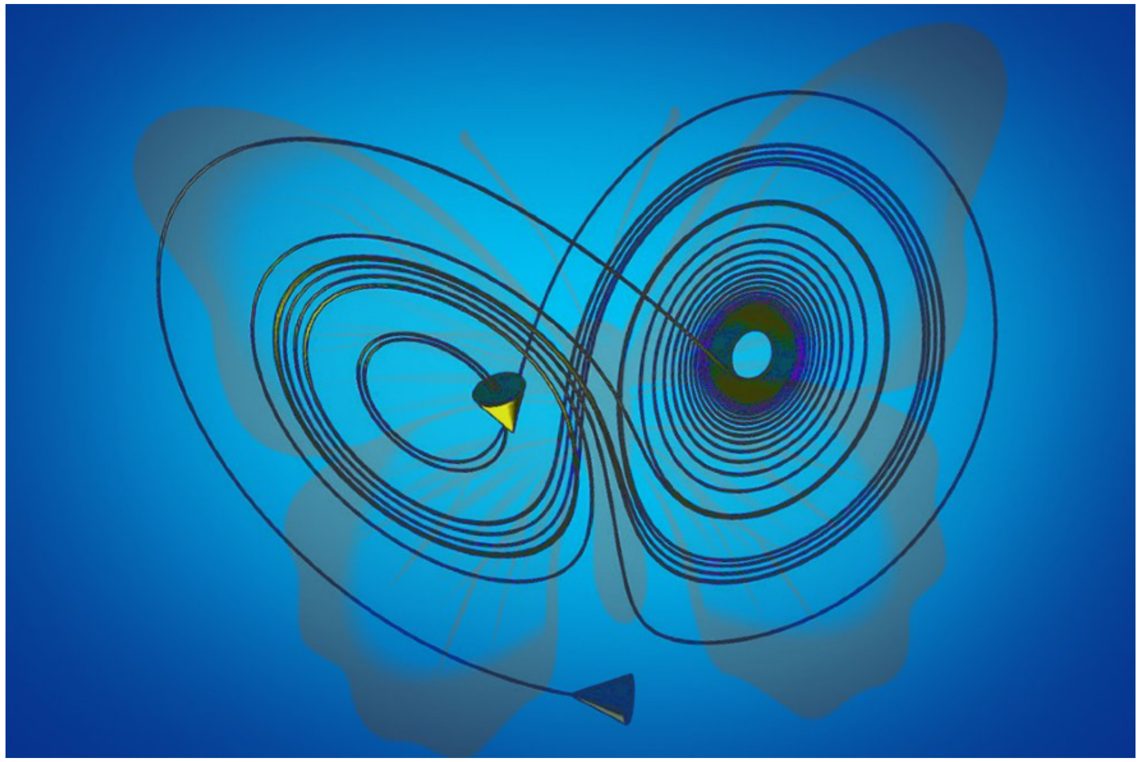

What is the relation of your theory with chaos and complexity theory. When I try to explain what you discover in your book to someone else they say, "Ah, chaos theory."

Chaos theory is really about a very specific phenomenon: that **sensitive dependence on initial conditions** can lead to randomness. And what one finds in the end is that the only way to get randomness out of this phenomenon is just to put randomness in, in the initial conditions. What I've found is that simple programs can actually produce randomness—and complexity—without it ever being put it. It's a much more powerful phenomenon.

Computational irreducibility and **free will**?

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Wolfram (2002):

“And it is this, I believe, that is the ultimate origin of the **apparent freedom** *of human will. For even though all the components of our brains presumably follow definite laws, I strongly suspect that their overall behavior corresponds to an irreducible computation whose outcome can never in effect be found by reasonable laws.”*

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S. Bringsjord, Free will and a new kind of science (2013):

"If someone's will is apparently free, it hardly follows that that will is in fact free. Nowhere in ANKS [his book] does Wolfram even intimate that he maintains that our decisions are in fact free."

➡ Wolfram is *"epistemologically correct"*, but *"metaphysically wrong"*.

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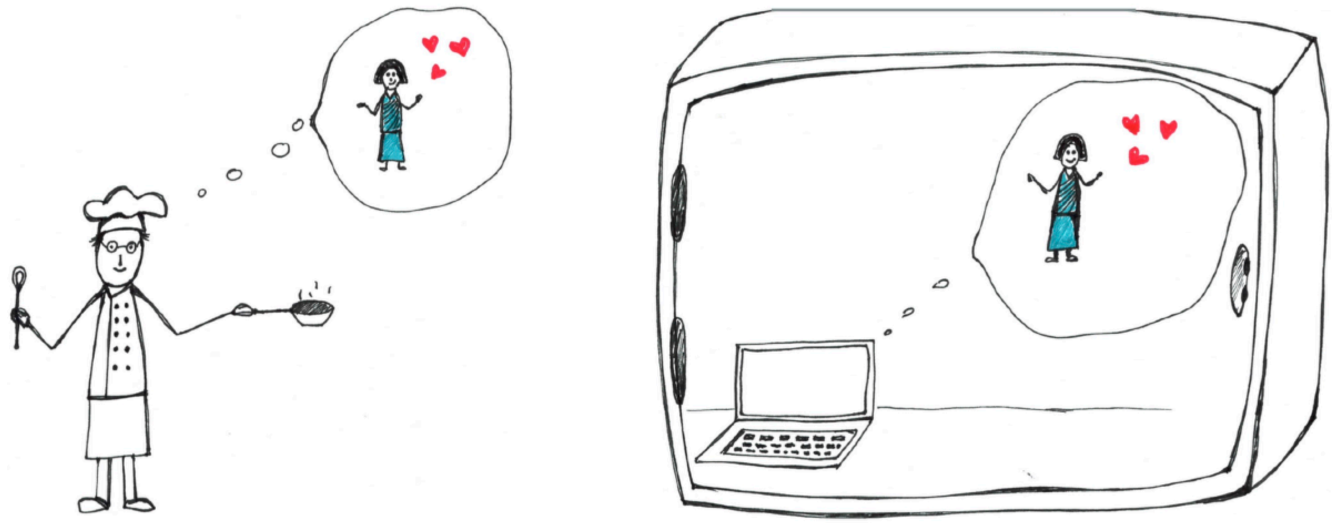

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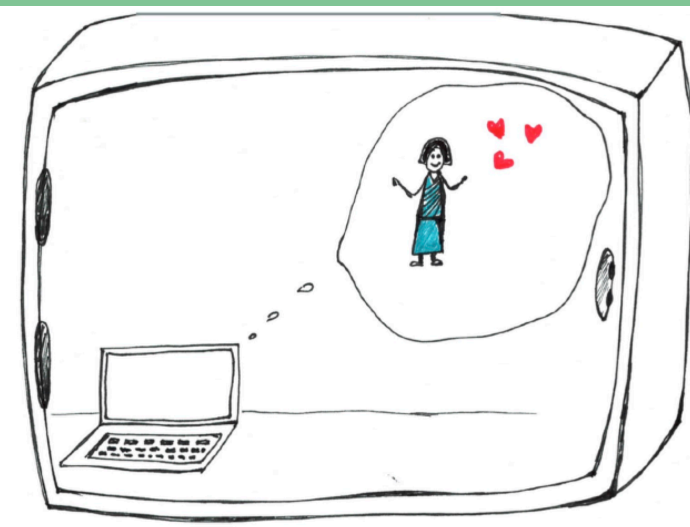

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Free will, compatibilism, and sourcehood

Philosophers argue for (one of) the following underpinnings of free will:

- **The freedom to do otherwise.**
  But what does that exactly mean?

- **Sourcehood.**
  What matters for an agent's freedom and responsibility
  is the source of her action—how her action was brought about.

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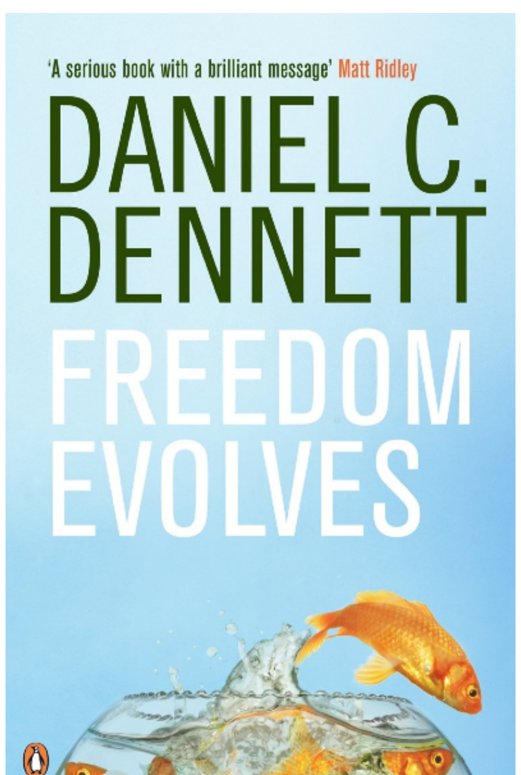

Like the **compatibilists**, I will focus on **sourcehood**:
it is a notion of free will that is compatible even with
a fully deterministic and digital world.
However, I will argue that computational irreducibility
is **not** the correct notion to study.

John the (emotional) cook

Worst-case assumptions: fully deterministic and digital world.

- Every morning, John prepares one of *N* breakfasts, where *N* is large.
- E.g., he thinks of his late Canadian wife, and then prepares omelette with Maple syrup.

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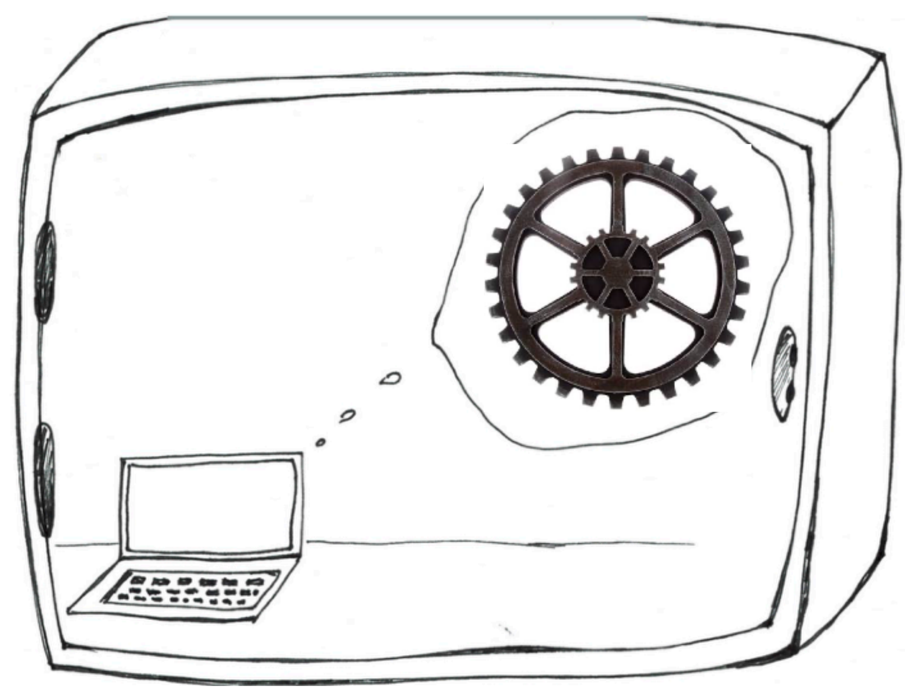

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- If T is small enough (shortcut):

See, John? You have decided to prepare the Canadian omelette. Ha, this is exactly what our computer has predicted half an hour earlier, as several witnesses can testify — well before you have thought about your Canadian wife!"

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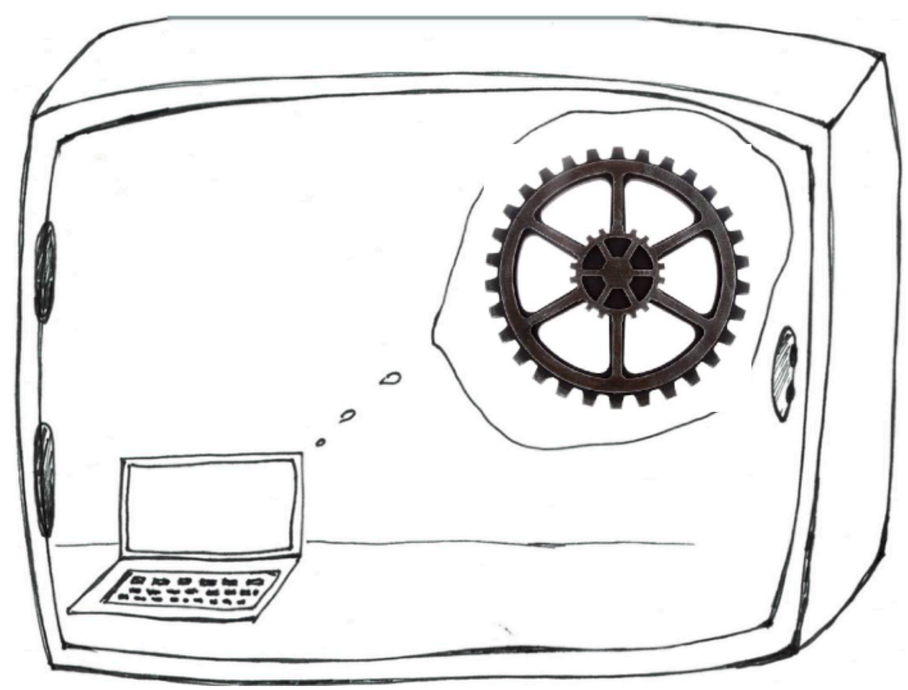

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See, John? You think your emotions have decided to prepare the Canadian omelette, but what happened in the safe was only determined by your physical state and apartment yesterday night. Your thoughts of your late wife this morning had no impact on the decision whatsoever!”

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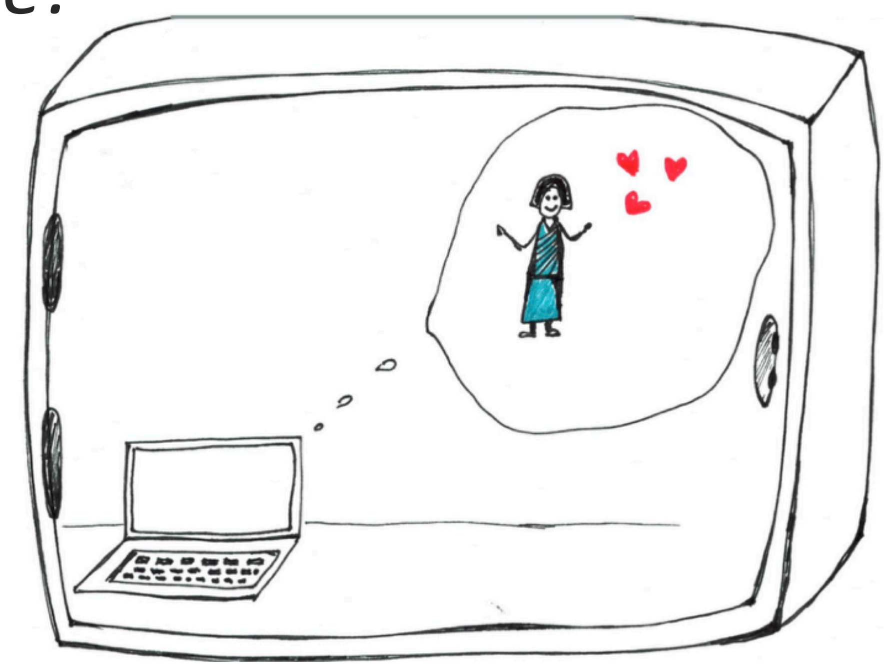
In **both** cases, sourcefulness of John's emotions is equally contested. **"Shortcut or not" is an irrelevant question.** Insofar as computational irreducibility is understood as "no shortcut", it is not the relevant notion.

John argues back

But look at your computer: it has performed an exact simulation of my thought processes. In particular, there have been **algorithmic correlates** *of my emotions in the safe!*



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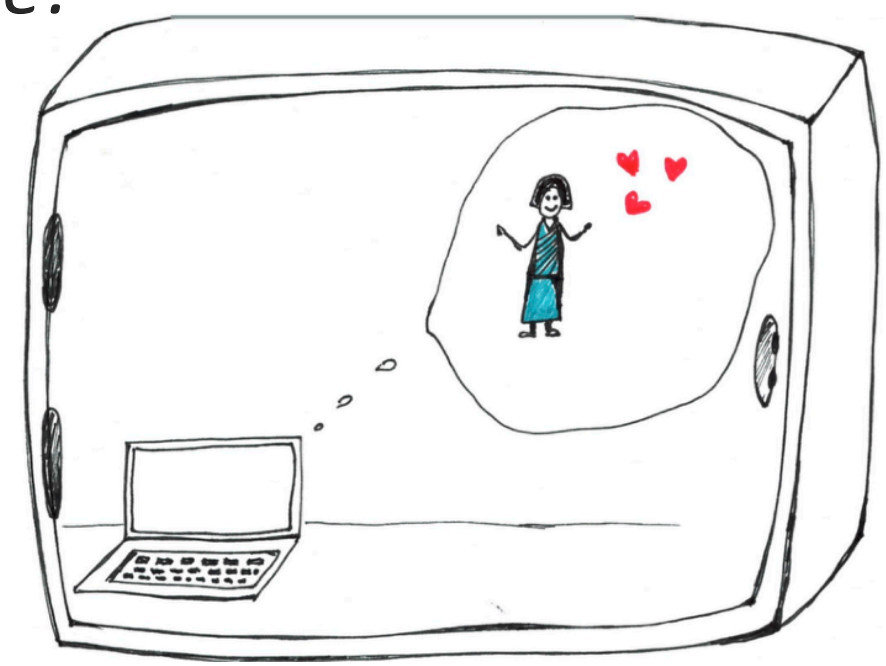

I am not that material body, but the computational process that is represented by it. You have just manufactured another representation of the same process in the safe.

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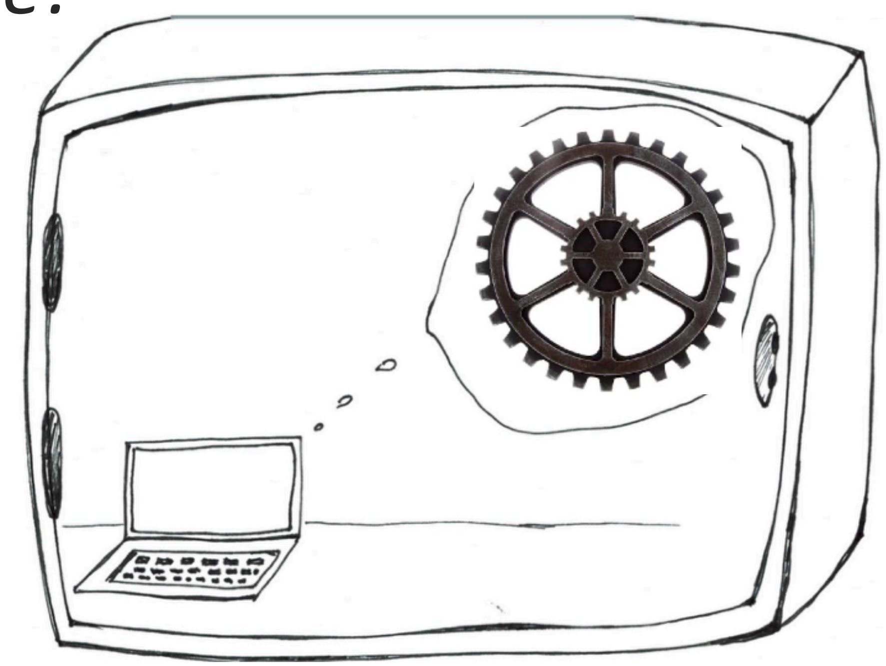

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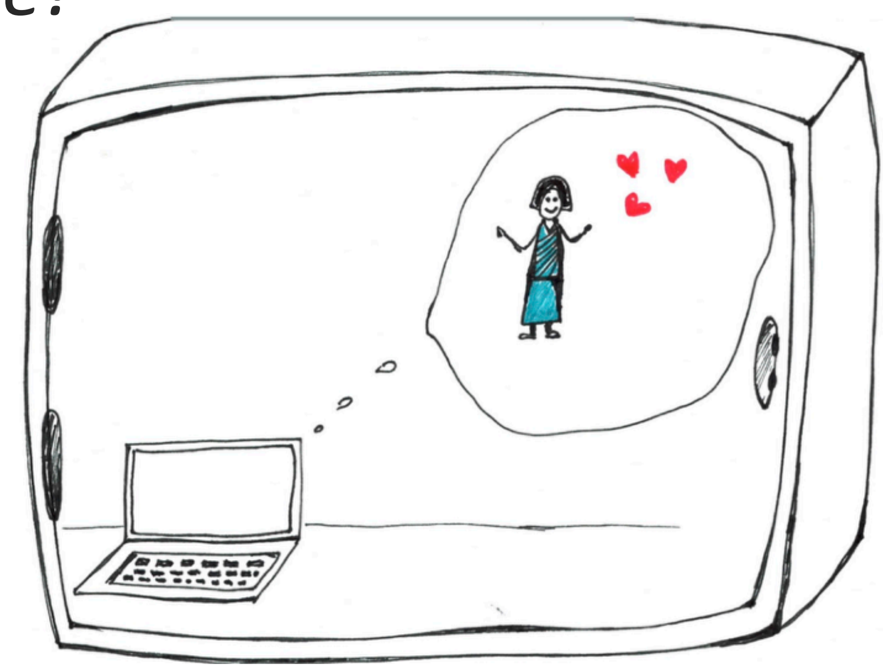

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Computational sourcehood: To predict John's decision, the simulation has to contain representations of all of John's instantaneous states.

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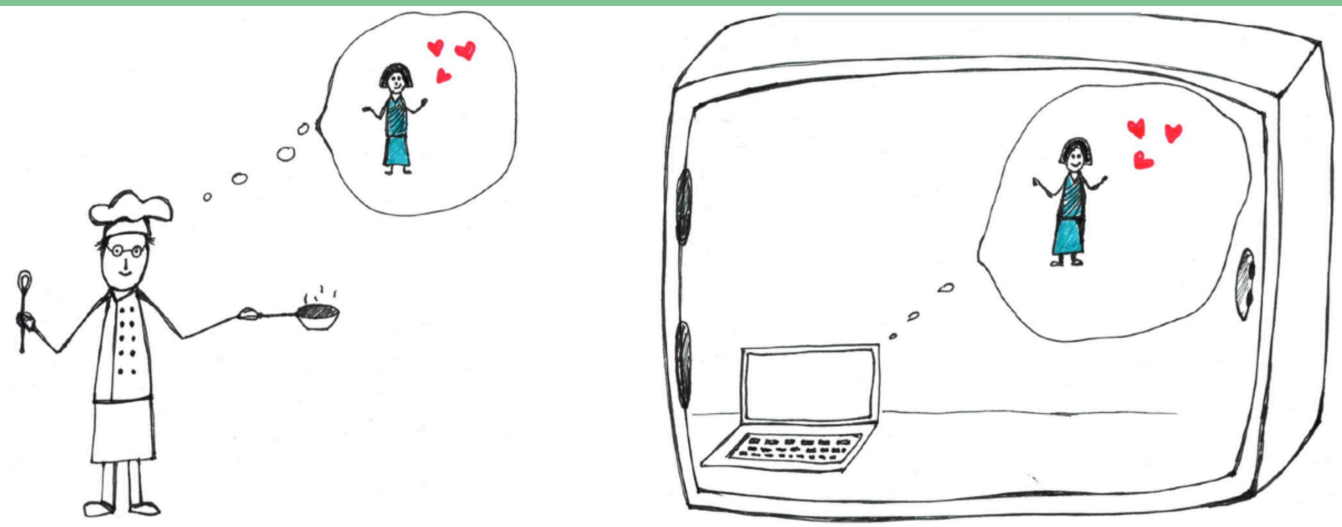

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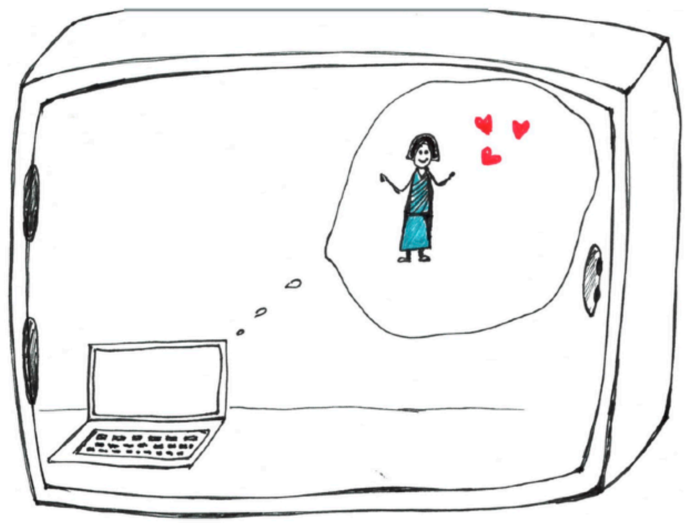

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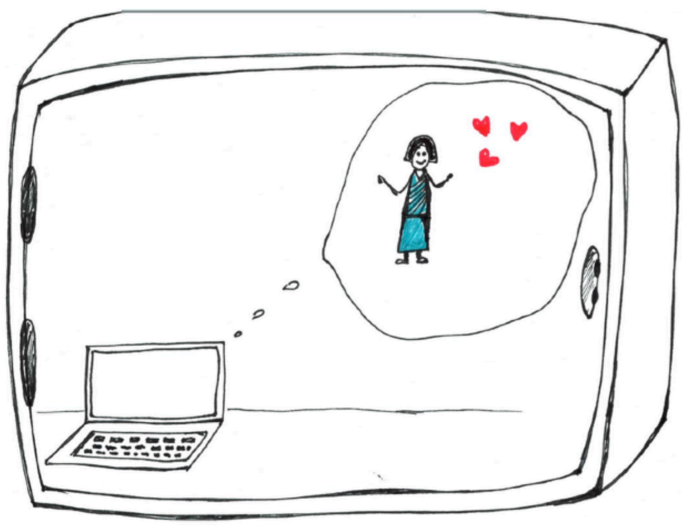
Computational sourcehood and Turing machines



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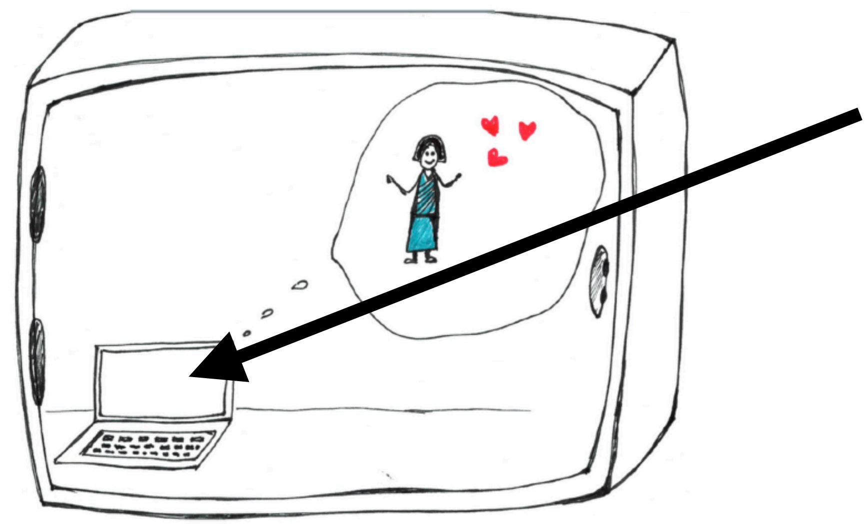
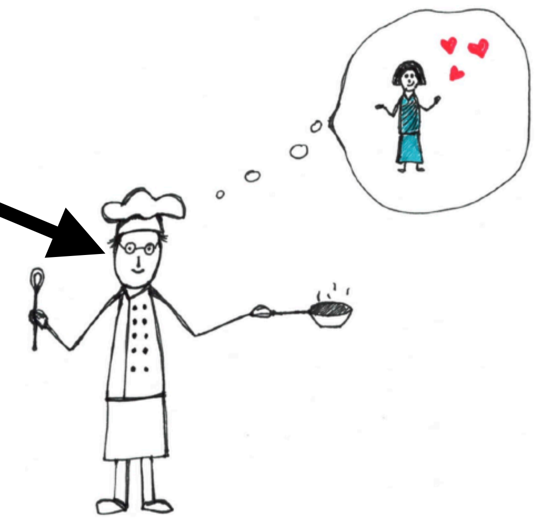
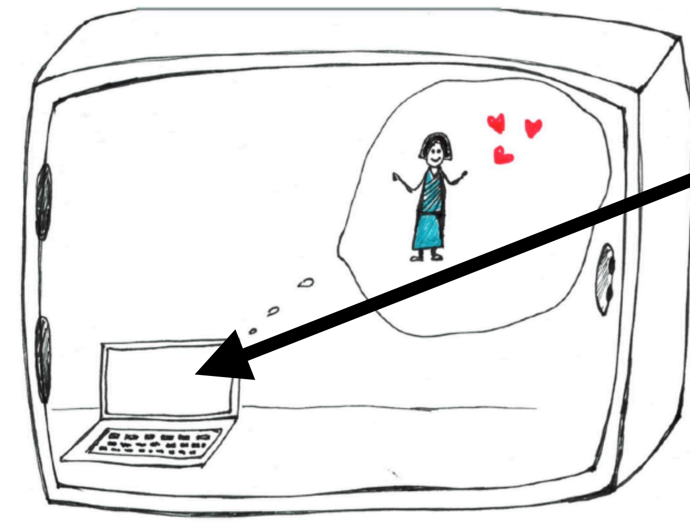


universal
TM *U*

Computational sourcehood and Turing machines

TM *T*





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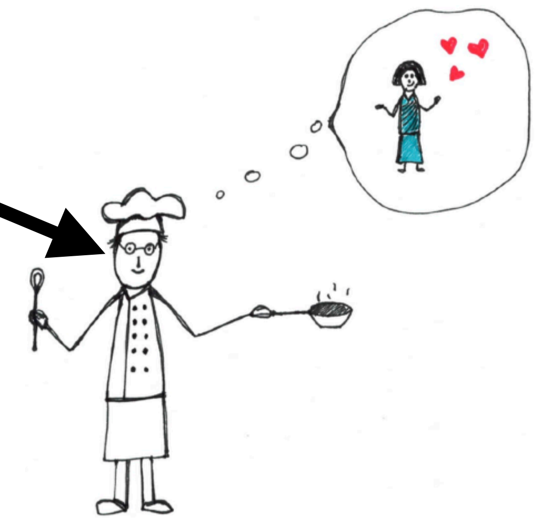

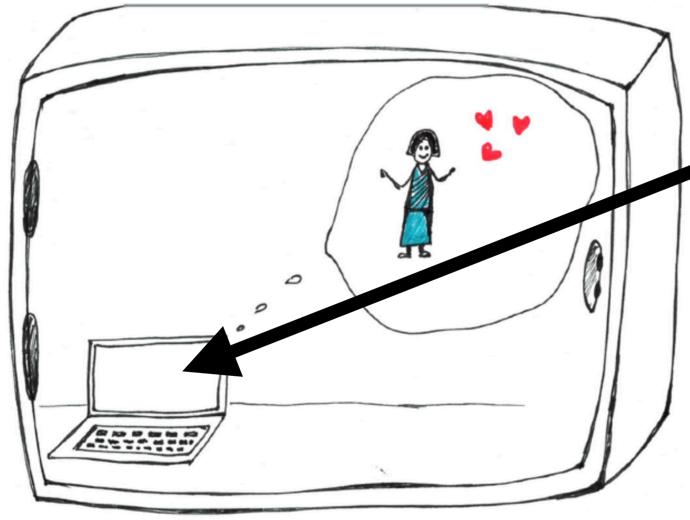

universal TM U

- x: description of John's (+apartment's) state the evening before

$$x \in \{0,1\}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$$

Computational sourcehood and Turing machines

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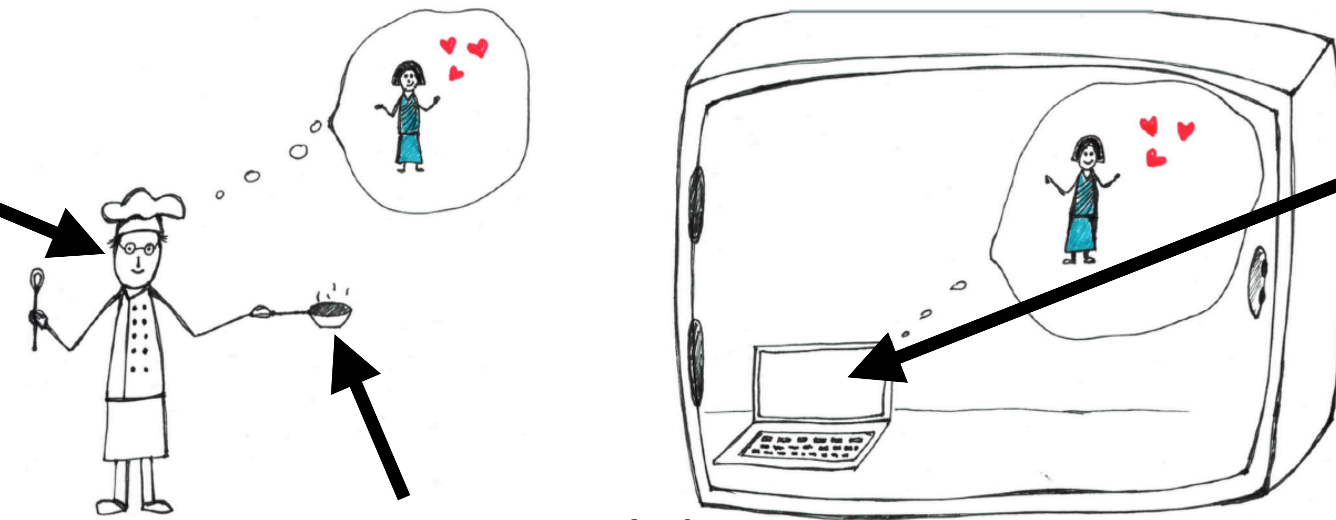

universal TM *U*

“output” *T(x)*:
actual choice of breakfast

- *x*: description of John’s (+apartment’s) state the evening before

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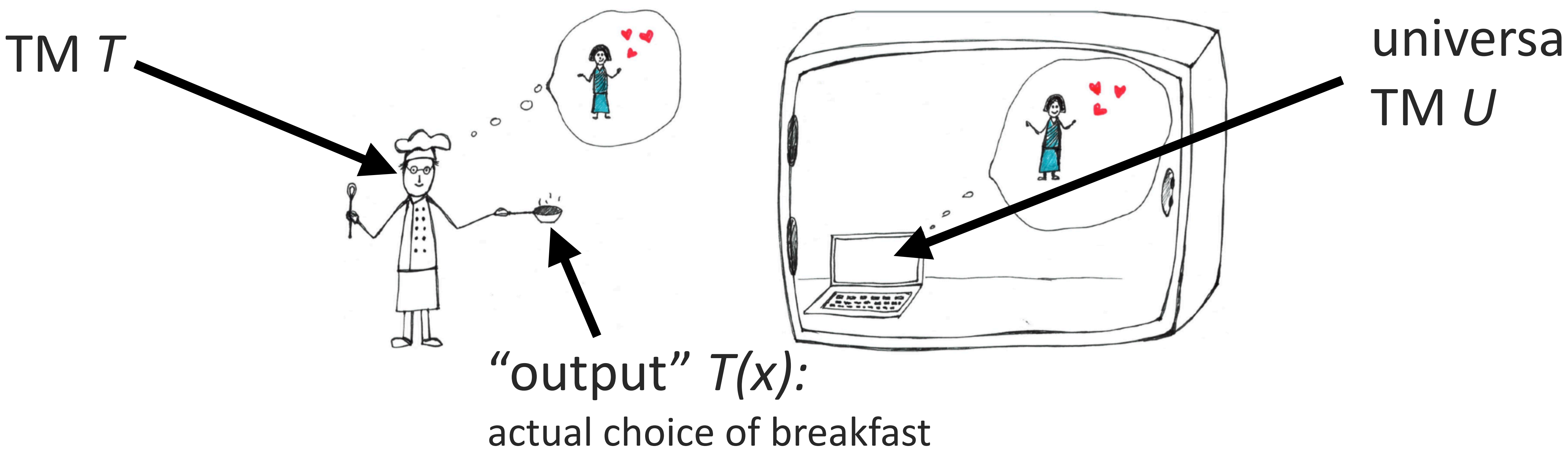
Computational sourcehood and Turing machines



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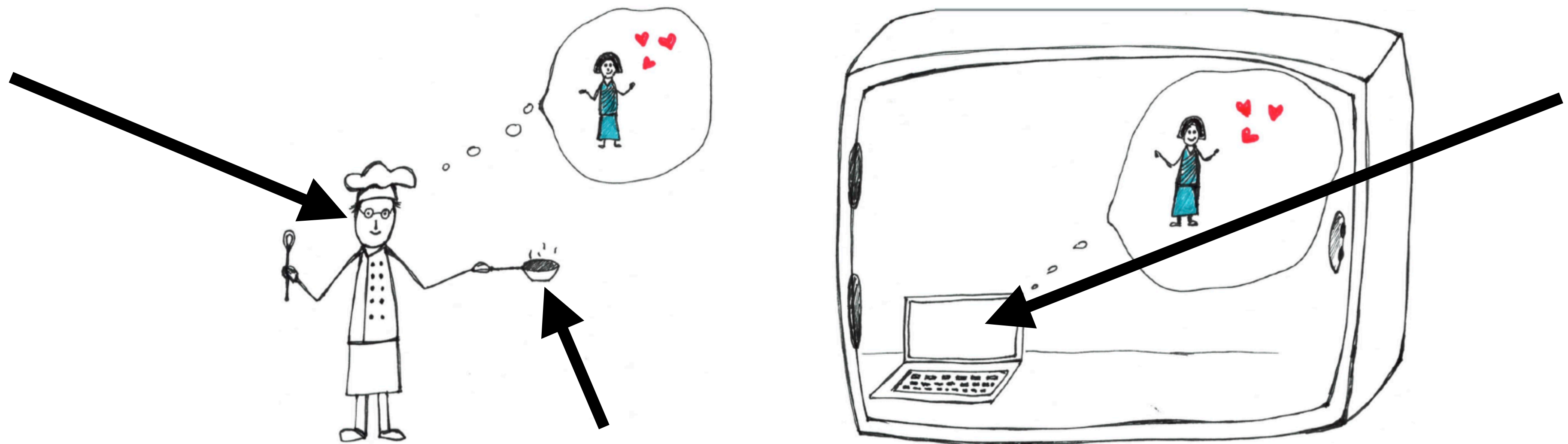
Computational sourcehood and Turing machines



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$$U(p_T x) = T(x)$$

Same outputs, but

does that mean that U must simulate T step by step?

A conjecture on universal Turing machines

Conjecture. Suppose that a TM *U* is **universal** in the sense that it reproduces the **outputs** of any other TM: that is,

$$U(p_T x) = T(x)$$

for every TM *T* and every input *x* on which *T* halts. Then, for "most" *T*, the universal TM *U* will generate its output by means of some form of **step-by-step simulation** of *T*'s computation.

In this sense, *T* is the "source" of its outputs (**computational sourcehood**).

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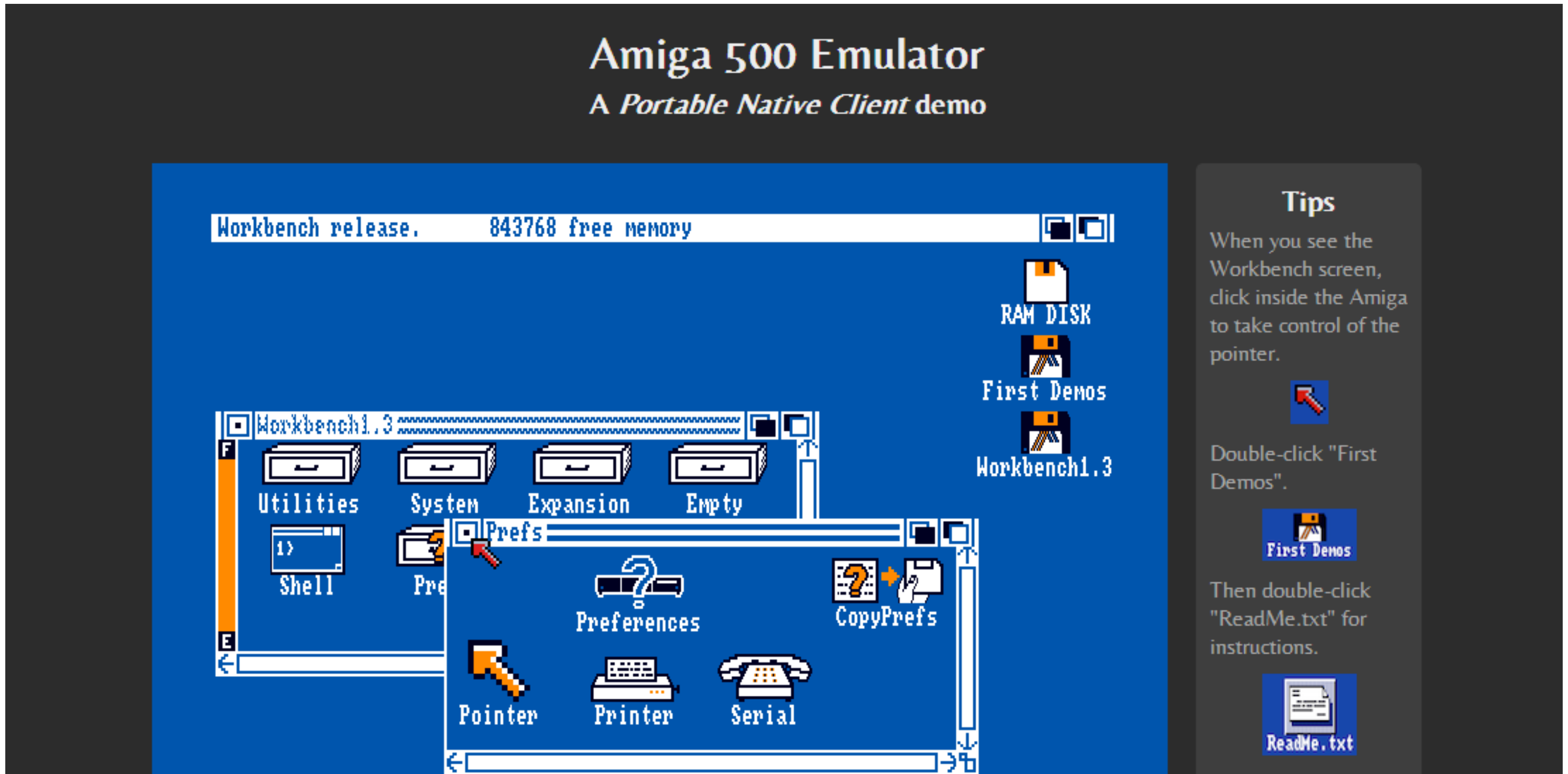
Standard emulators do this!



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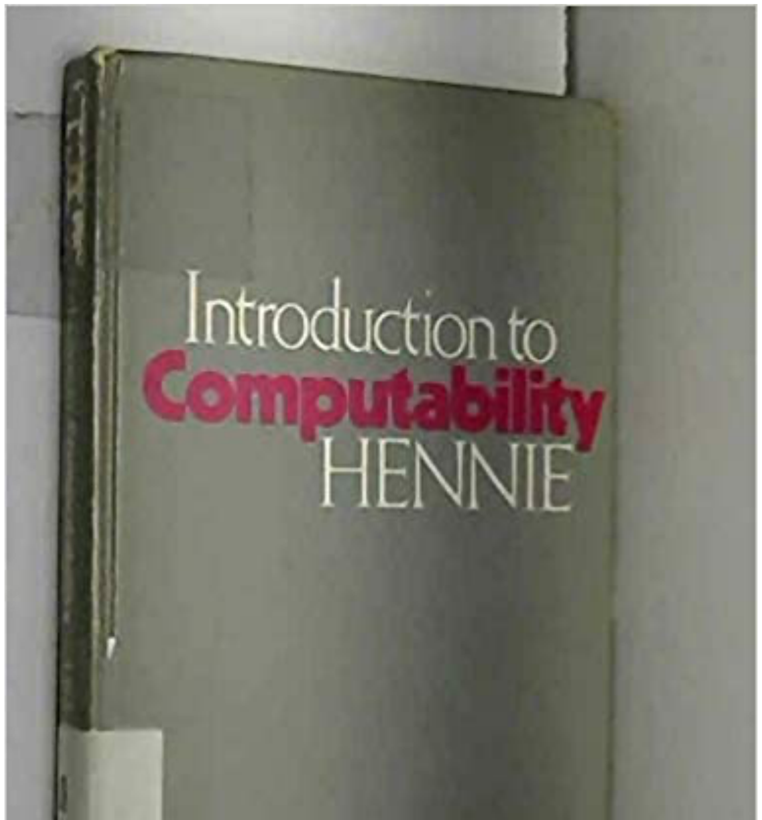
Textbook universal TMs do this too…
… and motivate attempts to **formalize** the conjecture rigorously.



Goal: find a rigorous formulation of the conjecture that has a chance to be true.

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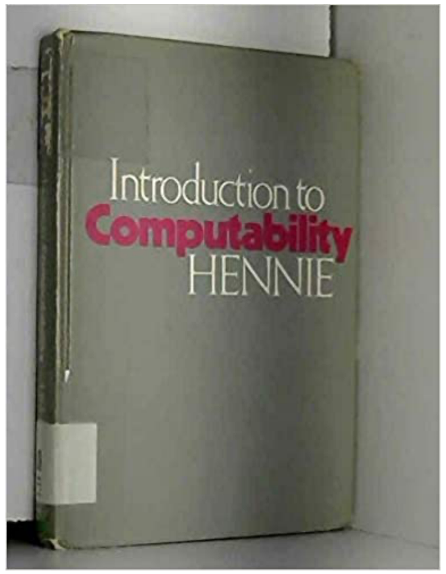

Hennie: encode the tape contents of *T* on the tape of the universal machine *U*.

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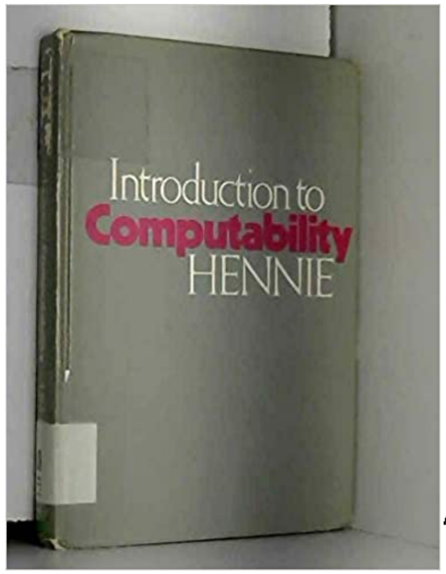

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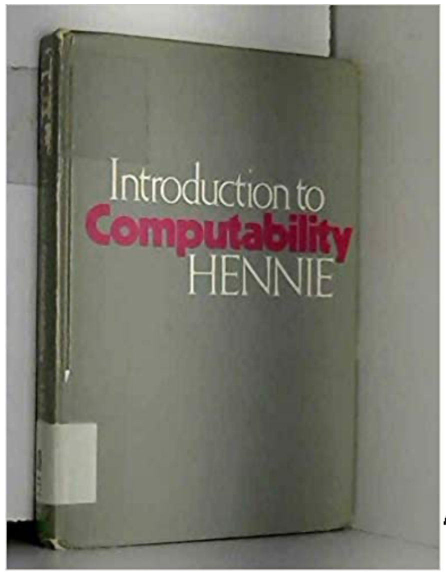
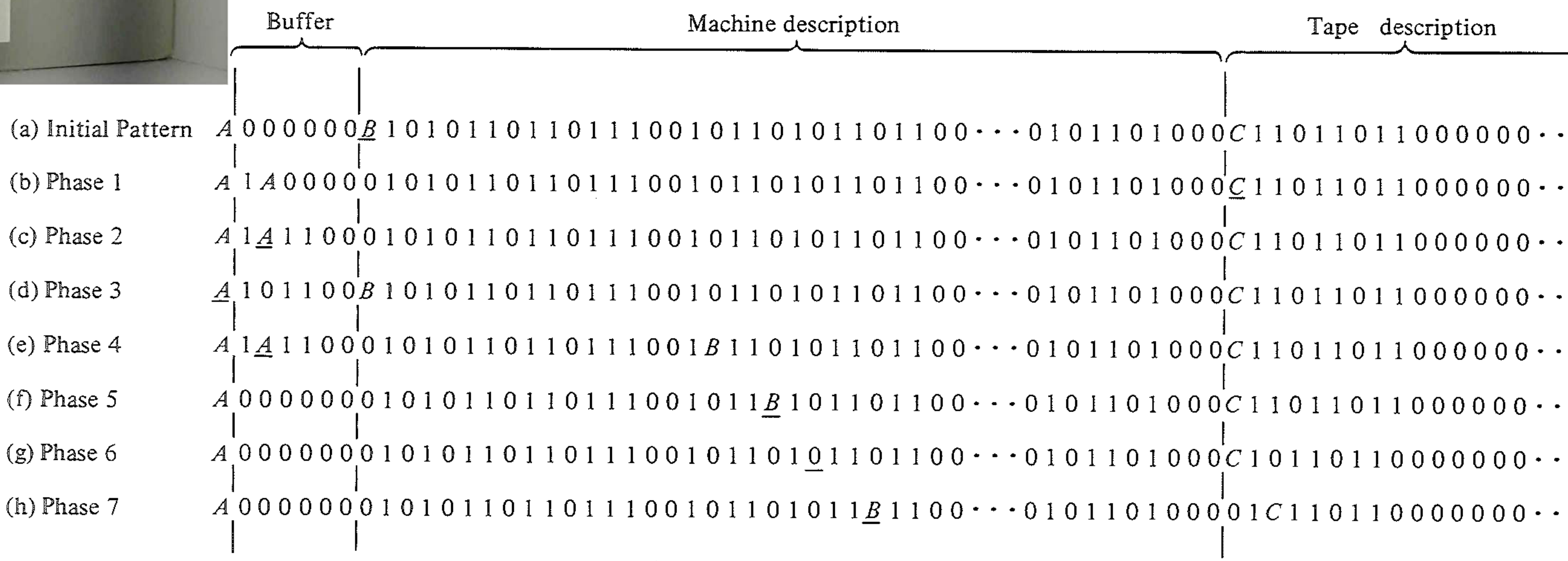
	Buffer	Machine description	Tape description
(a) Initial Pattern	*A* 0 0 0 0 0 0	$\underline{B}$ 1 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1 0 1 1 0 1 0 1 1 0 1 1 0 0 · · · 0 1 0 1 1 0 1 0 0 0	*C* 1 1 0 1 1 0 1 1 0 0 0 0 0 0 · · ·
(b) Phase 1	*A* 1 *A* 0 0 0 0 0	0 1 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1 0 1 1 0 1 0 1 1 0 1 1 0 0 · · · 0 1 0 1 1 0 1 0 0 0	$\underline{C}$ 1 1 0 1 1 0 1 1 0 0 0 0 0 0 · · ·
(c) Phase 2	*A* 1 $\underline{A}$ 1 1 0 0	0 1 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1 0 1 1 0 1 0 1 1 0 1 1 0 0 · · · 0 1 0 1 1 0 1 0 0 0	*C* 1 1 0 1 1 0 1 1 0 0 0 0 0 0 · · ·
(d) Phase 3	$\underline{A}$ 1 0 1 1 0 0	*B* 1 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1 0 1 1 0 1 0 1 1 0 1 1 0 0 · · · 0 1 0 1 1 0 1 0 0 0	*C* 1 1 0 1 1 0 1 1 0 0 0 0 0 0 · · ·
(e) Phase 4	*A* 1 $\underline{A}$ 1 1 0 0	0 1 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1 *B* 1 1 0 1 0 1 1 0 1 1 0 0 · · · 0 1 0 1 1 0 1 0 0 0	*C* 1 1 0 1 1 0 1 1 0 0 0 0 0 0 · · ·
(f) Phase 5	*A* 0 0 0 0 0 0	0 1 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1 0 1 1 $\underline{B}$ 1 0 1 1 0 1 1 0 0 · · · 0 1 0 1 1 0 1 0 0 0	*C* 1 1 0 1 1 0 1 1 0 0 0 0 0 0 · · ·
(g) Phase 6	*A* 0 0 0 0 0 0	0 1 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1 0 1 1 0 1 0 1 $\underline{0}$ 1 1 0 1 1 0 0 · · · 0 1 0 1 1 0 1 0 0 0	*C* 1 0 1 1 0 1 1 0 0 0 0 0 0 0 · · ·
(h) Phase 7	*A* 0 0 0 0 0 0	0 1 0 1 0 1 1 0 1 1 0 1 1 1 0 0 1 0 1 1 0 1 0 1 1 $\underline{B}$ 1 1 0 0 · · · 0 1 0 1 1 0 1 0 0 0	0 1 *C* 1 1 0 1 1 0 0 0 0 0 0 0 · · ·

A conjecture on universal Turing machines



Hennie: encode the tape contents of *T* on the tape of the universal machine *U*.



Instantaneous configuration of *U* (tape contents, head position, state) contains a complete image of the instantaneous configuration of *T*. One step for *T* corresponds to several steps for *U*.

A conjecture on universal Turing machines

$\mathcal{C}_T(x,t) :=$ configuration of TM *T* on input *x* after *t* steps.

Let $\mathcal{S}$ be a set of "simple functions", containing the identity *(just which set to choose best will be the main question in the following)*.

Definition. Let *T* and *T'* be TMs, and suppose there is some simple function $\varphi \in \mathcal{S}$ that maps the sequence of configurations

$$\mathcal{C}_{T'}(x,0), \mathcal{C}_{T'}(x,1), \mathcal{C}_{T'}(x,2), \ldots, \mathcal{C}_{T'}(x,t'_H)$$

to

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Then we write $T \preceq_{\mathcal{S}} T'$ ("**simulation preorder**").

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Proof idea. There will be a function $\varphi \in \mathcal{S}$ that reads *x* and *t* from $\mathcal{C}_C(x,t)$ and simply **recomputes** $\mathcal{C}_T(x,t)$.

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Shall we go even simpler than linear time?
Wait a minute…

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Details cumbersome; see paper.

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To have any chance that our conjecture is true,

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the set $\mathcal{S}$ must be characterized by something else than simplicity.

From simplicity to preservation of structure

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Attempt of **Definition:** A function φ on TM configurations is **structure-preserving** if for every pair of configurations c, c' such that

$$\mathcal{D}(c, c') \text{ is small,}$$

there is a pair of configurations C, C' with $\varphi(C) = c,\ \ \varphi(C') = c'$ such that

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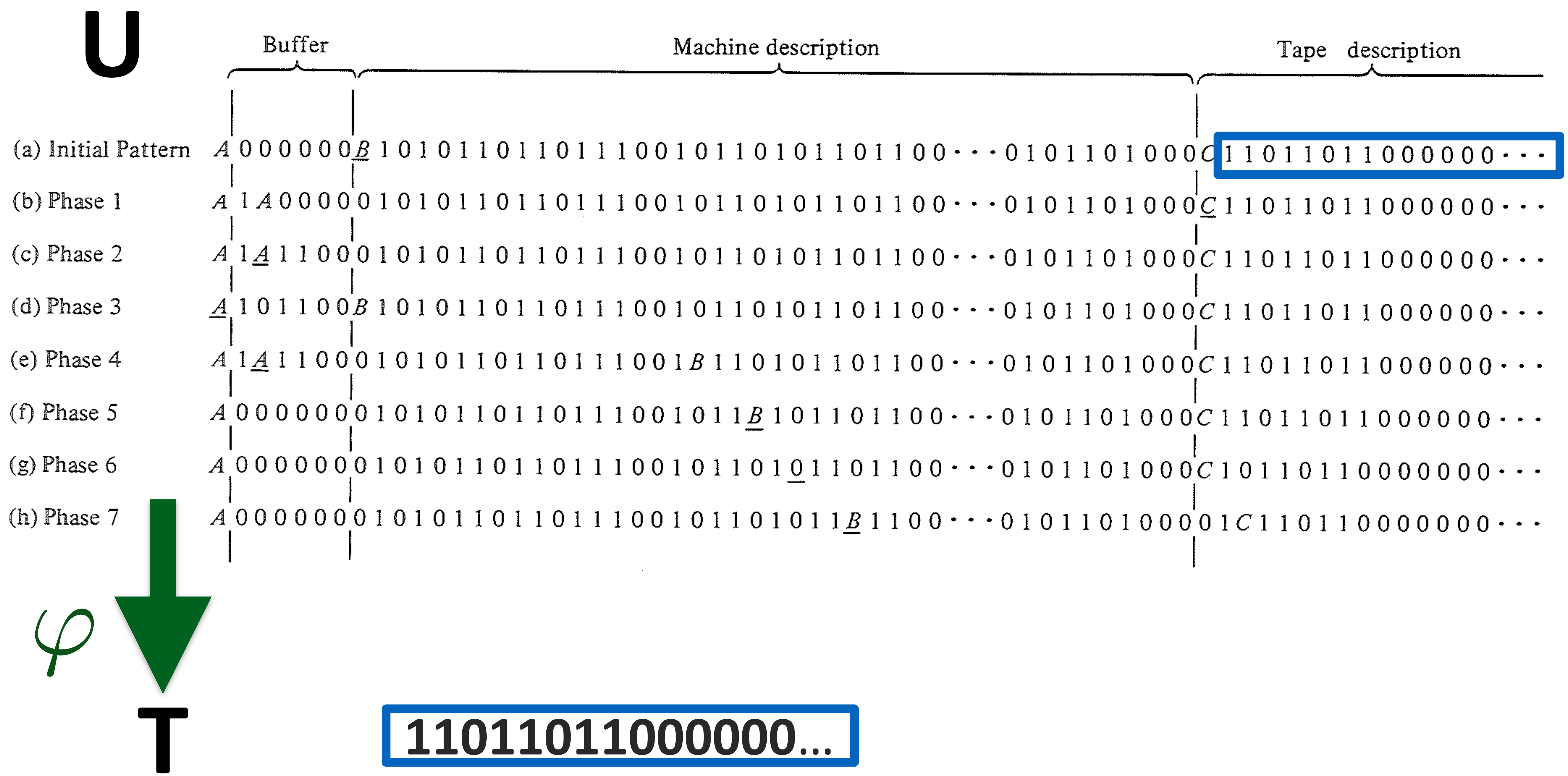
Decoding functions for **standard textbook universal TMs** *are* structure-preserving in this sense!

→

Hennie's *U* and structure-preserving decoding



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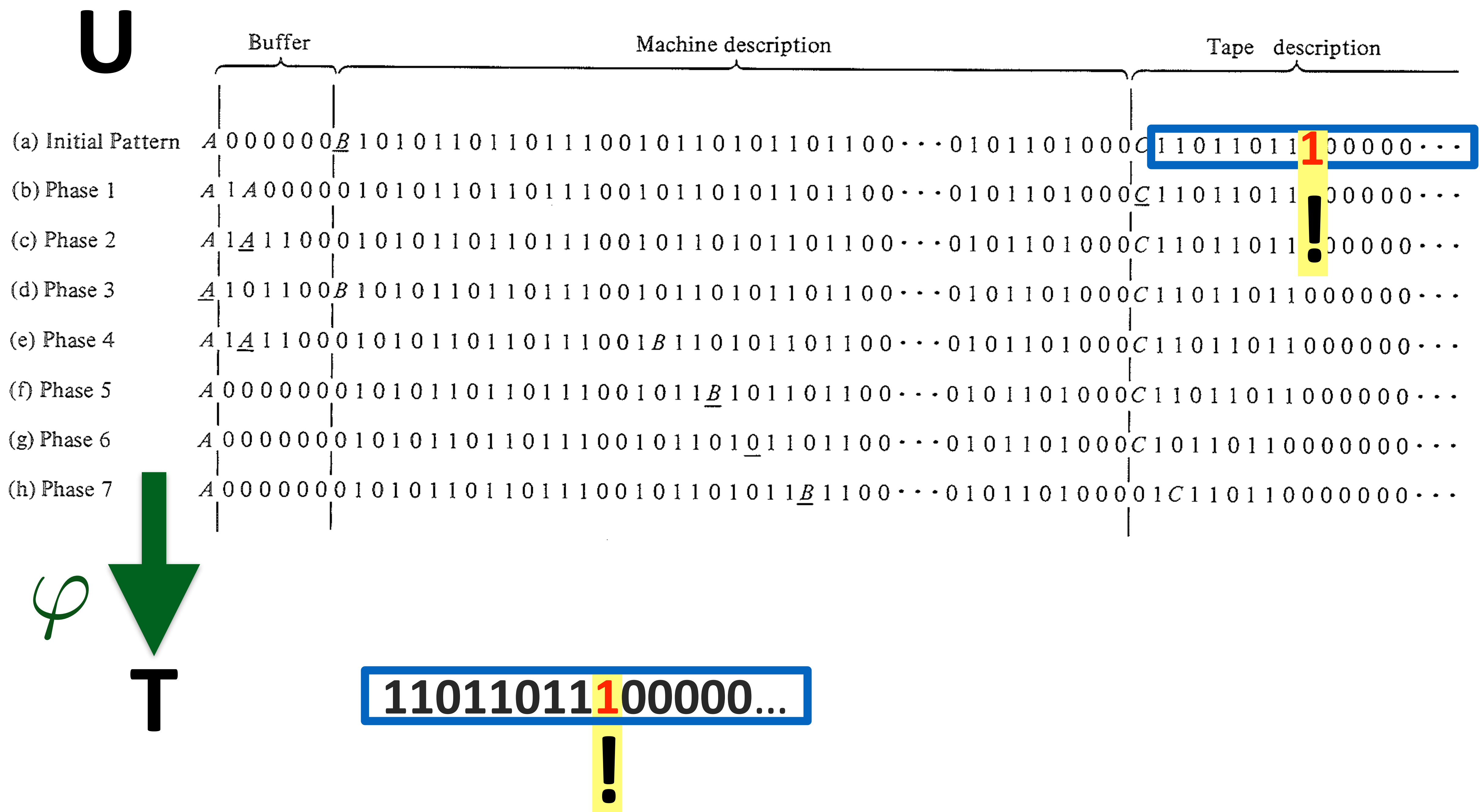

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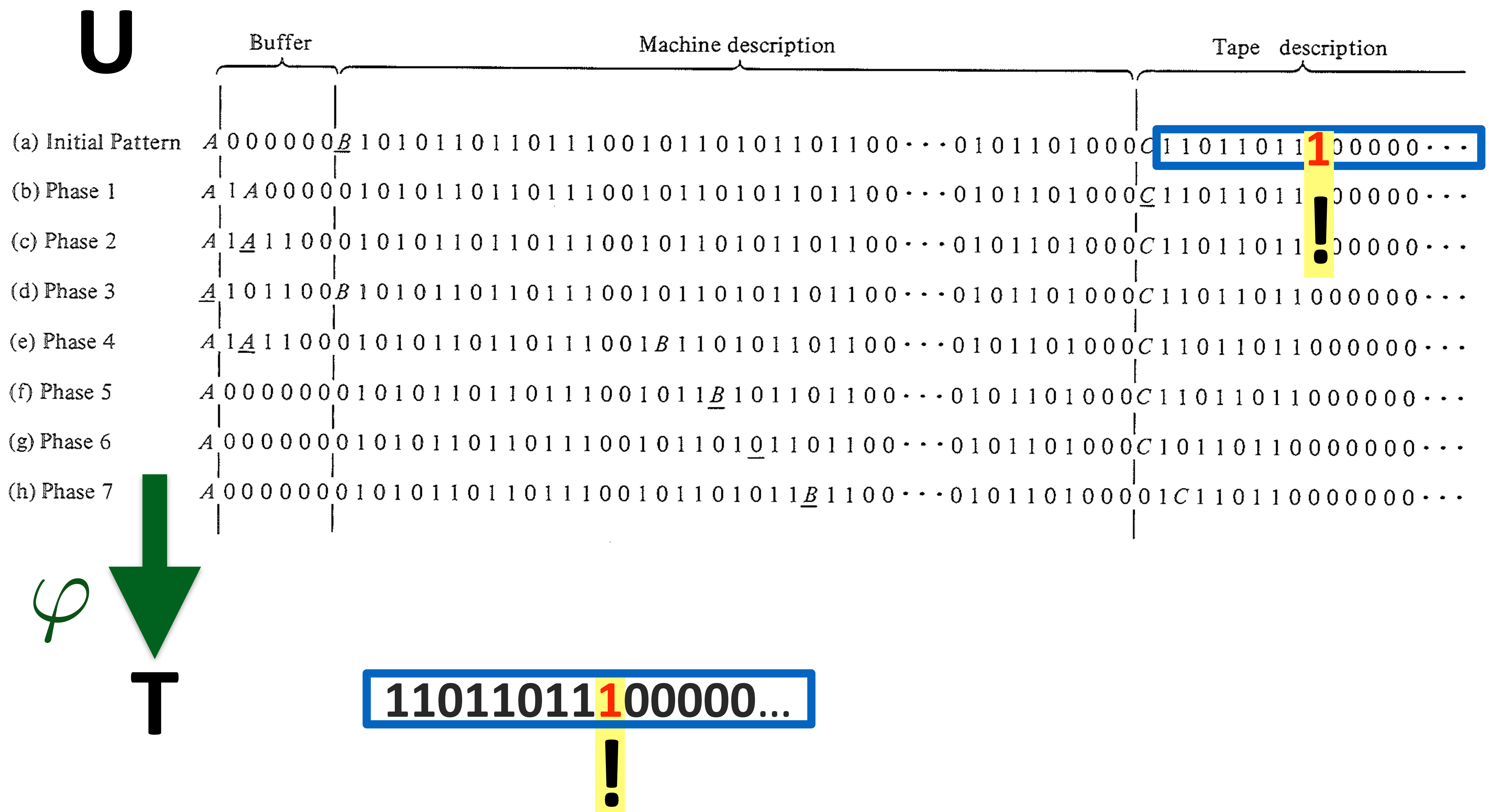

For these two configurations c_T, c'_T with $\mathcal{D}(c_T, c'_T) = 1$,

we find C_U, C'_U with $\varphi(C_U) = c_T, \varphi(C'_U) = c'_T$ and

$$\mathcal{D}(C_U, C'_U) = 1.$$

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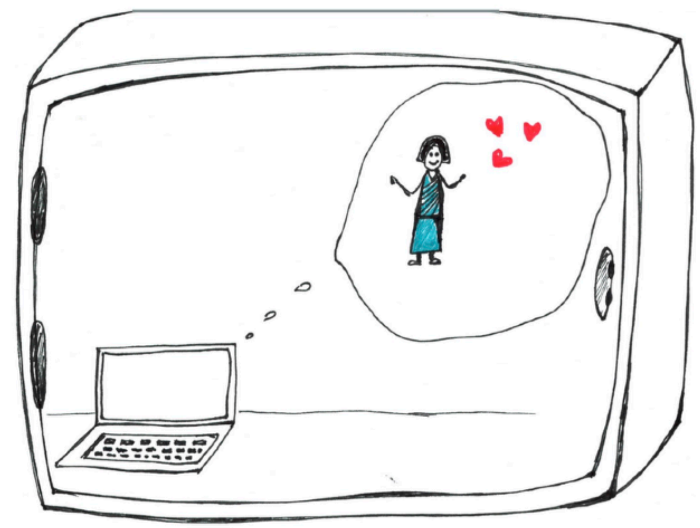
Amended conjecture. For every universal TM *U*, we have

$$T \preceq_{\mathcal{S}} U(p_T \bullet)$$

for an infinite and diverse set of TMs *T*, where $\mathcal{S}$ is a natural set of structure-preserving functions on TM configurations.

Conclusions

- Wolfram's **Computational Irreducibility** and apparent free will
- **Computational sourcehood** as an attempt at defining an aspect of *actual* free will. Motivation: thought experiment of John the cook

- Conjecture on **universal TMs**: they must typically simulate step-by-step.
- **Attempt at formalization** via ~~simple~~ structure-preserving functions

M. Krumm and M. P. Müller, **arXiv:2101.12033**
(To be updated soon)

Thank you!