Operational interpretation of entropy and free energy without the thermodynamic limit

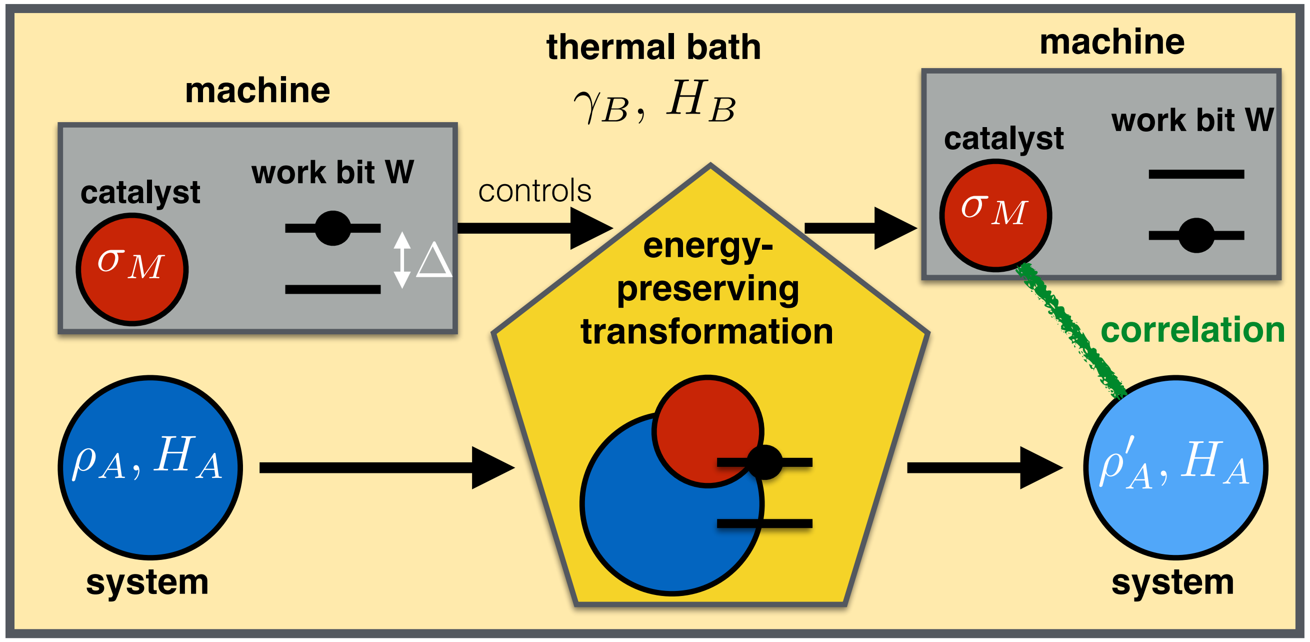
Markus P. Müller[1,2,4], Matteo Lostaglio[3],
Michele Pastena[4], and Jakob Scharlau[4]

[1] Institute for Quantum Optics and Quantum Information, Vienna
[2] Perimeter Institute for Theoretical Physics, Waterloo, Canada
[3] ICFO - Institut de Ciencies Fotoniques, Barcelona, Spain
[4] Department of Theoretical Physics, University of Heidelberg, Germany

Outline

1. Standard view: thermodynamic limit

2. Thermodynamics as a resource theory



3. A new one-shot interpretation of free energy

4. Similar results in quantum information?

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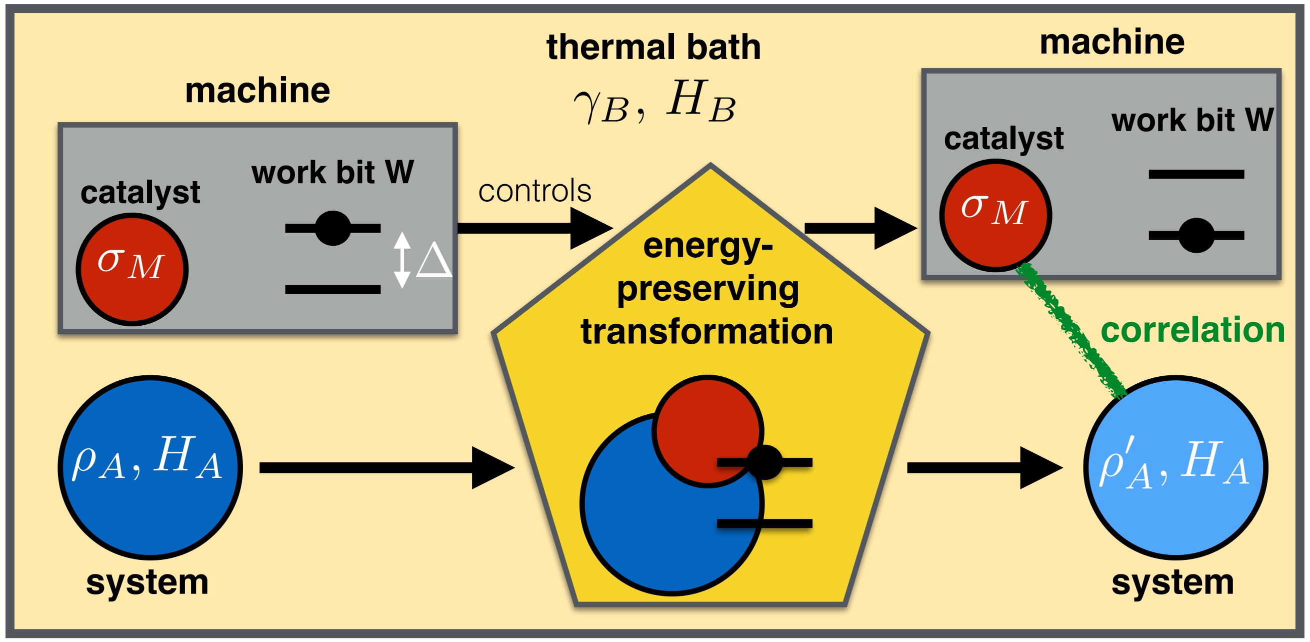

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Folklore: spontaneous processes have

$$\Delta F \leq 0 \qquad \text{(2nd law)},$$

where $F = U - TS$.
If this is negative, then we can extract $|\Delta F|$ of work from the system.

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But this is a statement **on average**, since "work" is a random variable.

$$e^{-\Delta F/k_B T} = \langle e^{-W/k_B T} \rangle \;\Rightarrow\; \Delta F \leq \langle W \rangle.$$

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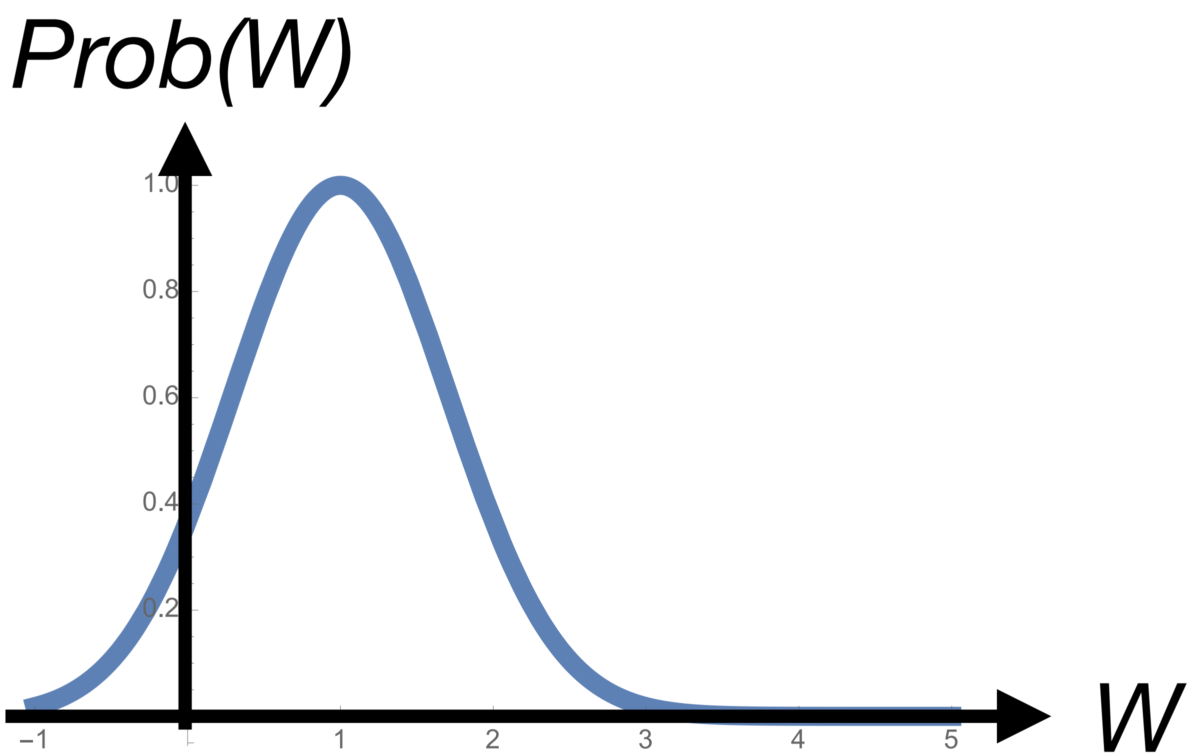
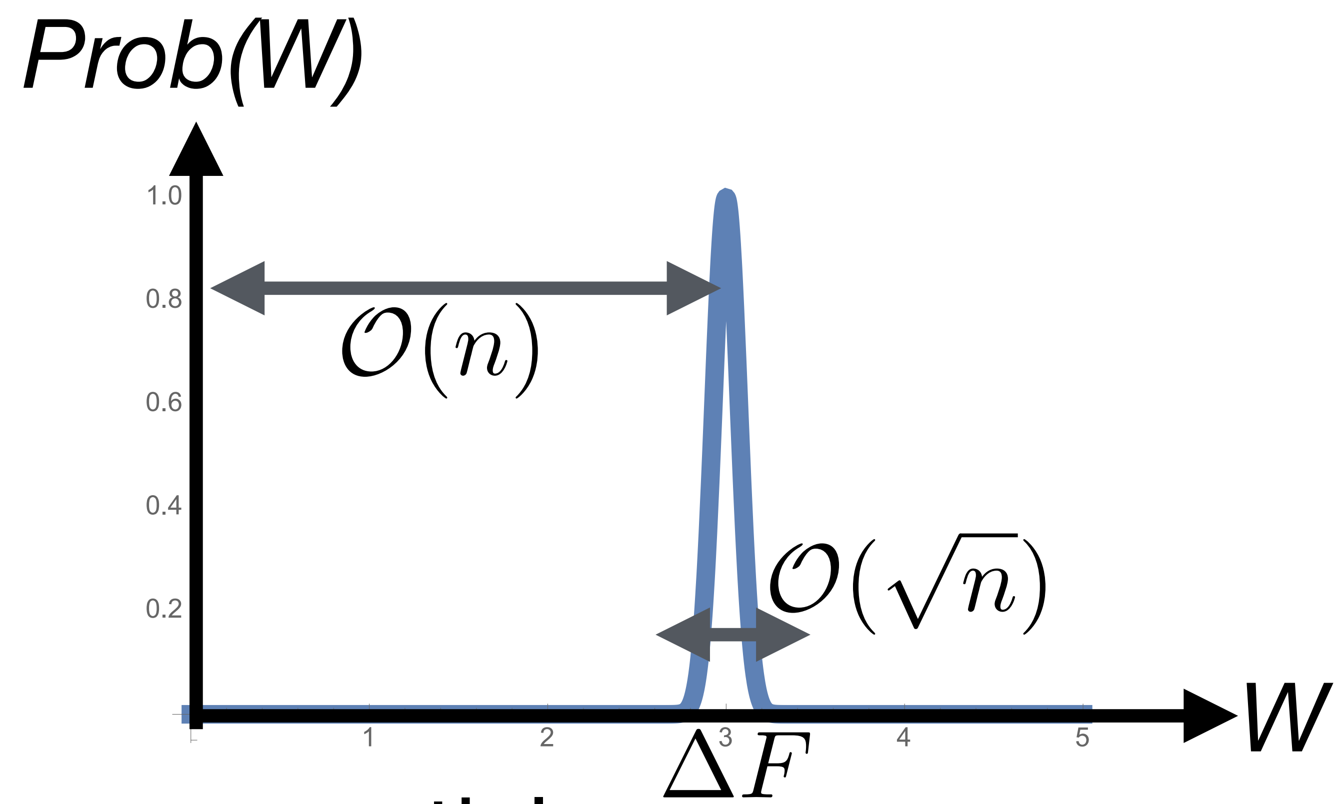


small # of particles



n particles

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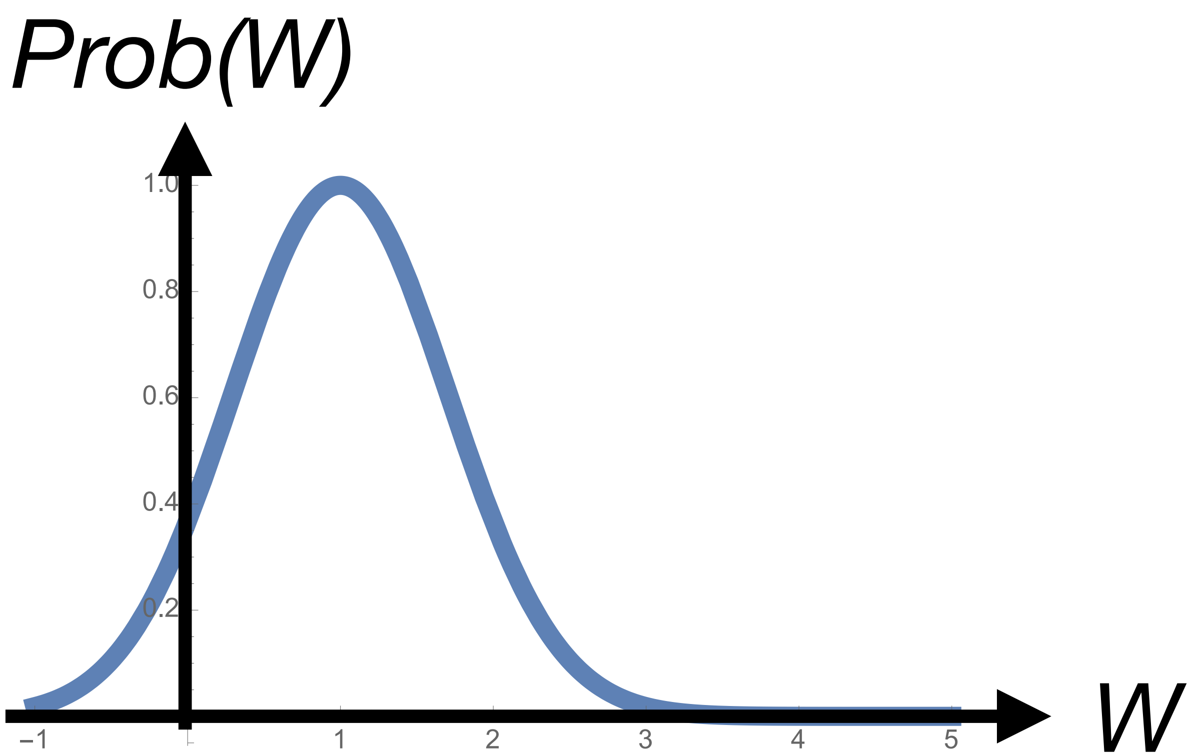

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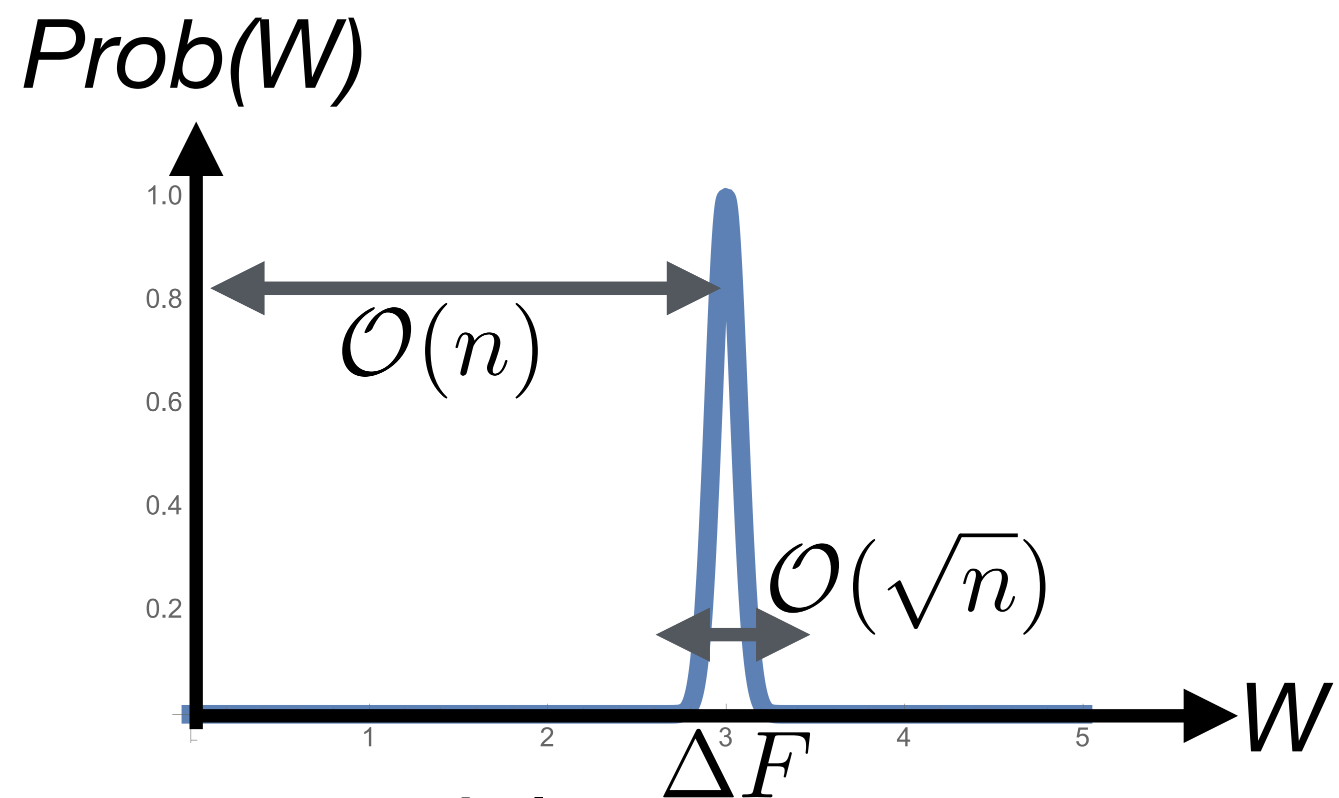

n particles

Extractable work “is” (optimally) ΔF :
only true in the thermodynamic limit $n \to \infty$
when fluctuations become irrelevant (law of large numbers).

Standard view: thermodynamic limit

But what do we do for "small" (quantum?) or strongly correlated systems? Work $\approx$ its fluctuations ⟶ reliability?

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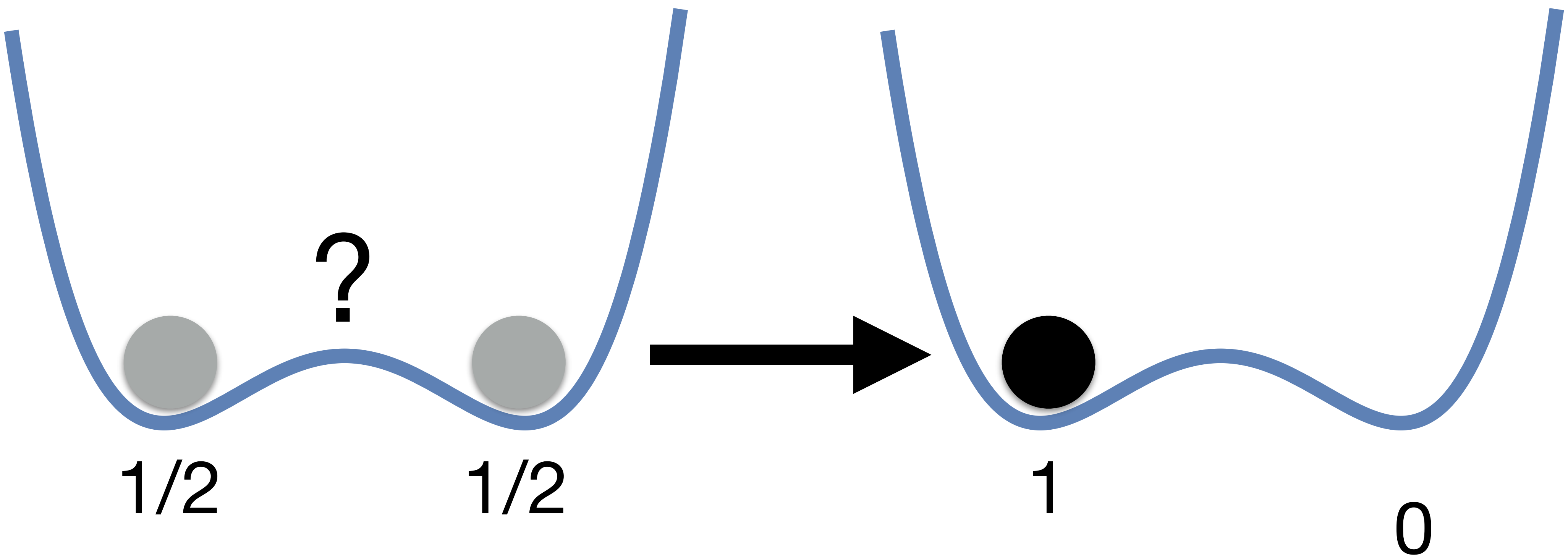
Landauer erasure:



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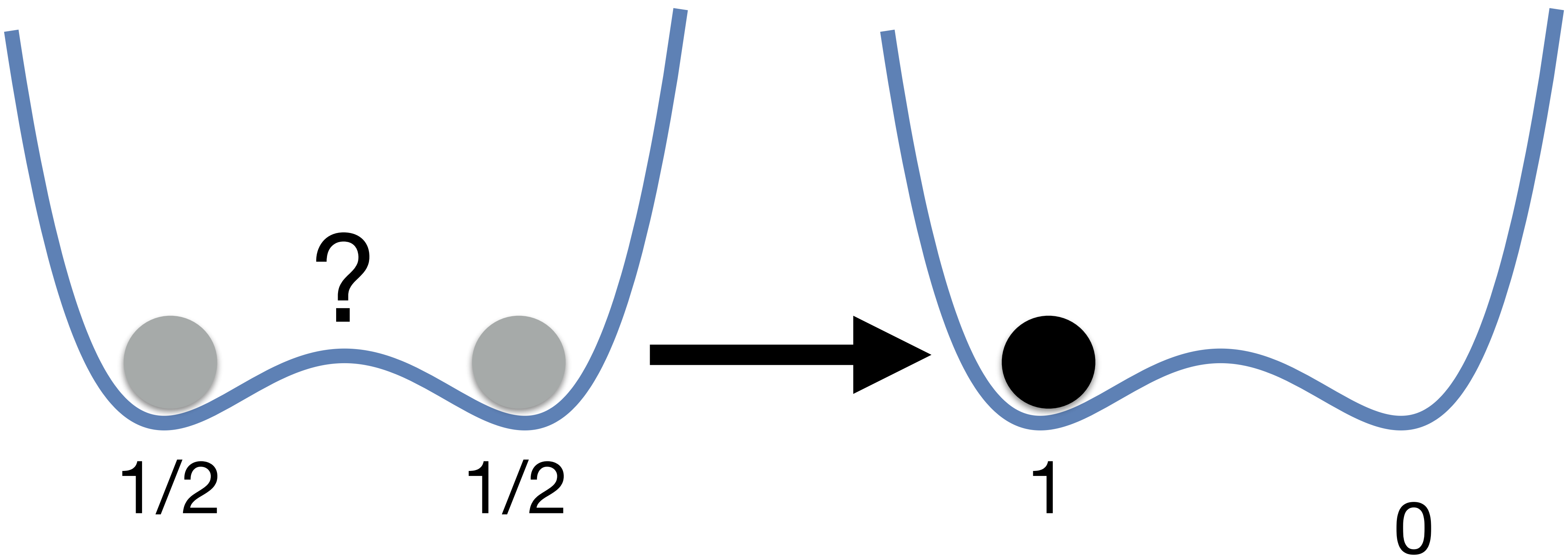

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Landauer

Free energy *F* determines possibility of state transitions **only in the thermodynamic limit.** For "small" systems, **resource theory formulation** will give more stringent constraints (and solve Bennett's puzzle). More soon.

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Schumacher compression:

Given n copies of a quantum state ρ,
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such that

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S determines compression rate **in the limit** $n \to \infty$.

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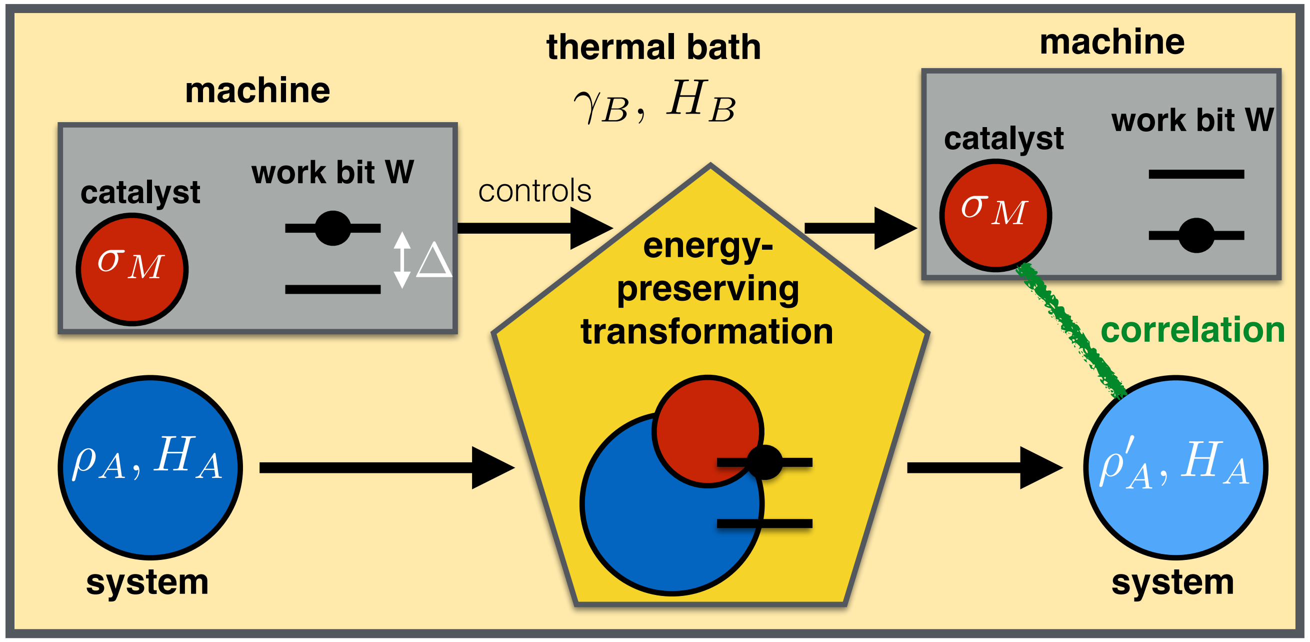

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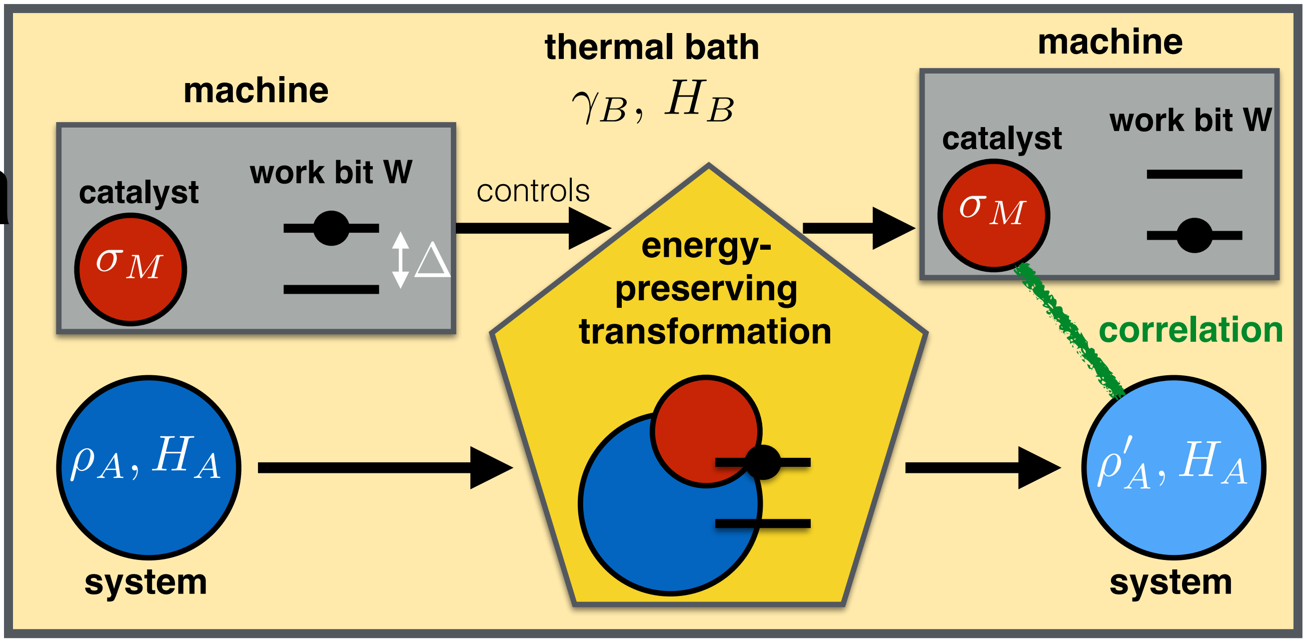

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Thermodynamics as a resource theory

The second laws of quantum thermodynamics

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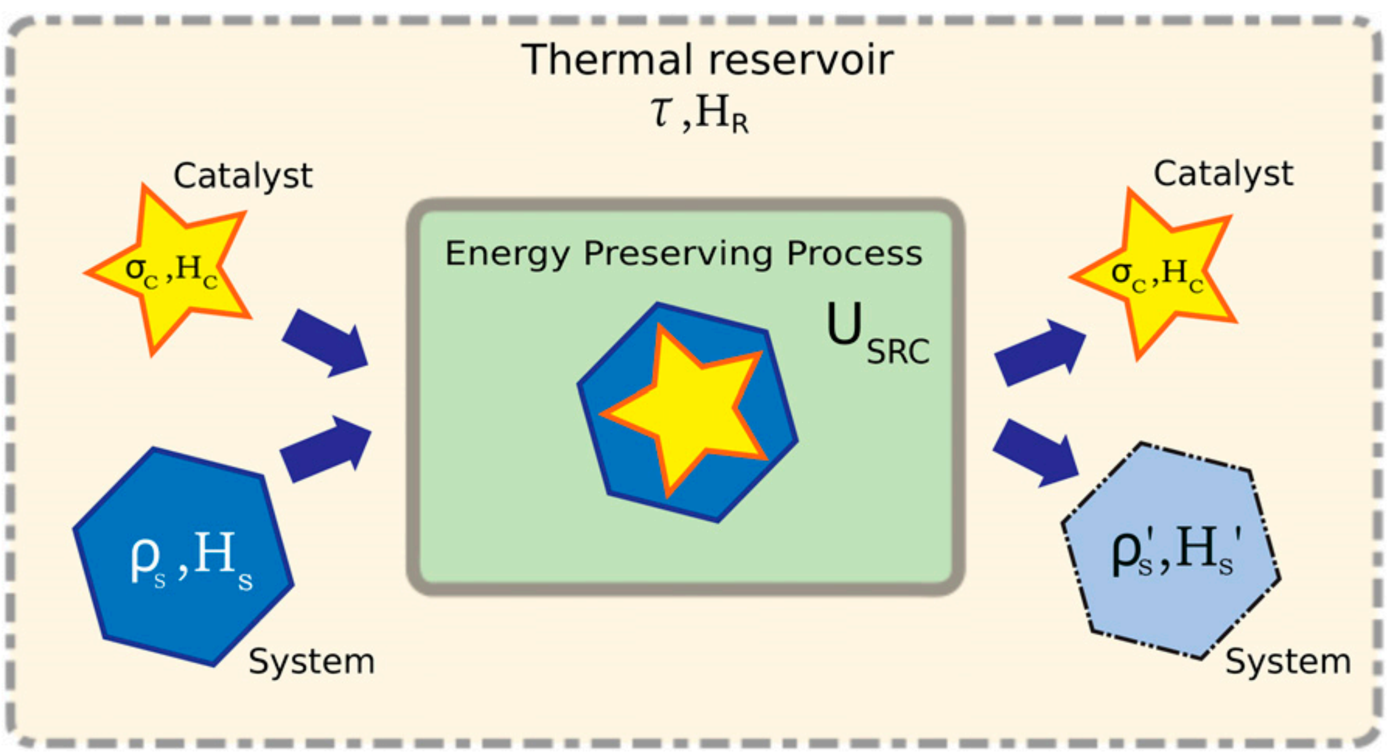

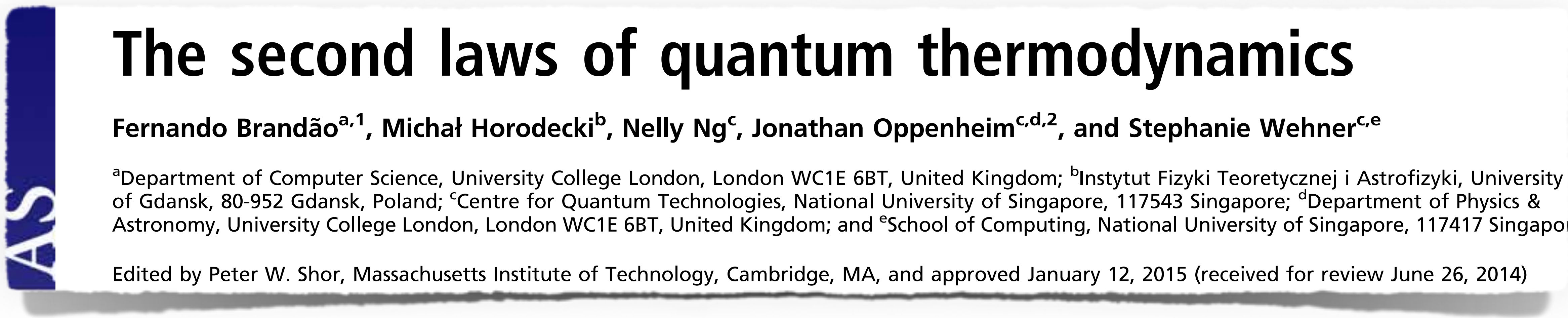

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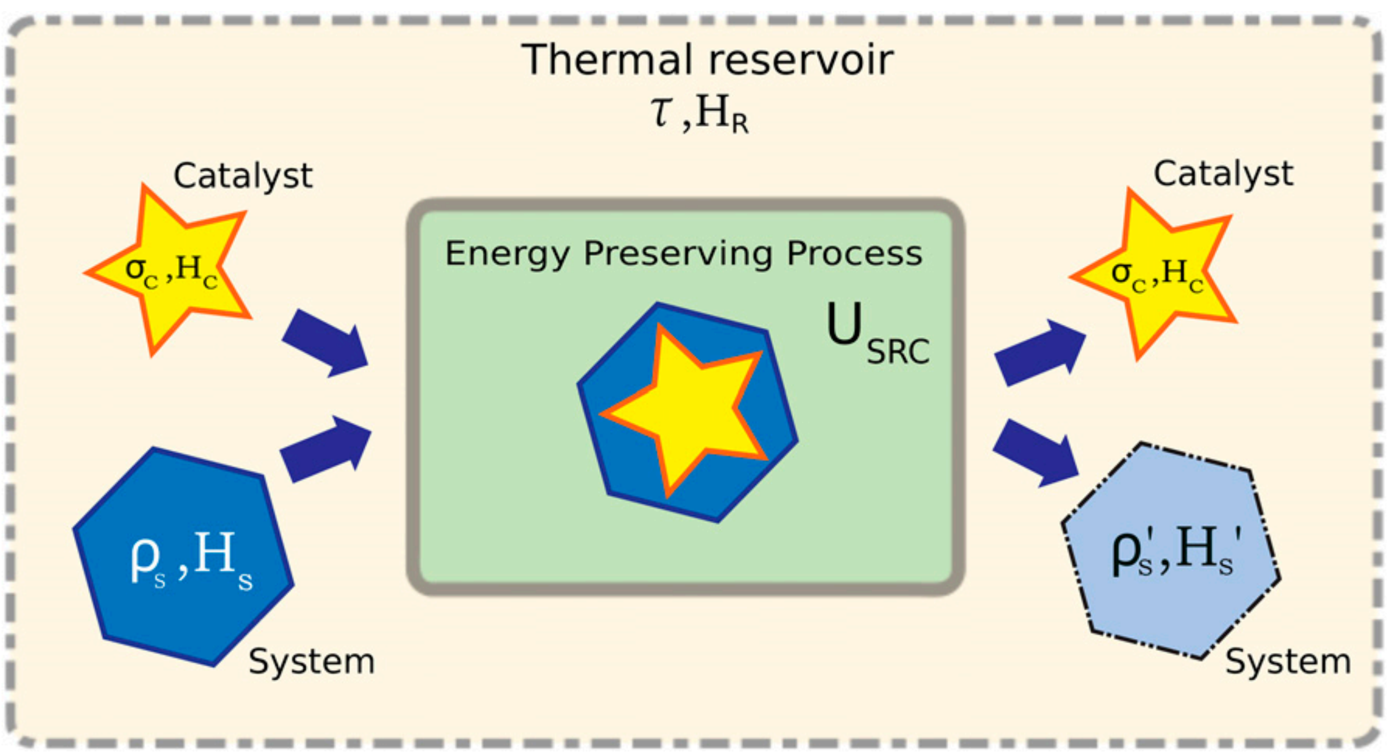

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$$=\rho'_S\otimes\sigma_C$$

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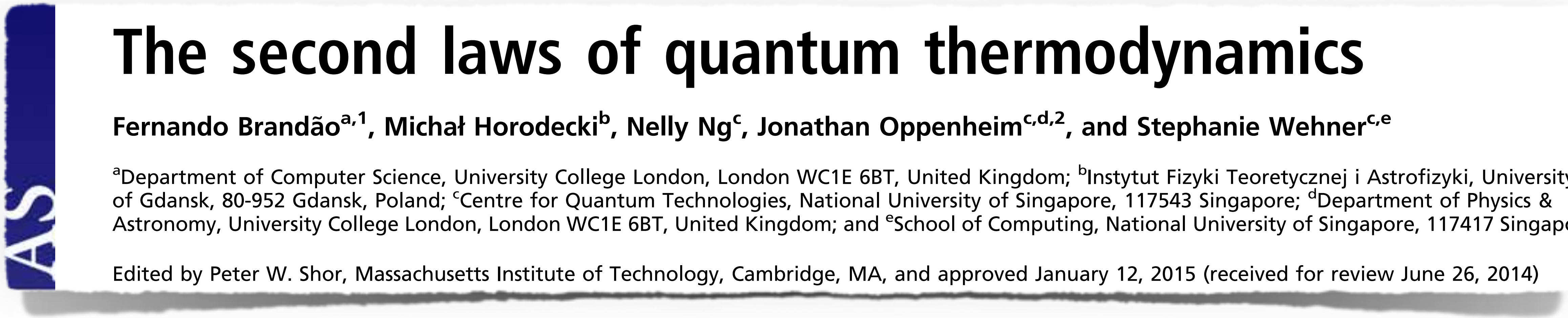

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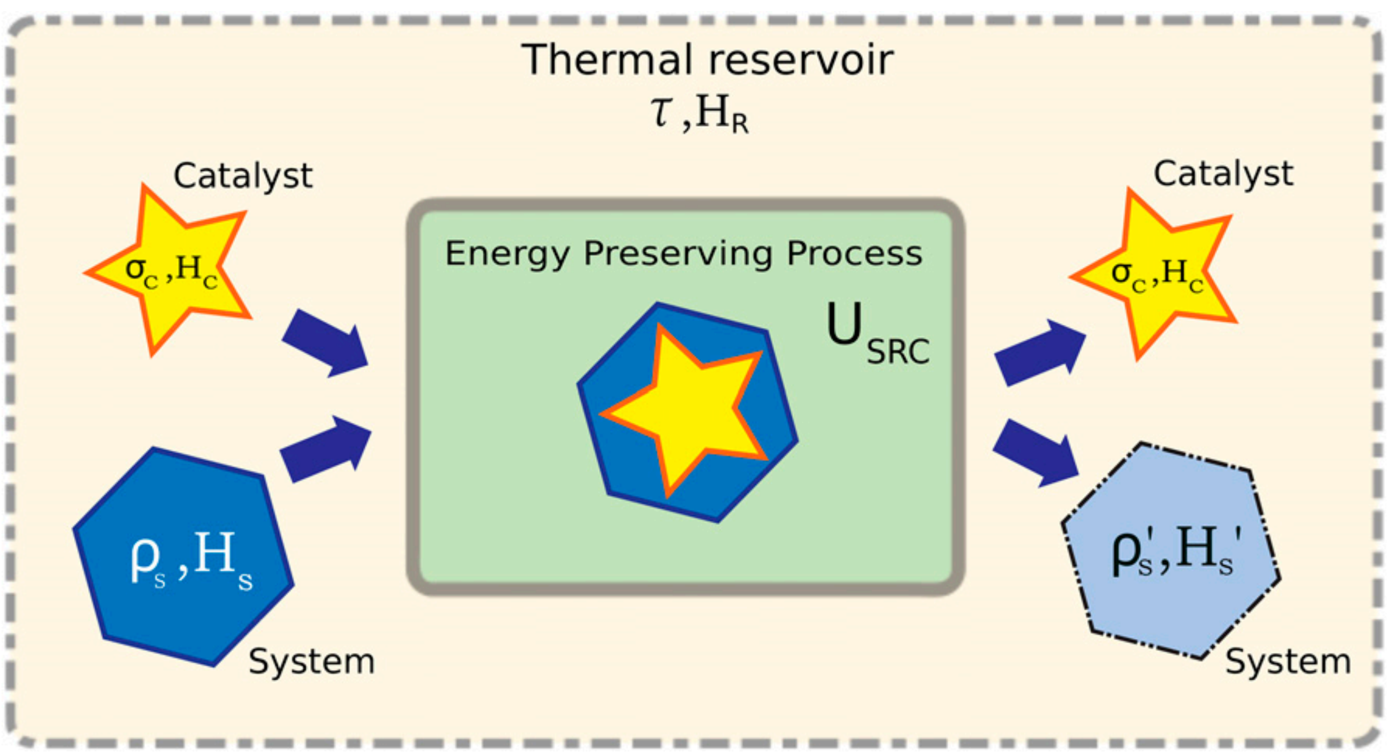

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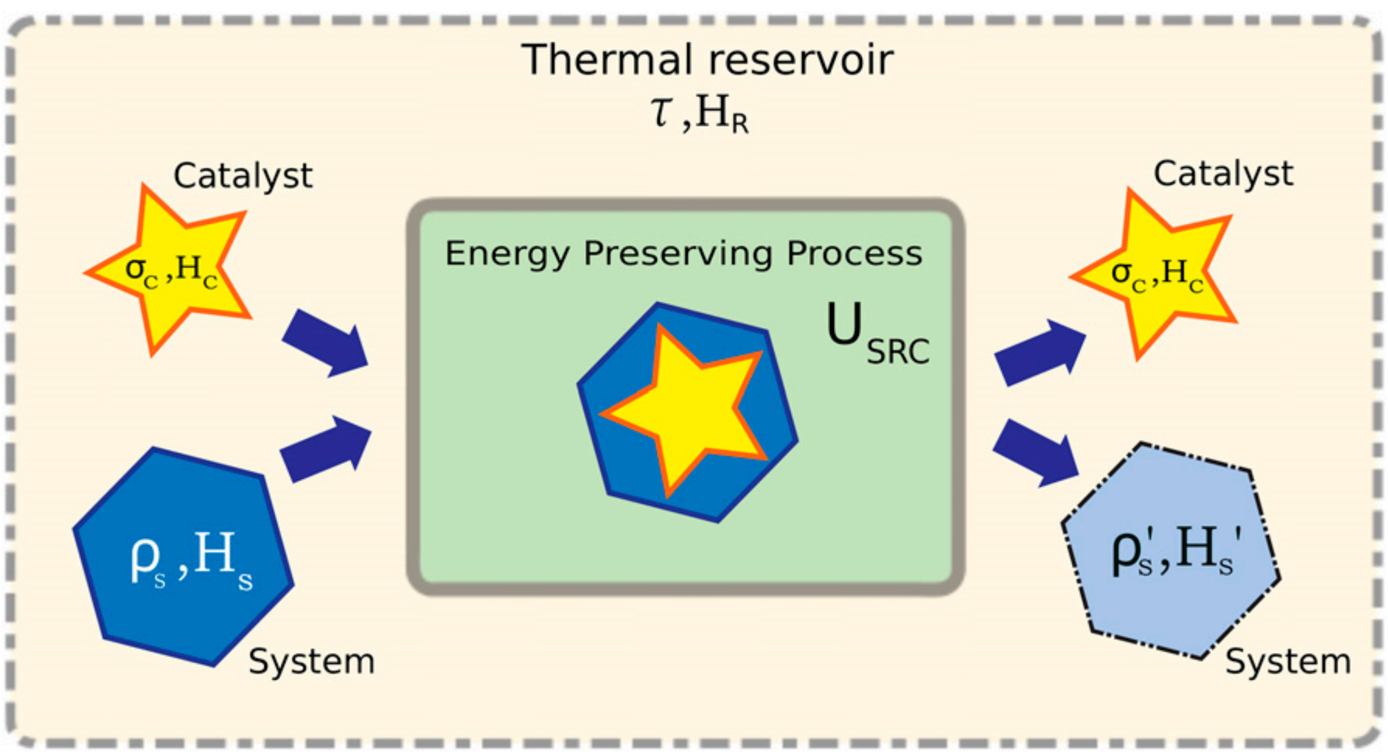

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thermal reservoir

$$[U_{SRC},H_S+H_R+H_C]=0$$

(energy strictly preserved)

The rules of the game:

- It is "free" to bring in any system *B* in its thermal state $\gamma_B = \exp(-H_B/(k_B T))$,
- strictly energy-preserving unitaries are free,
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Mathematically completely rigorous.
Also, nice insights: if any non-thermal state was free then the resource theory would be trivial (**all** transitions possible).

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$$F(\rho_A) \equiv F_1(\rho) = \mathrm{tr}(\rho_A H_A) - k_B T S(\rho_A),$$

$$F_\alpha(\rho) = k_B T S_\alpha(\rho\|\gamma) + F_\alpha(\gamma).$$

Rényi divergence

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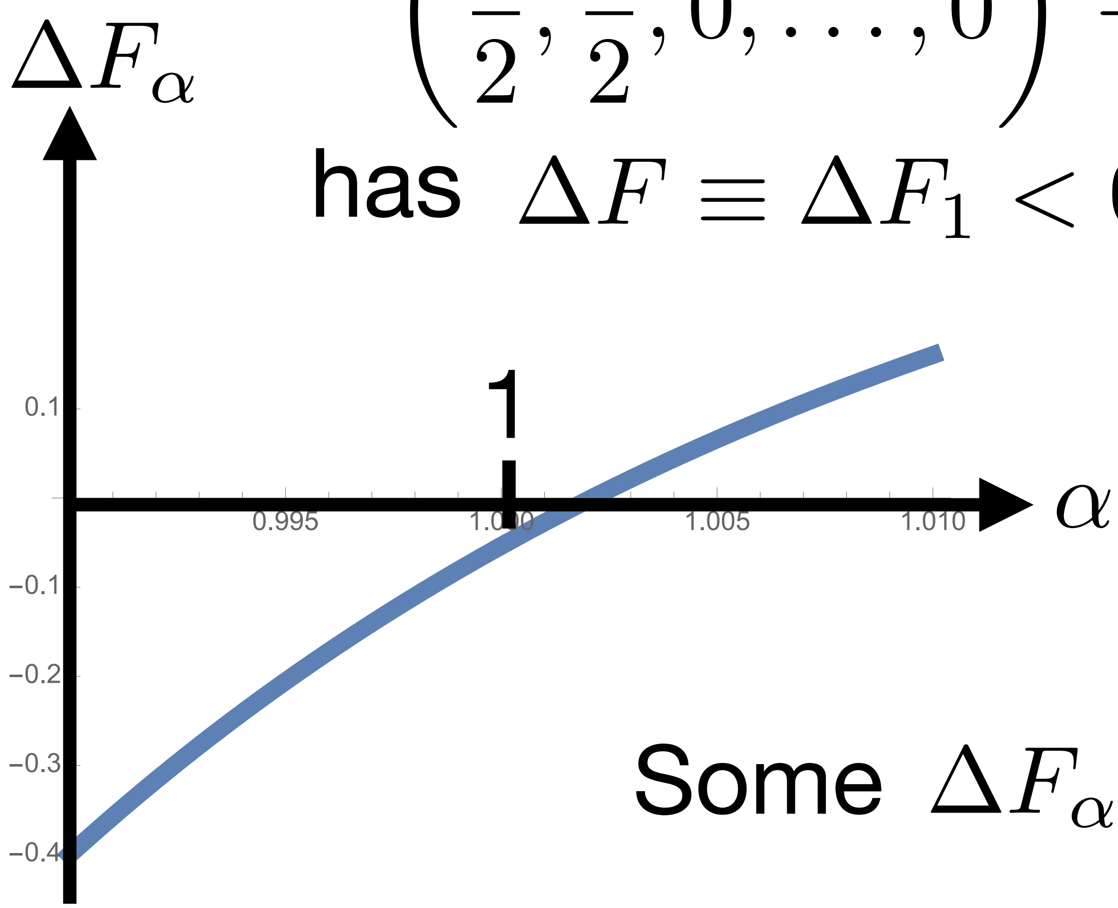

$$\epsilon = \frac{1}{100}, \quad N = 10^{30}.$$

Some $\Delta F_\alpha > 0$ hence impossible.

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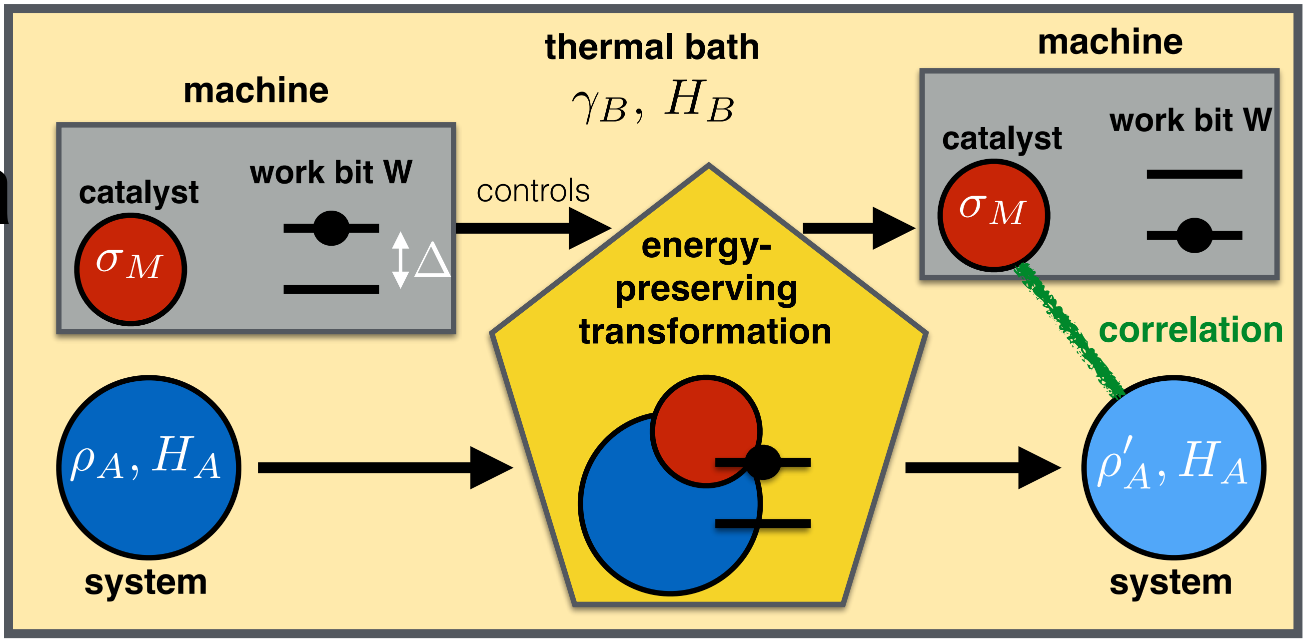

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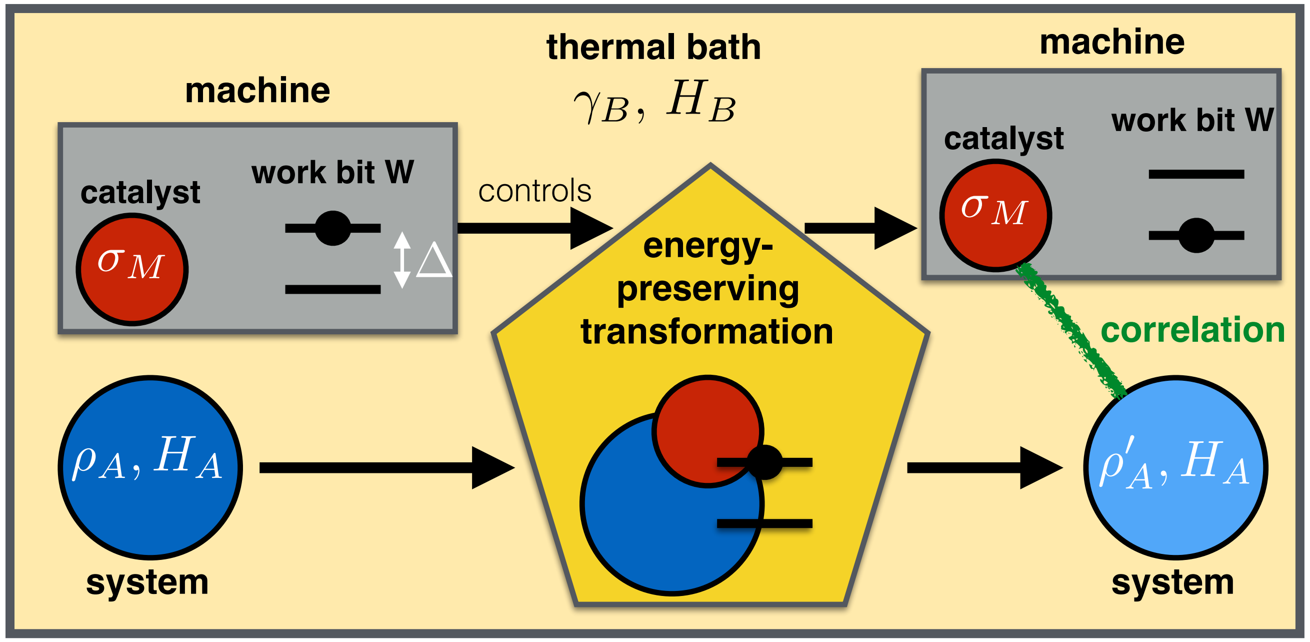

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MM, *Correlating thermal machines and the second law at the nanoscale*, arXiv:1707.03451

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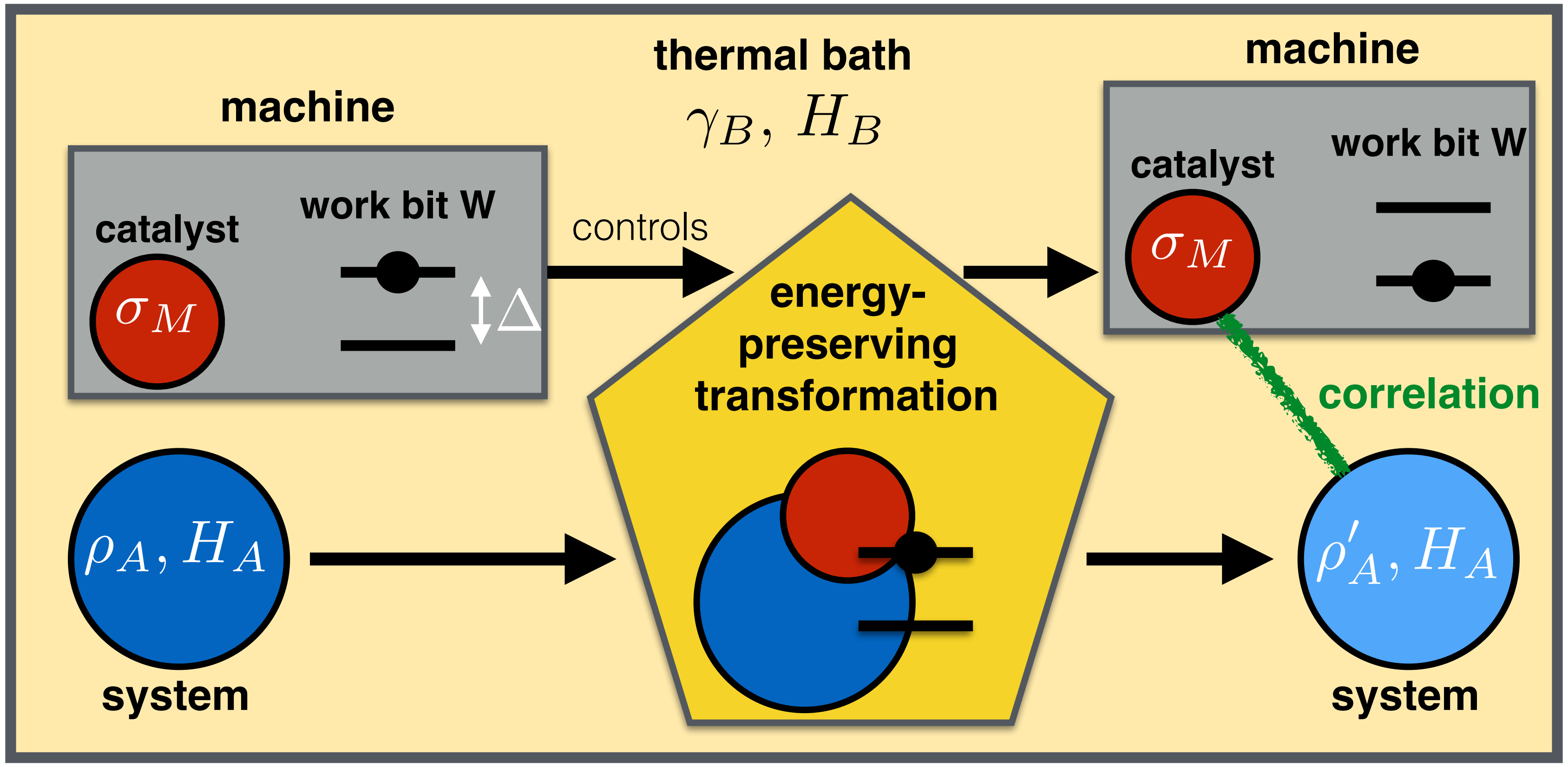

Preserve catalyst exactly, but allow **correlations** to build up.

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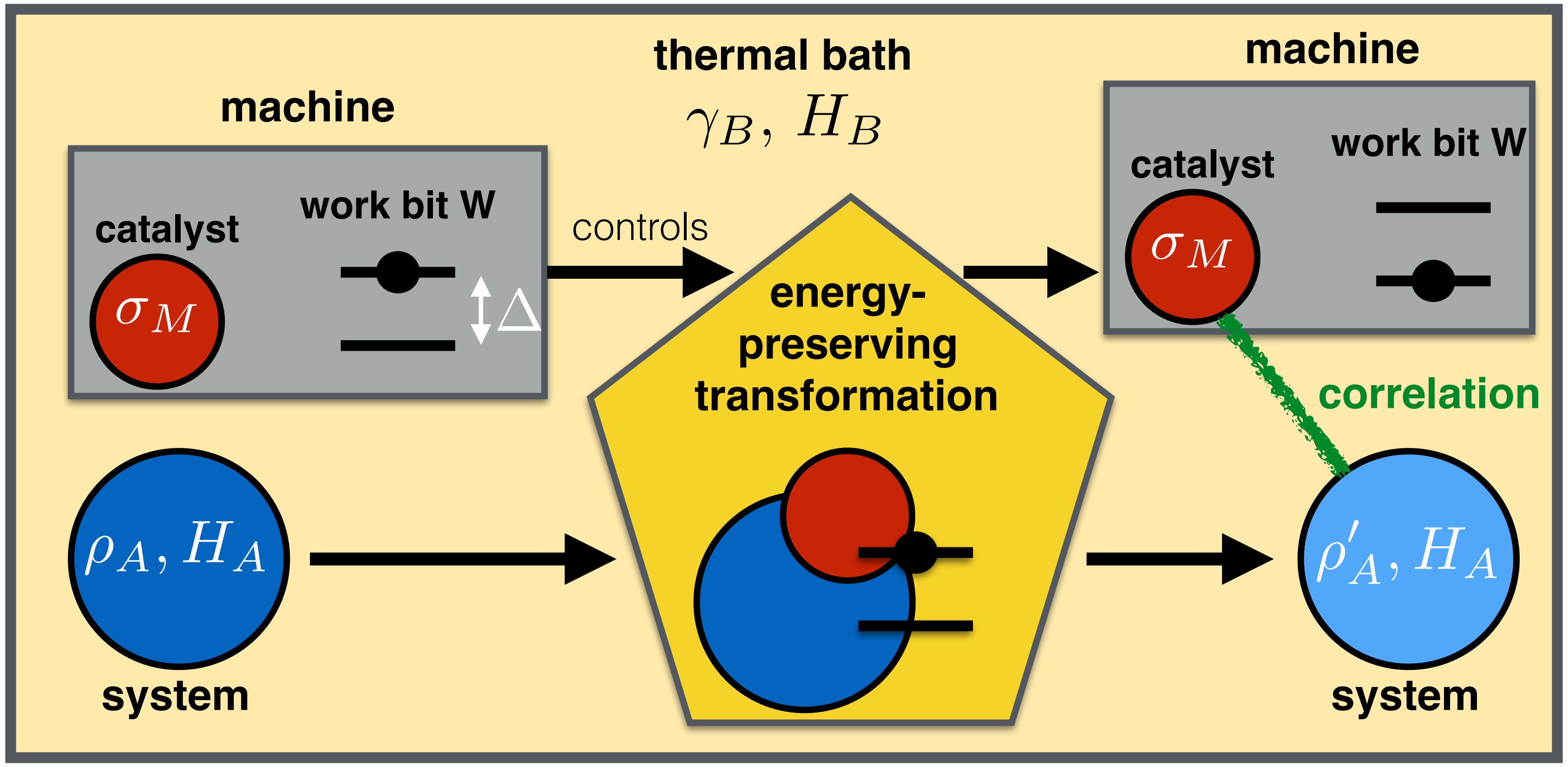
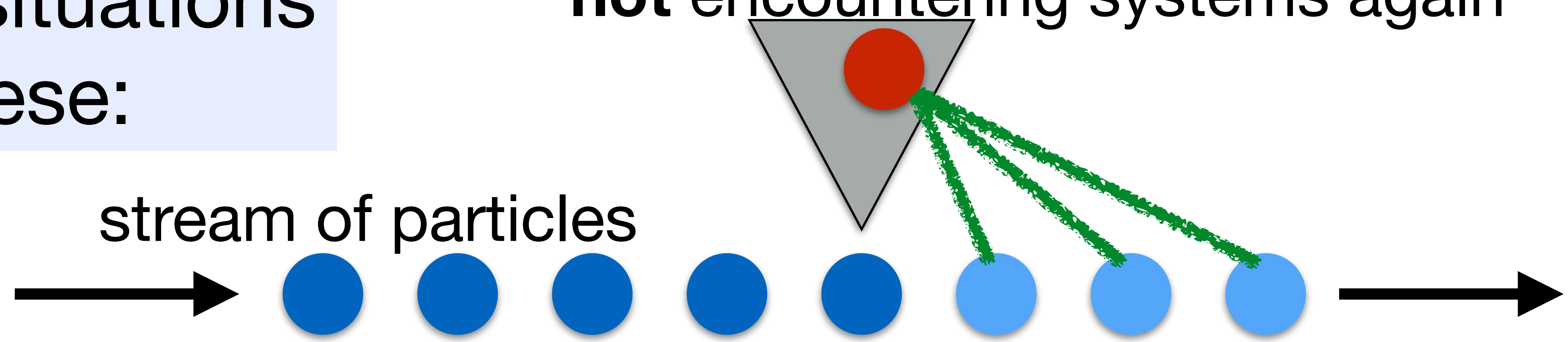


Preserve catalyst exactly, but allow **correlations** to build up.

Applies to situations like these:

thermal machine,
acting single-shot,
not encountering systems again



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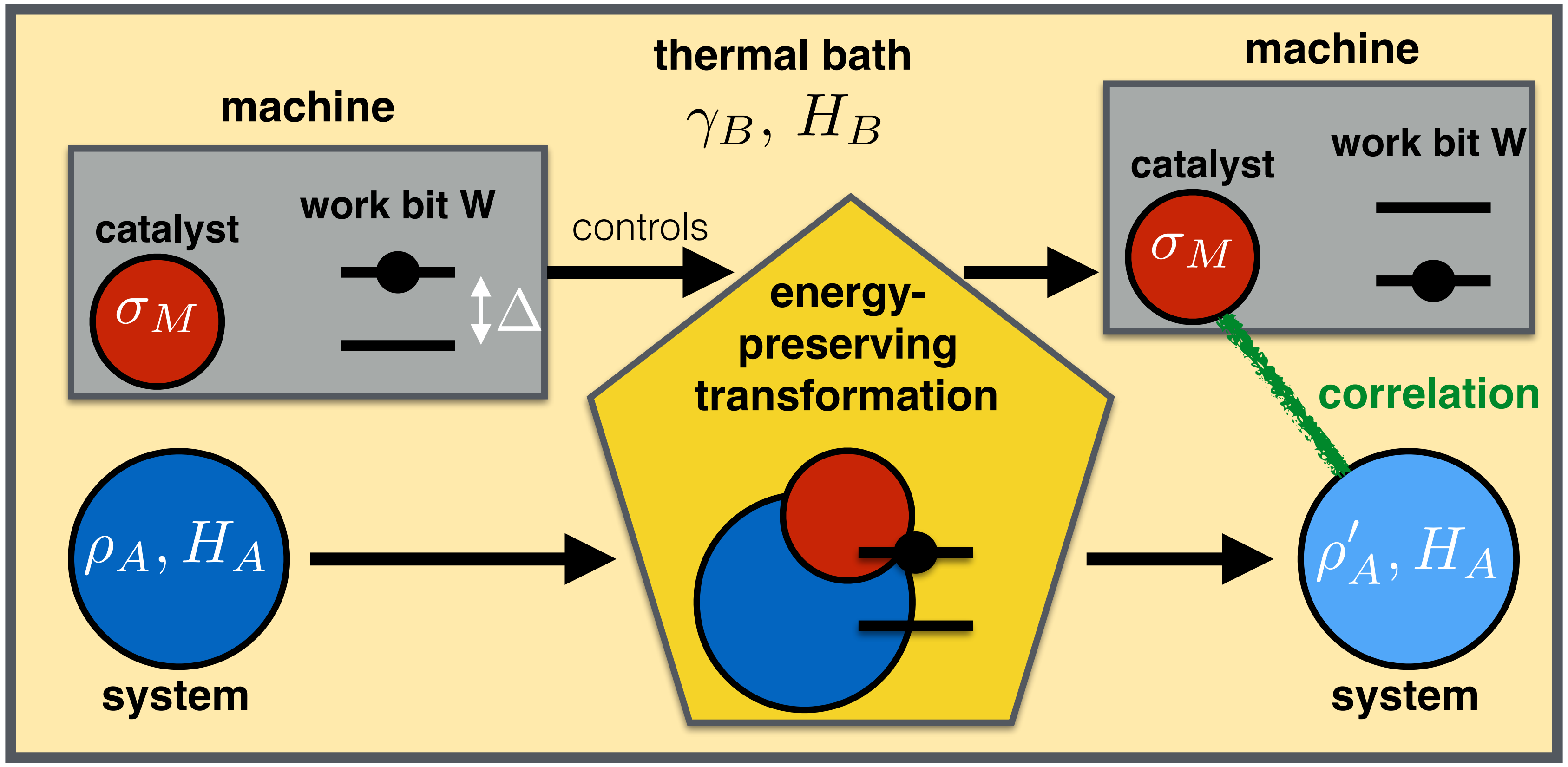

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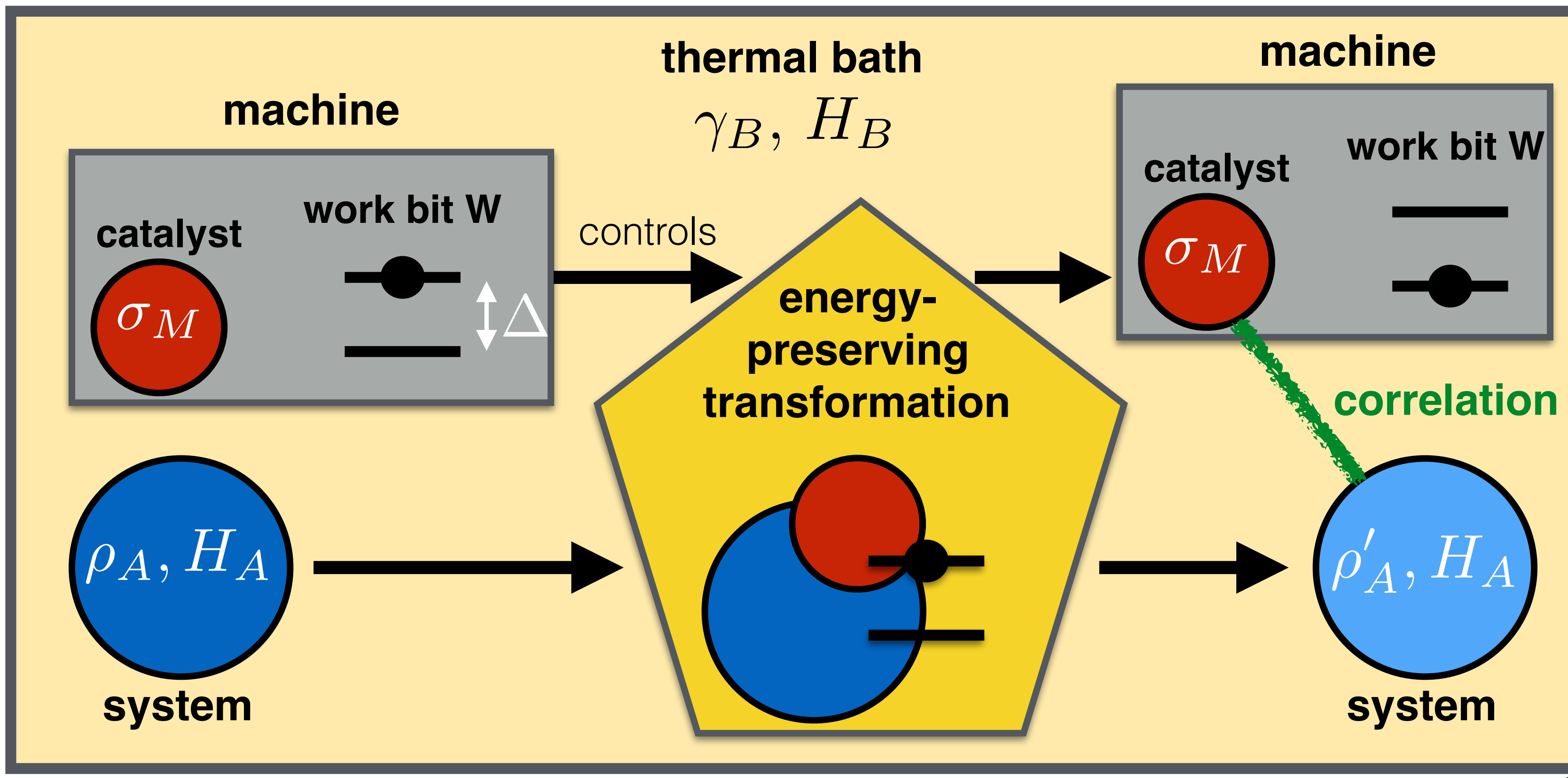

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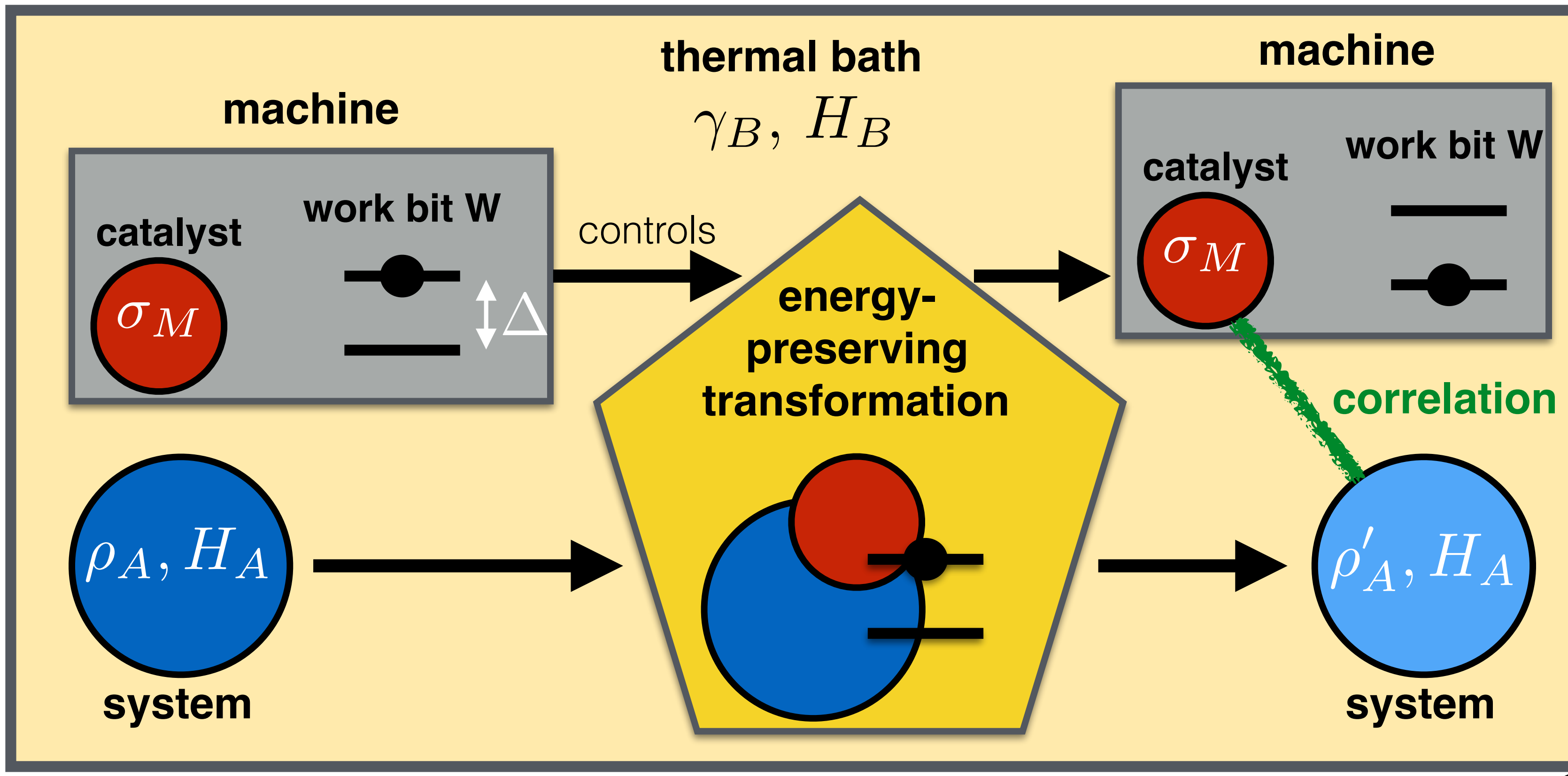

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One-shot operational interpretation of Helmholtz free energy!

Second laws ⟶ second law!

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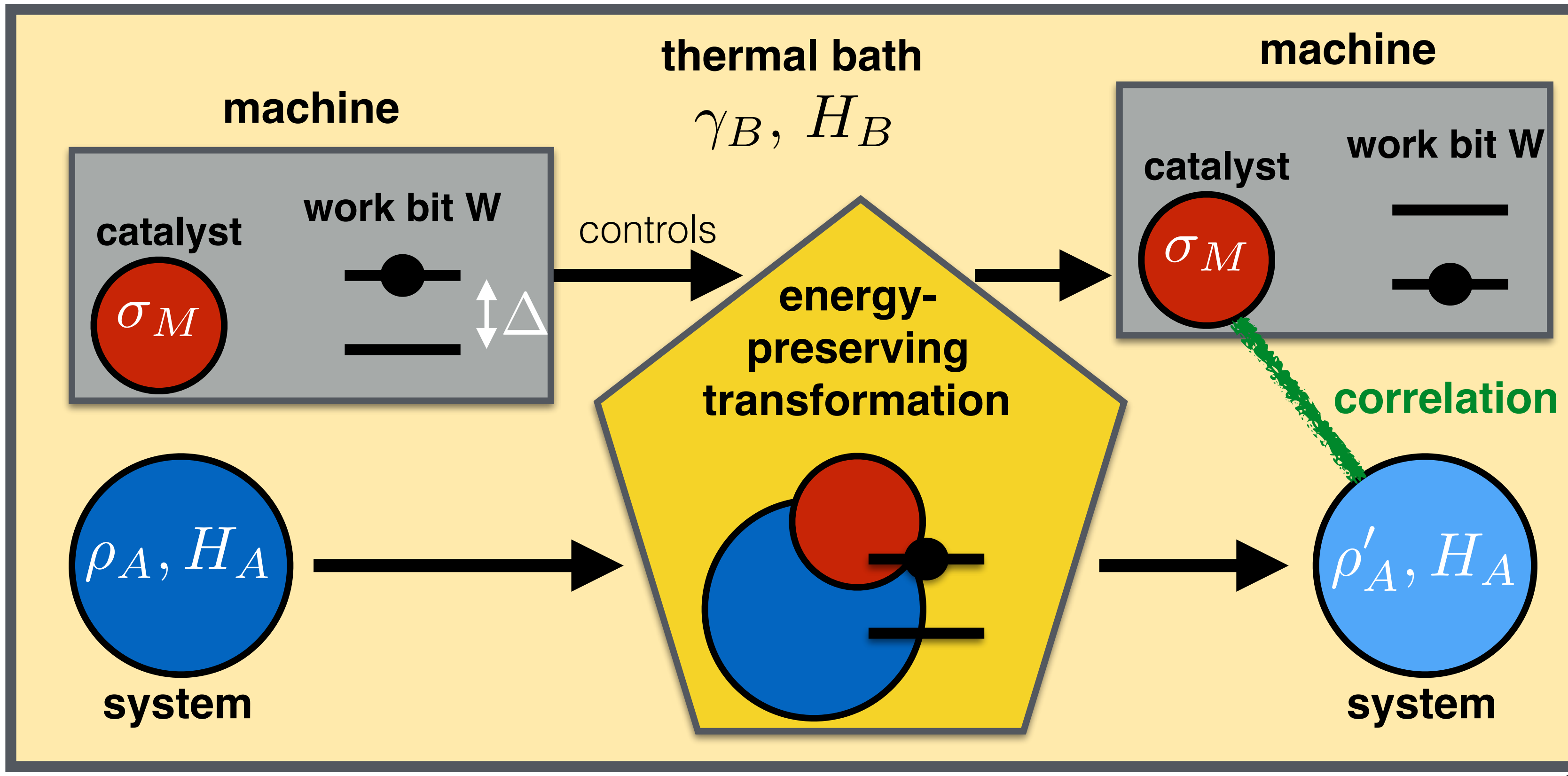

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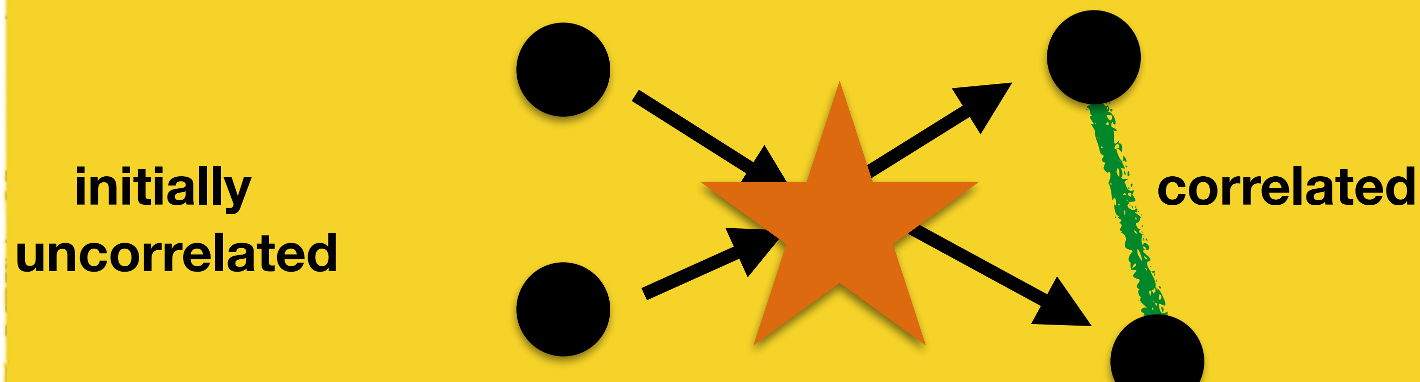
In spirit somewhat close to Boltzmann's original **Stoßzahlansatz** (1896):



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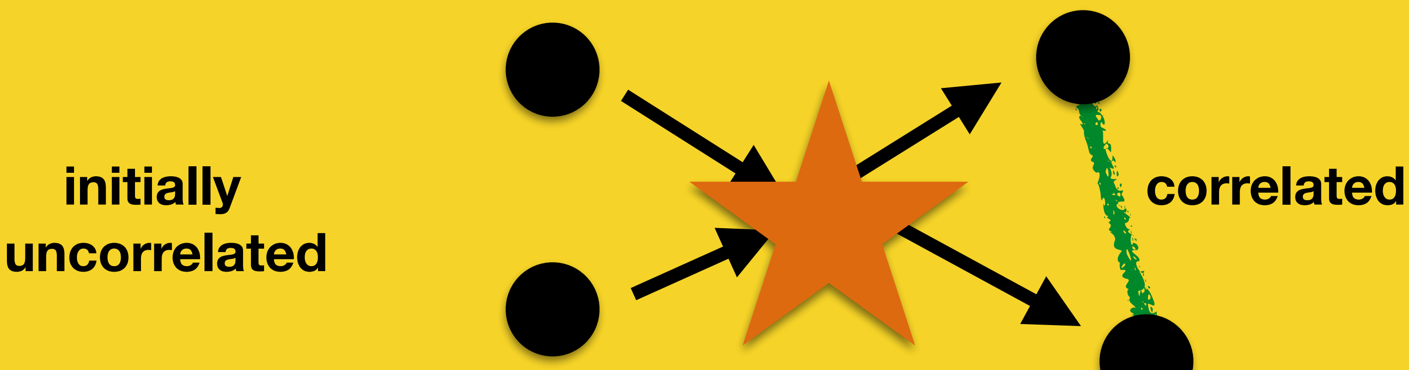
MM, *Correlating thermal machines and the second law at the nanoscale*, arXiv:1707.03451

In spirit somewhat close to Boltzmann's original **Stoßzahlansatz** (1896):



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p.

arbitrary
ccuracy $\varepsilon > 0$.

Theore onal states) iff
involved).

Conjecture. Result also true in the presence of quantum coherence.

One-sh ltz free energy!

Extractable work / work of formation…

MM, *Correlating thermal machines and the second law at the nanoscale*, arXiv:1707.03451

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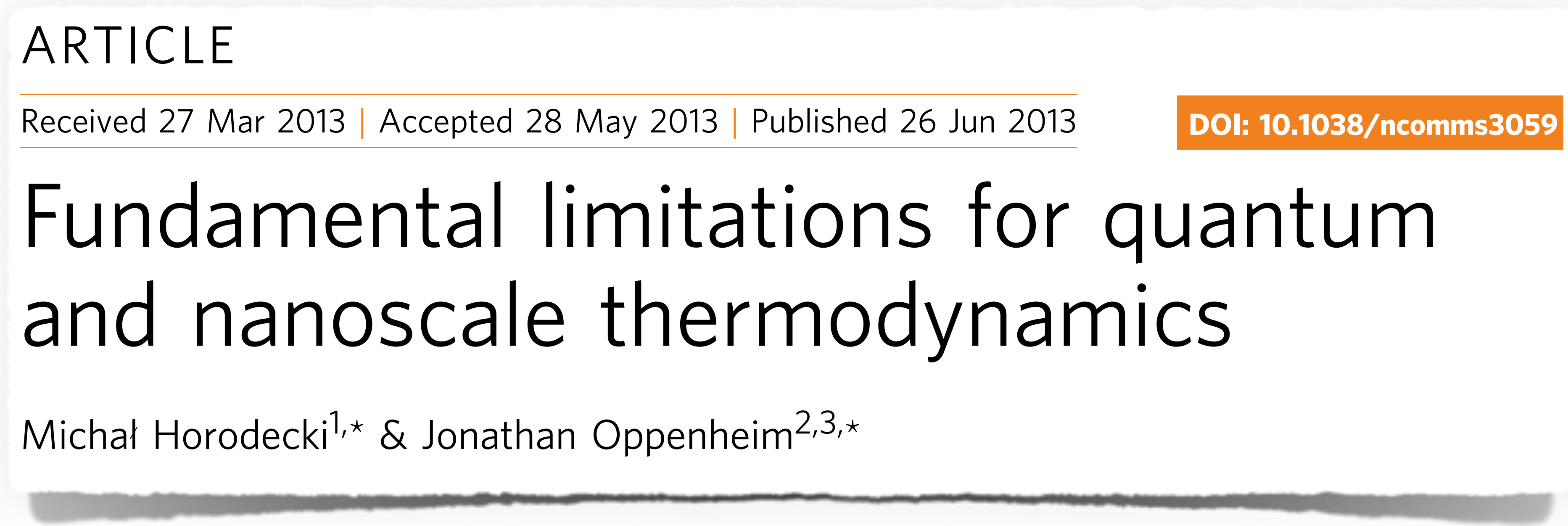
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Fundamental limitations for quantum and nanoscale thermodynamics

Michał Horodecki[1,*] & Jonathan Oppenheim[2,3,*]

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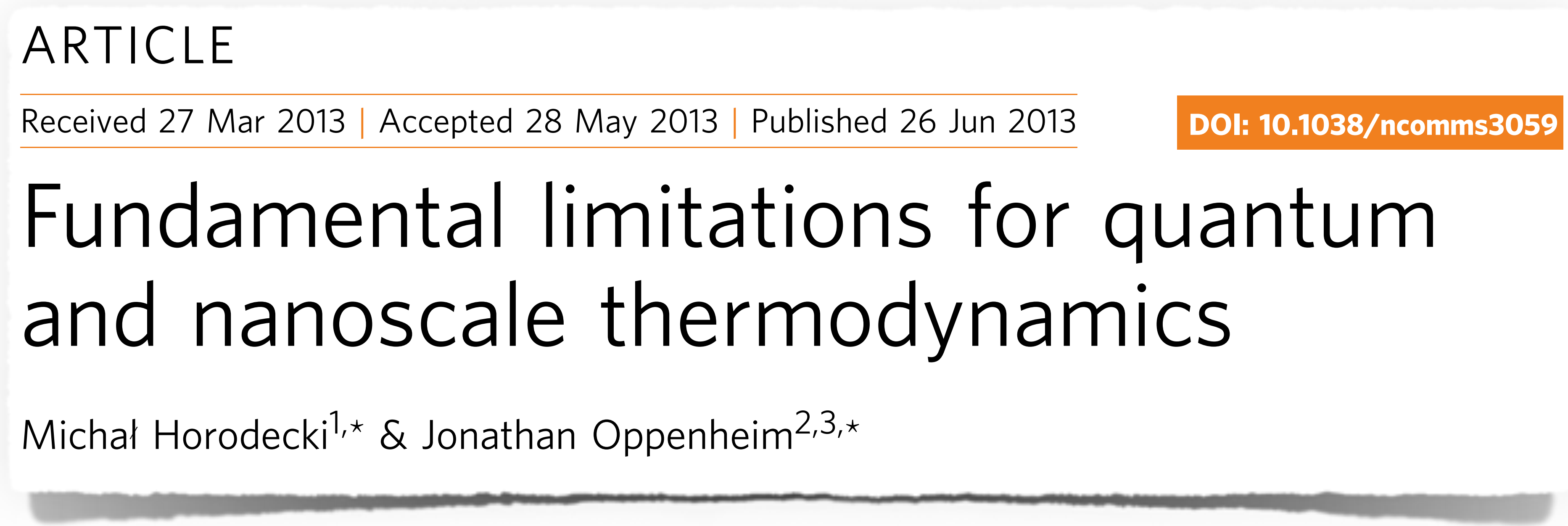

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- extractable work: $F_0^{\varepsilon}(\rho) - F(\gamma)$
- work of formation: $F_{\infty}^{\varepsilon}(\rho) - F(\gamma)$

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$$\lim_{n\to\infty}\frac{1}{n}F^{\varepsilon}_{0/\infty}(\rho^{\otimes n}) = F(\rho).$$

Work characterized by *F* only in the thermodynamic limit.

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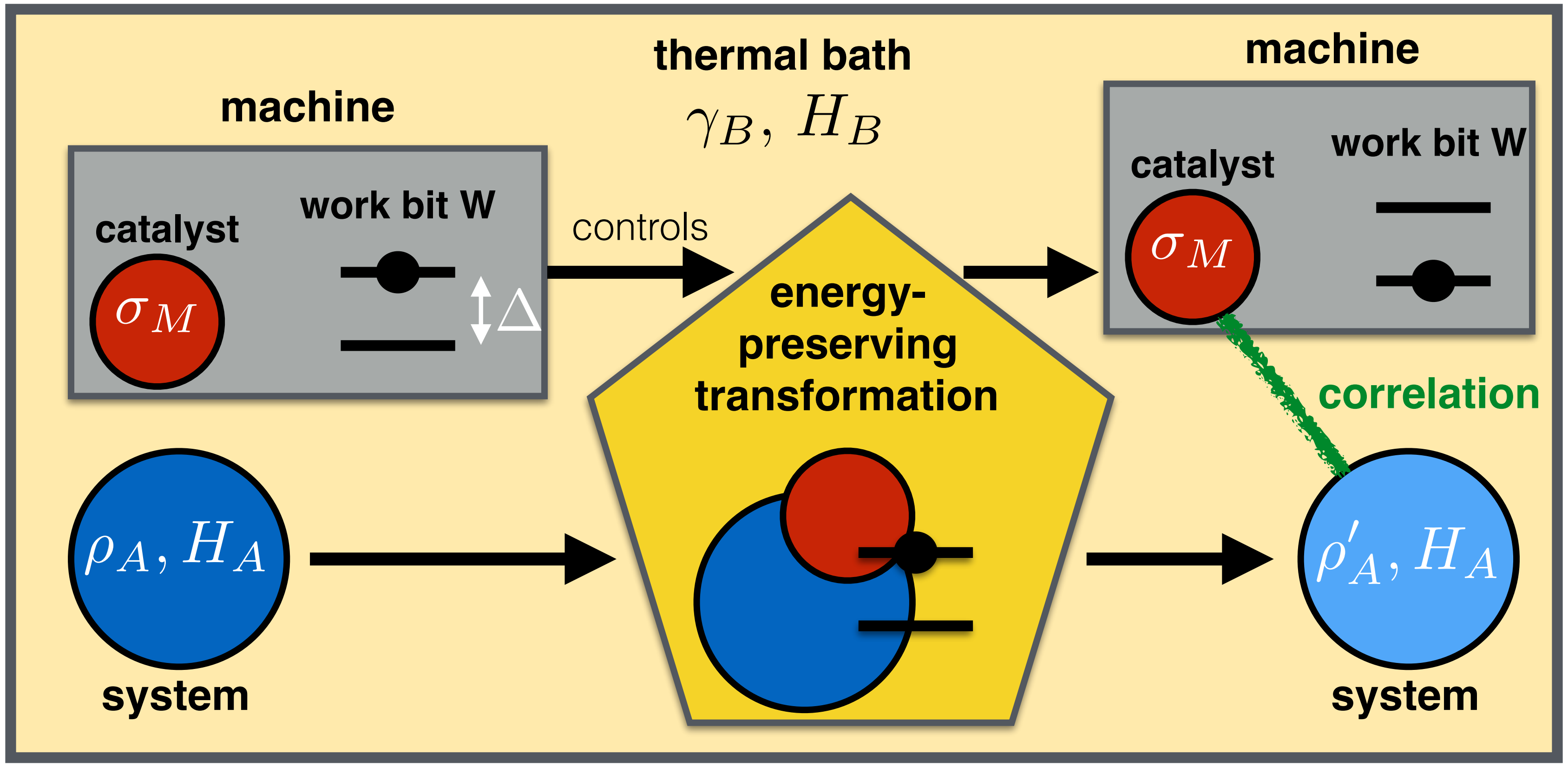

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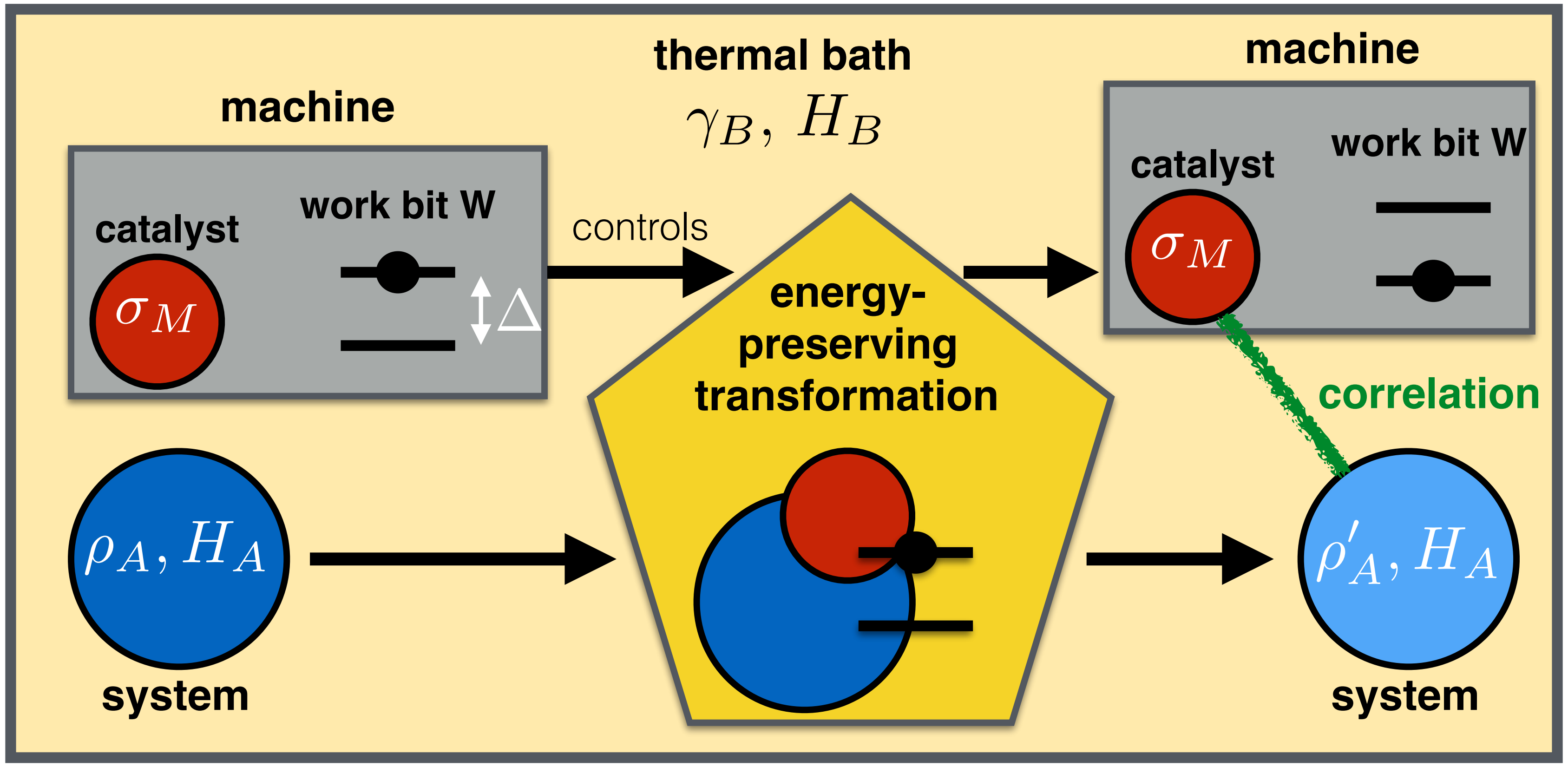

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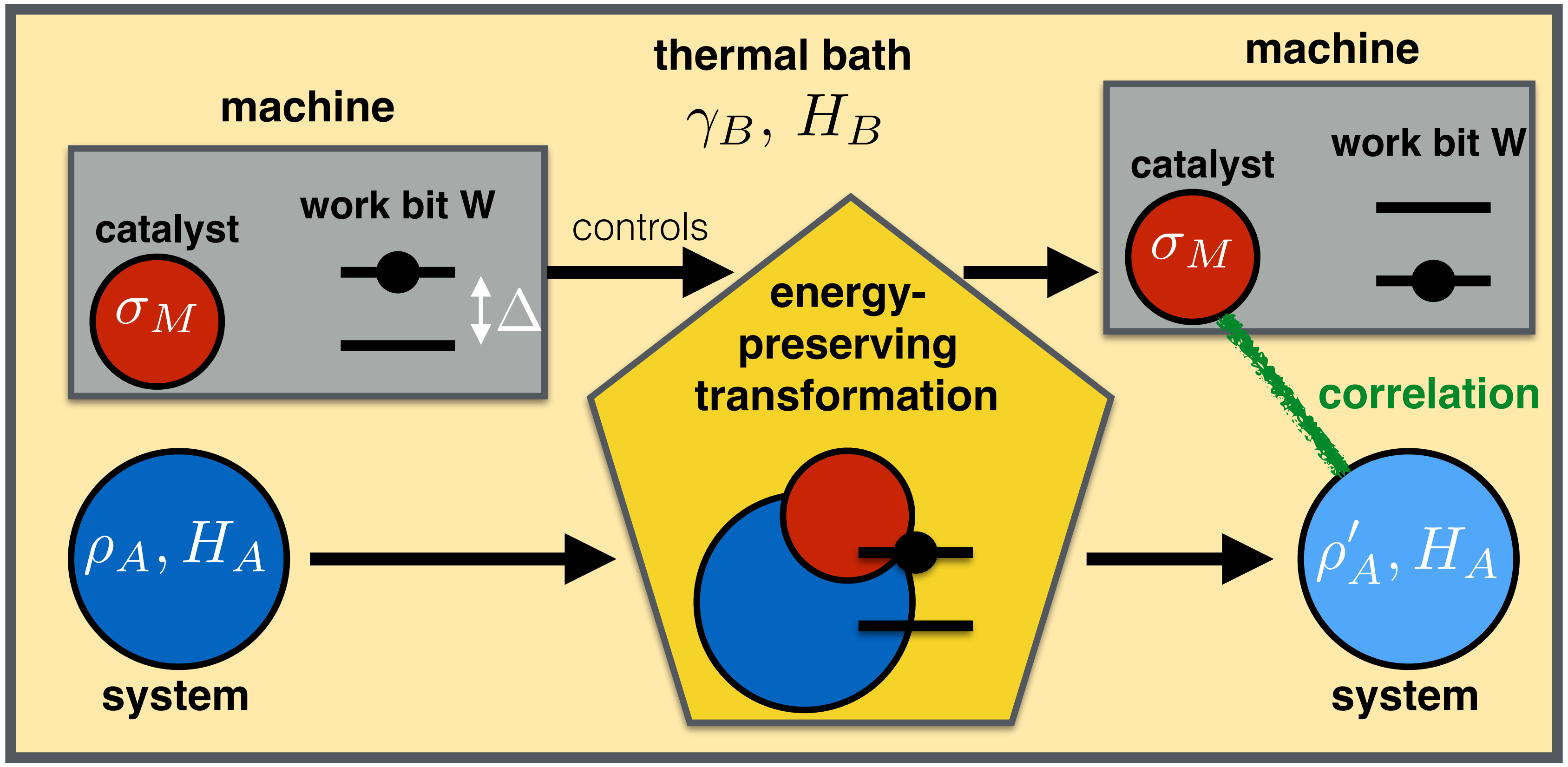

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$$\rho_A \otimes \sigma_M \otimes |e\rangle\langle e|_W$$
$$\mapsto$$
$$\sigma_{AM} \otimes |g\rangle\langle g|_W$$

Theorem. *Fix any initial state ρ_A and target state ρ'_A, both block-diagonal, such that $F(\rho'_A) \geq F(\rho_A)$. Using a work bit W with some energy gap Δ larger than, but arbitrarily close to $F(\rho'_A) - F(\rho_A)$, the transition*

$$\rho_A \otimes \sigma_M \otimes |e\rangle\langle e|_W \mapsto \sigma_{AM} \otimes |g\rangle\langle g|_W$$

can be achieved by a thermal operation, where $\sigma_A := \mathrm{Tr}_M \sigma_{AM}$ is arbitrarily close to ρ'_A.

The state σ_M is exactly identical before and after the transformation, M is finite-dimensional, and the resulting correlations between A and M can be made arbitrarily small.

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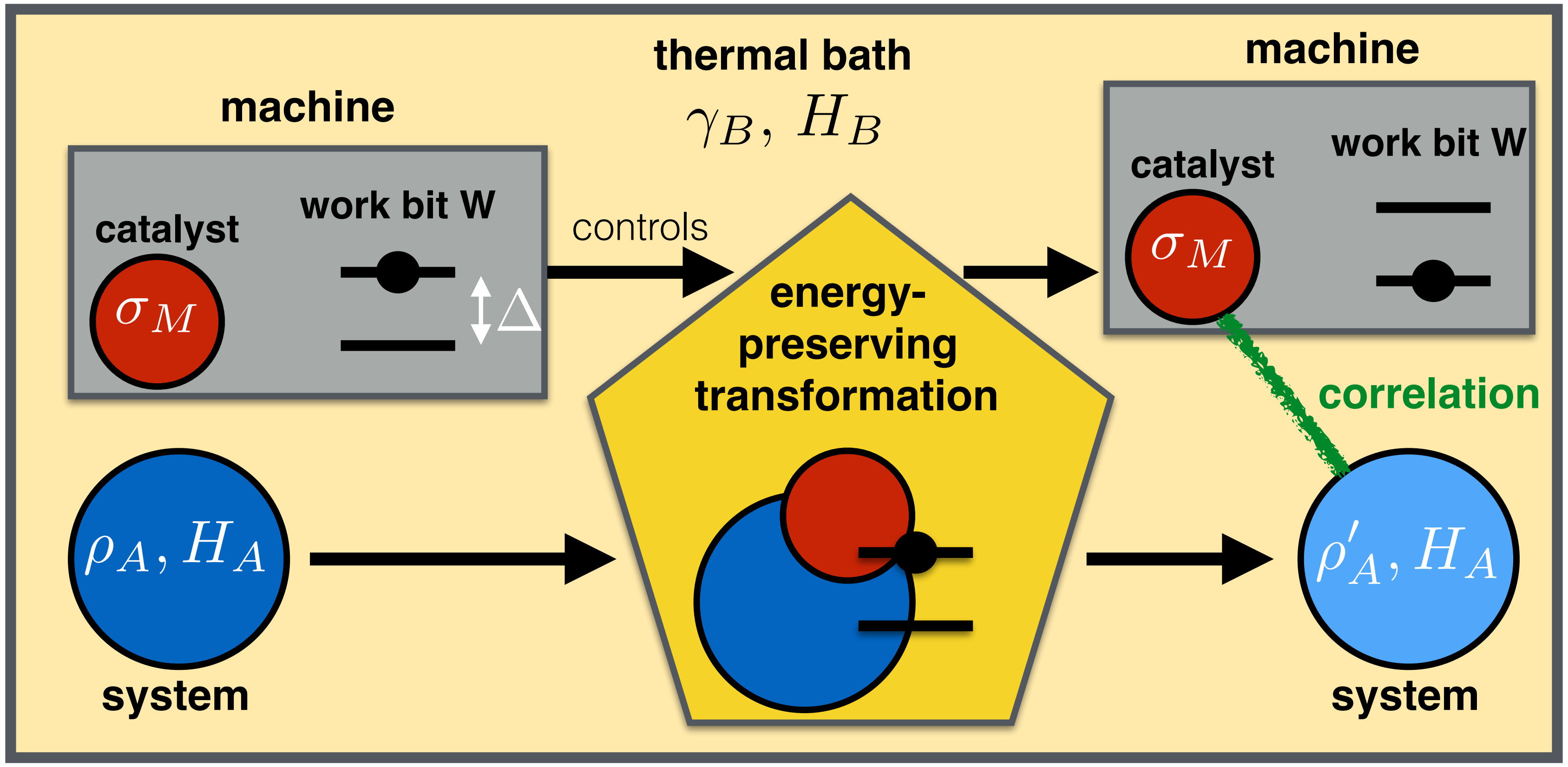

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$$F(\rho'_A) \geq F(\rho_A)$$

$$\Delta > F(\rho'_A) - F(\rho_A)$$

σ_A as close to ρ'_A as you like,

σ_M exactly preserved, $\dim M < \infty$.

Extractable work (when allowing correlations)…

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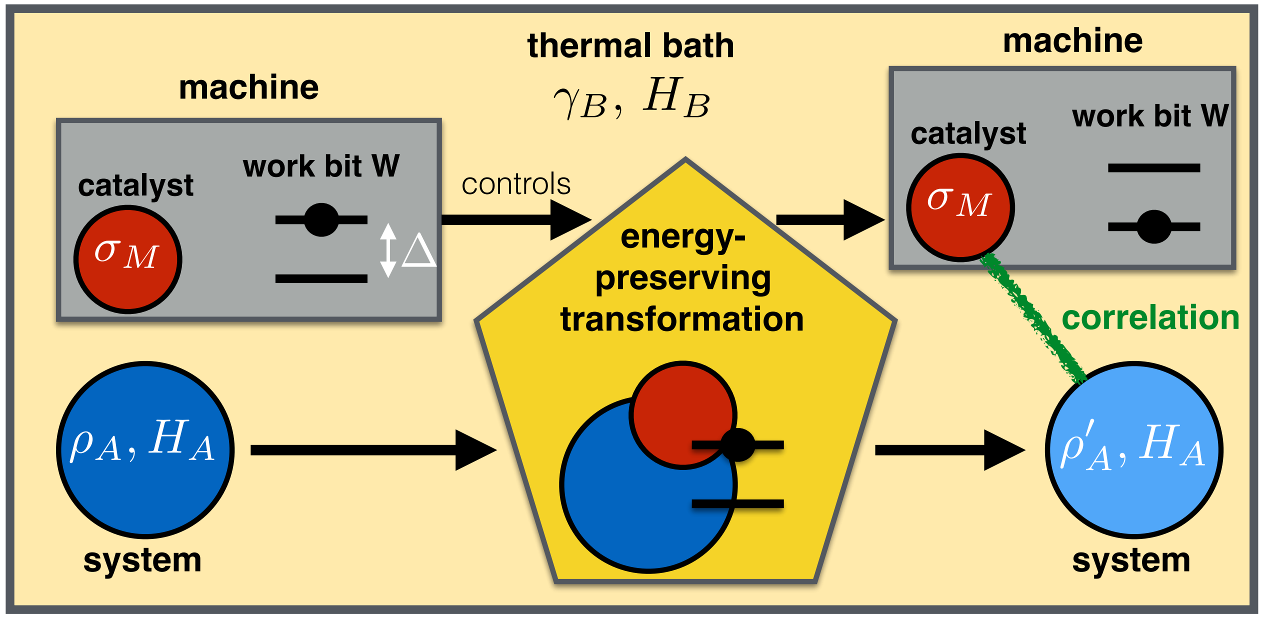

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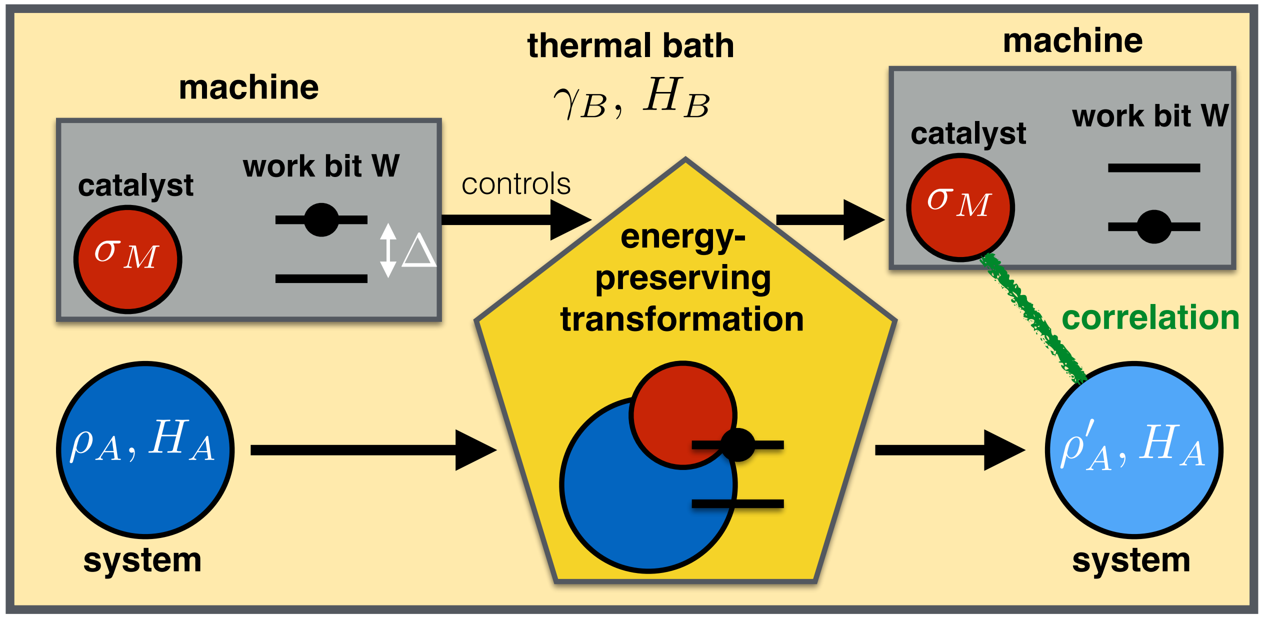

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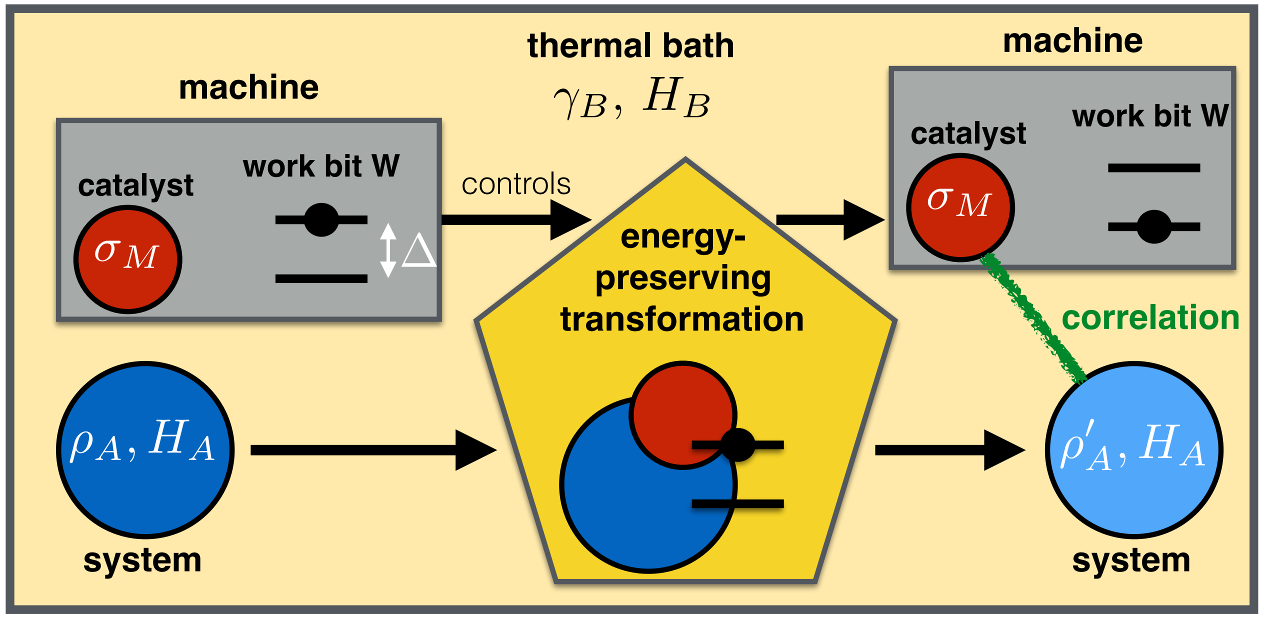

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$$\rho_A \otimes \sigma_M \otimes (1, 0, \ldots, 0) \otimes |g\rangle\langle g|_W$$

$$\big\downarrow$$

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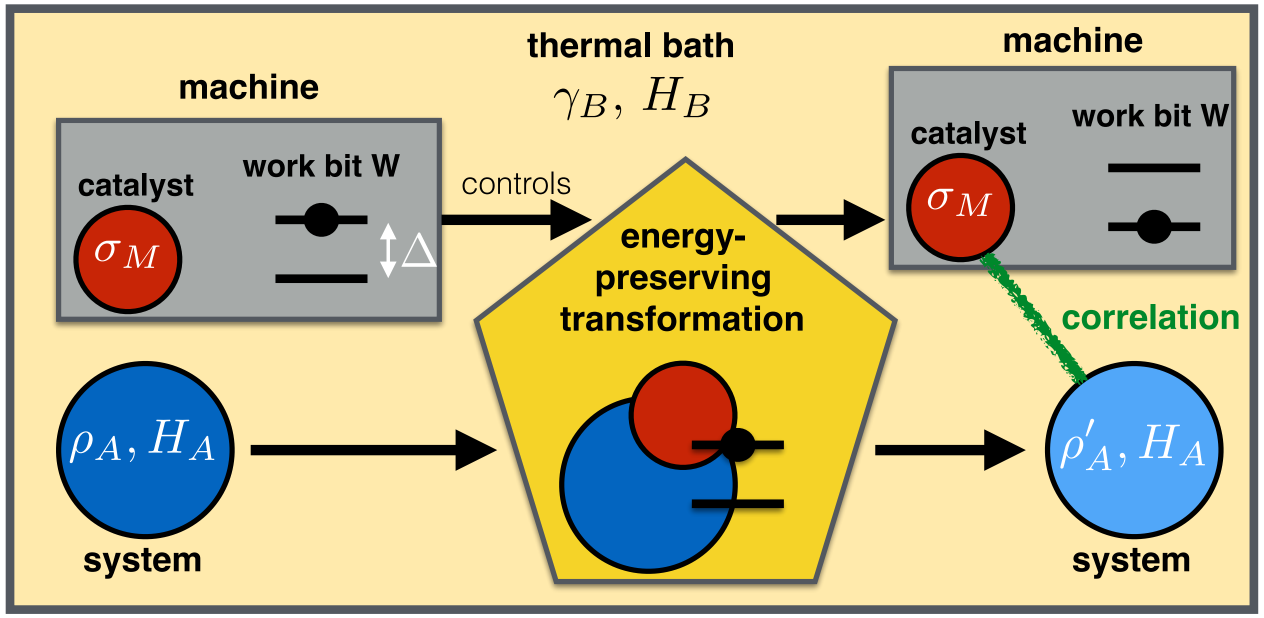

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$$\rho_A \otimes \sigma_M \otimes \tau_S^{(m,n)} \otimes |g\rangle\langle g|_W \mapsto \sigma_{AMS} \otimes |e\rangle\langle e|_W.$$

Here $\sigma_M = \mathrm{Tr}_{AS}\sigma_{AMS}$ *remains identical during the transformation,* $\sigma_S = \tau_S^{(m,n,\varepsilon)}$*, and* σ_A *is as close to* ρ'_A *as we like. This can be achieved for any choice of* $\varepsilon > 0$*, as long as* n/m *is large enough.*

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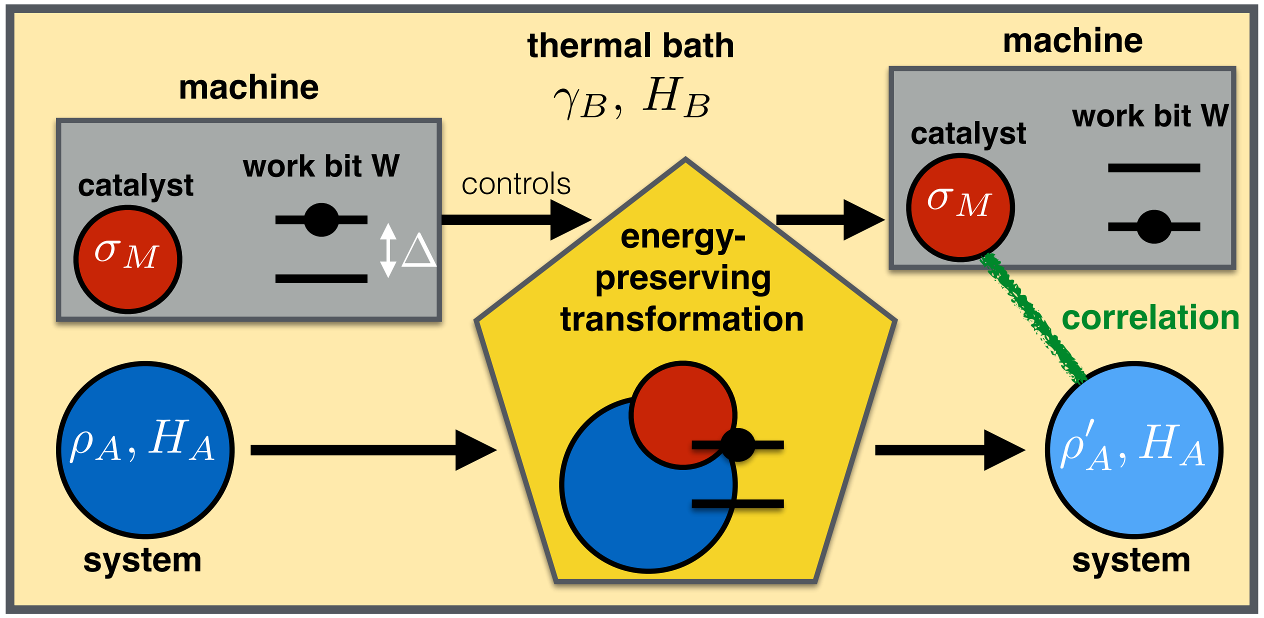

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I.e. can make fluctuations arbitrarily small (but not zero).

Mathematical background

MM, *Correlating thermal machines and the second law at the nanoscale*, arXiv:1707.03451

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These results rely heavily on the following new theorem:

Main Theorem. Let $p, p' \in \mathbb{R}^m$ be probability distributions with $p^\downarrow \neq p'^\downarrow$. Then there exists an extension p'_{XY} of $p' \equiv p'_X$ such that

$$p_X \otimes p'_Y \succ p'_{XY} \tag{6}$$

if and only if $H_0(p) \leq H_0(p')$ and $H(p) < H(p')$. Moreover, for every $\varepsilon > 0$, we can choose Y and p'_{XY} such that the mutual information is $I(X:Y) \equiv S(p'_{XY} \| p'_X \otimes p'_Y) < \varepsilon$.

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Majorization: prob. vectors $p = (p_1, \ldots, p_n), \quad q = (q_1, \ldots, q_n)$

$$p \succ q \quad \Leftrightarrow \quad \sum_{i=1}^k p_i^\downarrow \geq \sum_{i=1}^k q_i^\downarrow \qquad (k = 1, \ldots, n).$$

Stochastic independence as a resource

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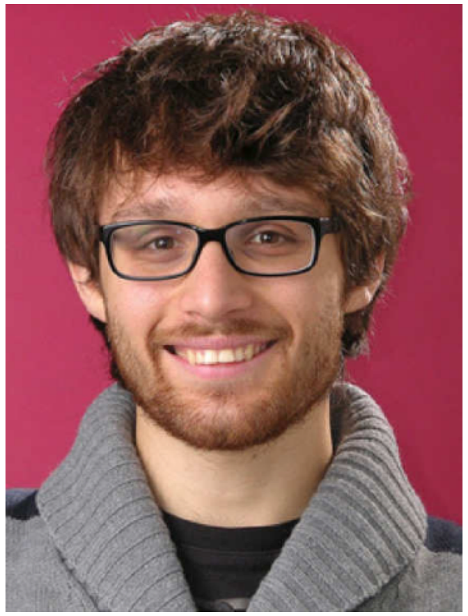
M. Lostaglio, **MM**, and M. Pastena, *Stochastic independence as a resource in small-scale thermodynamics*, Phys. Rev. Lett. **115**, 150402 (2015).



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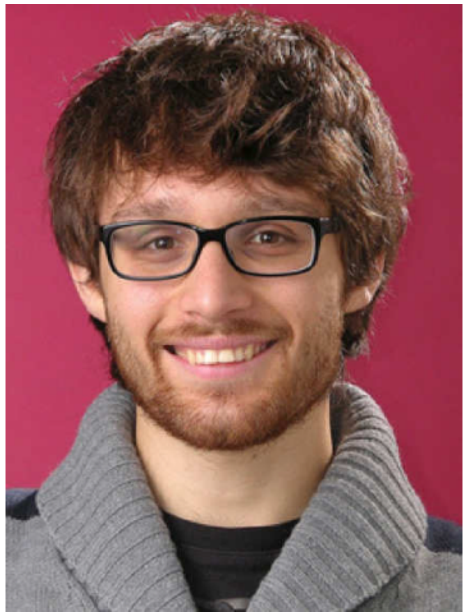

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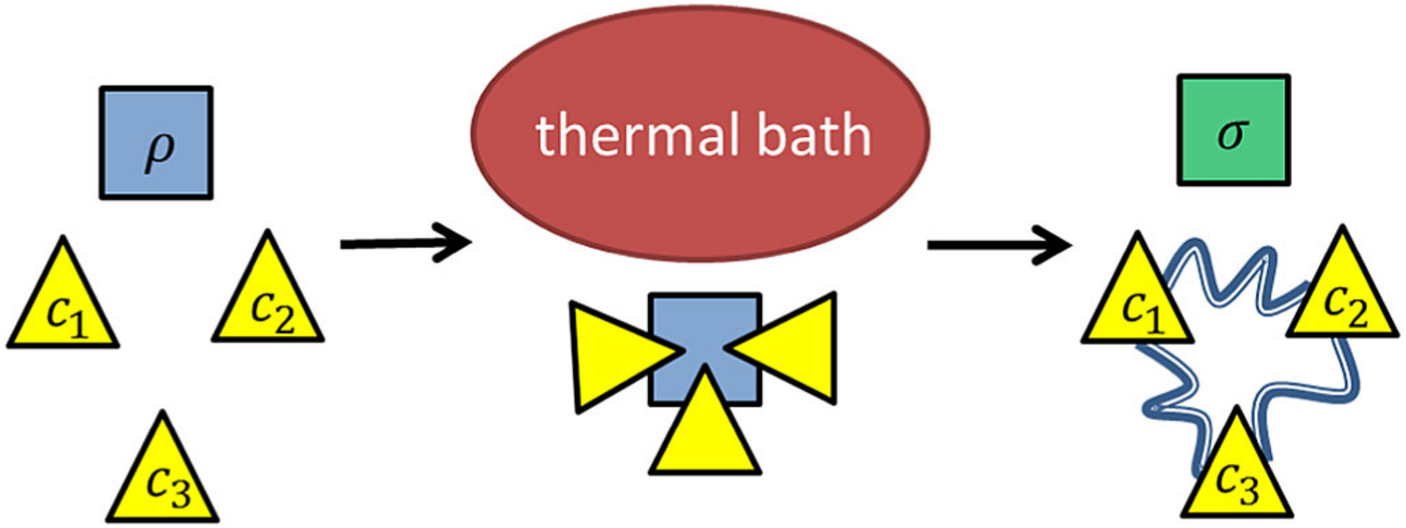

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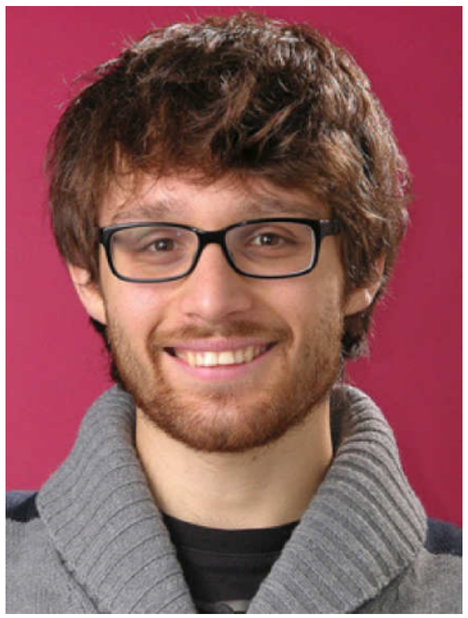

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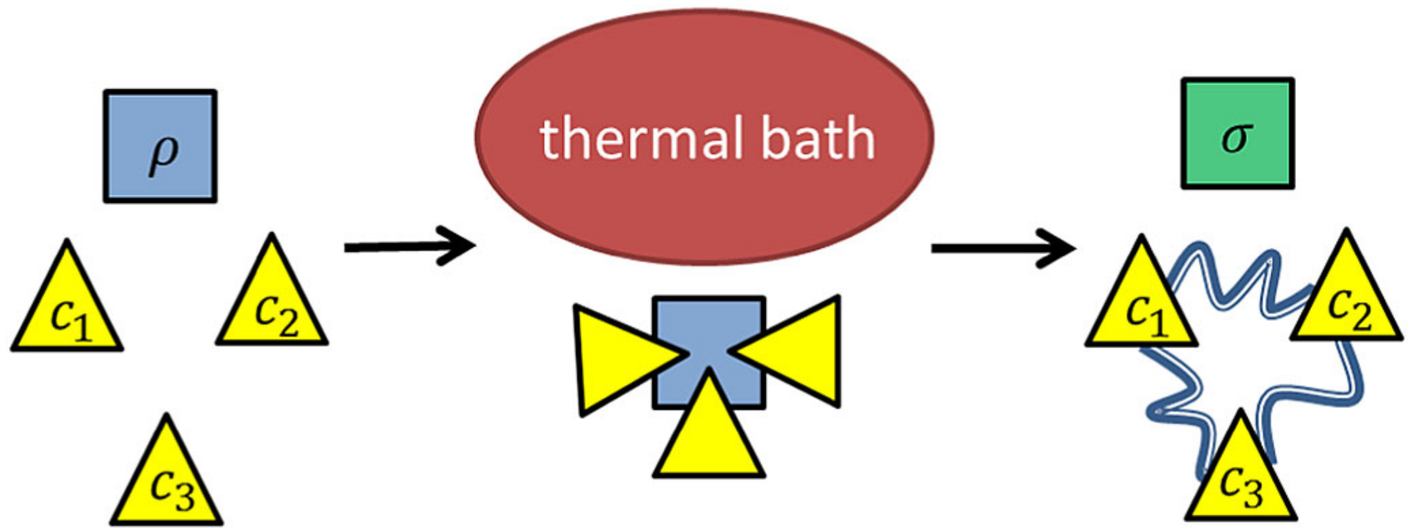

Correlating external systems can allow otherwise impossible state transitions.

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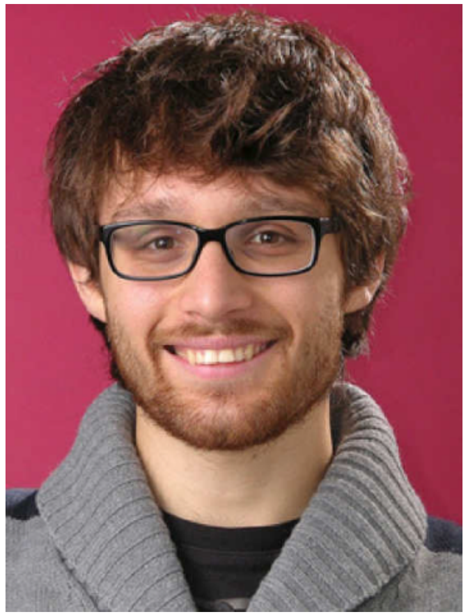

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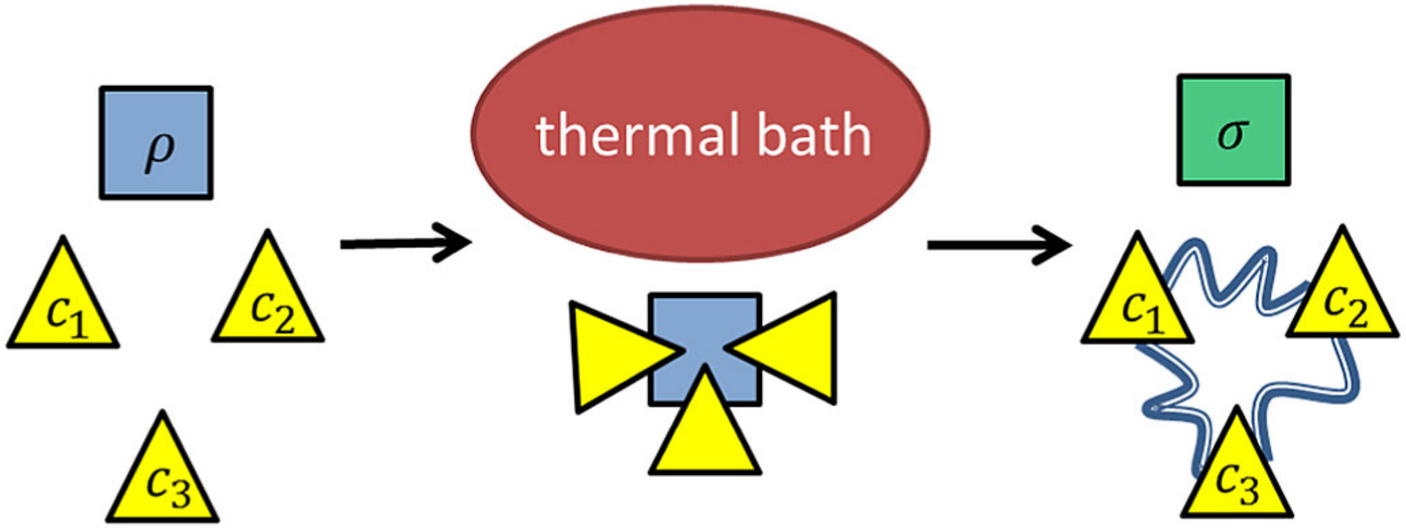

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The exact opposite of what one would expect from standard thermodynamics!

$$F(\rho_{AB}) \geq F(\rho_A \otimes \rho_B).$$

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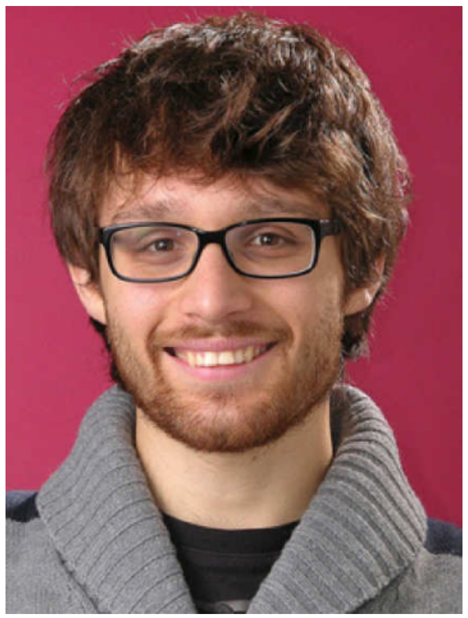

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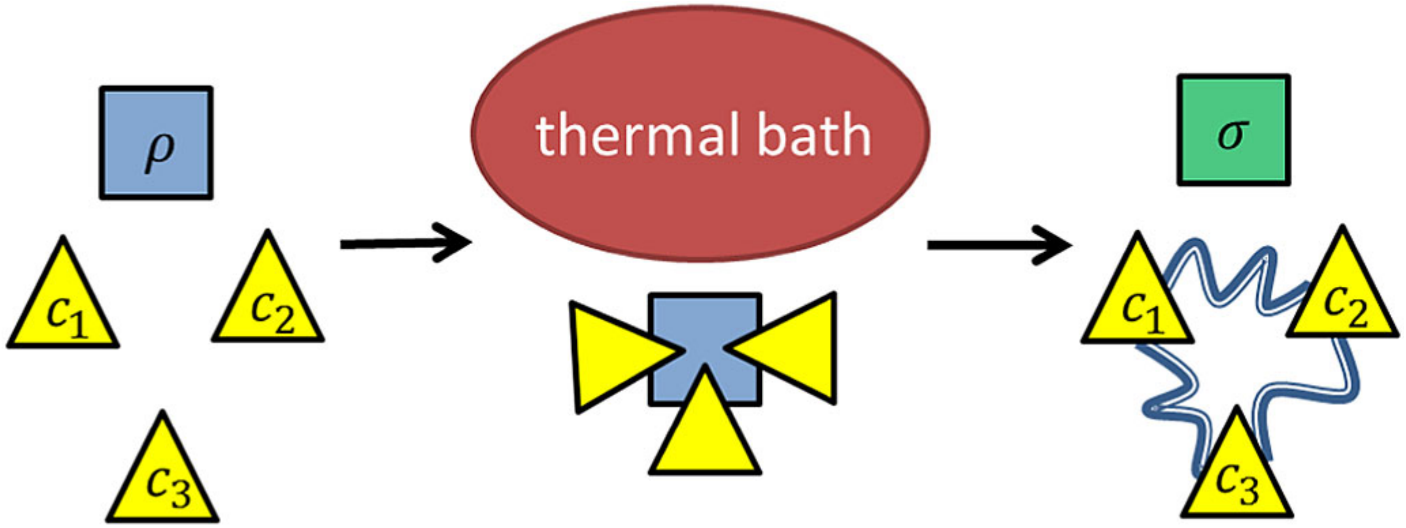

Correlating external systems can allow otherwise impossible state transitions. **“Trade fluctuations for correlations.”**

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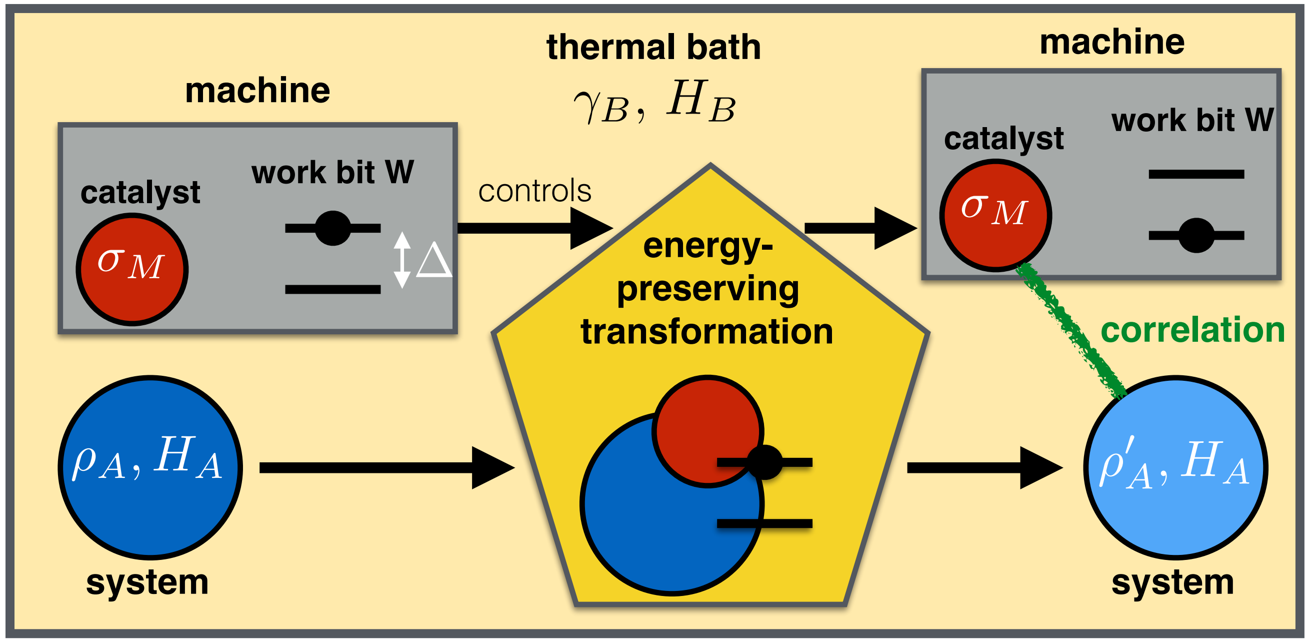
Outline

1. Standard view: thermodynamic limit

2. Thermodynamics as a resource theory



3. **A new one-shot interpretation of free energy**

4. Similar results in quantum information?

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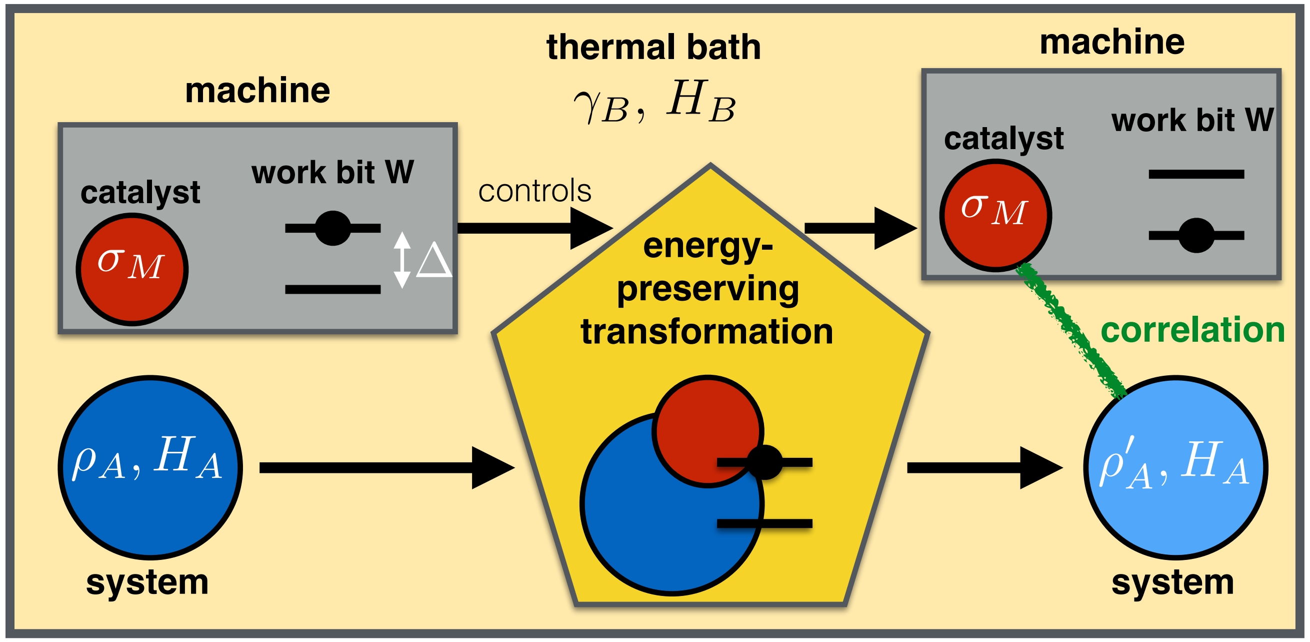

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PRL **118,** 080503 (2017) PHYSICAL REVIEW LETTERS week ending 24 FEBRUARY 2017

Catalytic Decoupling of Quantum Information

Christian Majenz,[1,*] Mario Berta,[2] Frédéric Dupuis,[3] Renato Renner,[4] and Matthias Christandl[1]

[1]*Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø*

[2]*Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, California 91125, USA*

[3]*Faculty of Informatics, Masaryk University, Brno, Czech Republic*

[4]*Institute for Theoretical Physics, ETH Zurich, 8093 Zürich, Switzerland*

(Received 24 May 2016; published 23 February 2017)

Similar results in quantum information?

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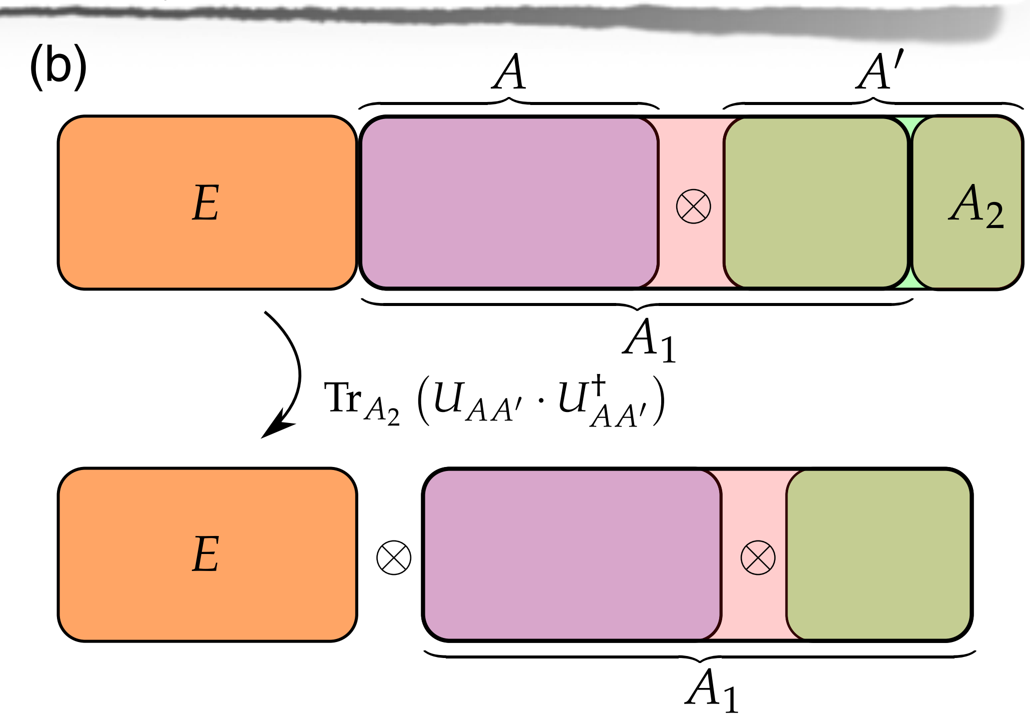
[3]*Faculty of Informatics, Masaryk University, Brno, Czech Republic*
[4]*Institute for Theoretical Physics, ETH Zurich, 8093 Zürich, Switzerland*
(Received 24 May 2016; published 23 February 2017)

Theorem 1: (Catalytic decoupling) For any bipartite quantum state ϱ_{AE} and $0 < \delta \leq \varepsilon \leq 1$, we have:

$$R_c^\varepsilon(A;E)_\varrho \lesssim \frac{1}{2} I_{\max}^{\varepsilon-\delta}(E;A)_\varrho, \tag{11}$$

where $\lesssim$ stands for smaller or equal up to terms $\mathcal{O}(\log\log|A| + \log(1/\delta))$. We also have the converse

$$R_c^\varepsilon(A;E)_\varrho \geq \frac{1}{2} I_{\max}^{\varepsilon}(E:A)_\varrho. \tag{12}$$



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Interesting, for example, because standard entropies have **dual spacetime interpretations**:

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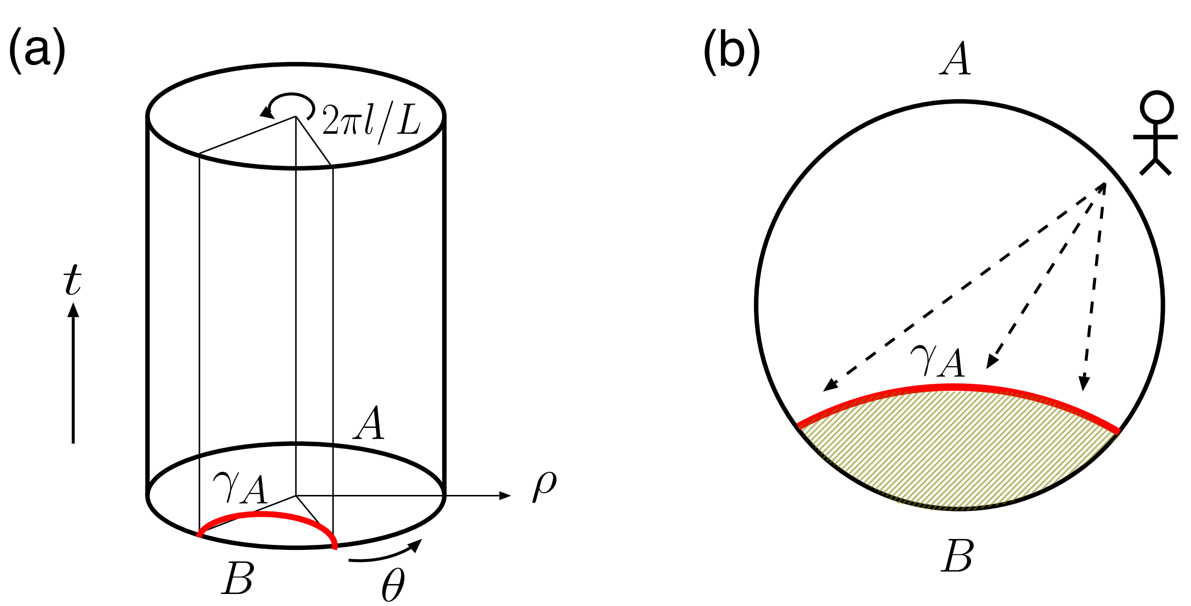
S. Ryu and T. Takayanagi, *Holographic Derivation of Entanglement Entropy from AdS/CFT*, Phys. Rev. Lett. **96**, 181602 (2006).

$$S_A = \frac{\text{Area of } \gamma_A}{4G_N^{(d+2)}}$$



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But here's an example, following from the above:

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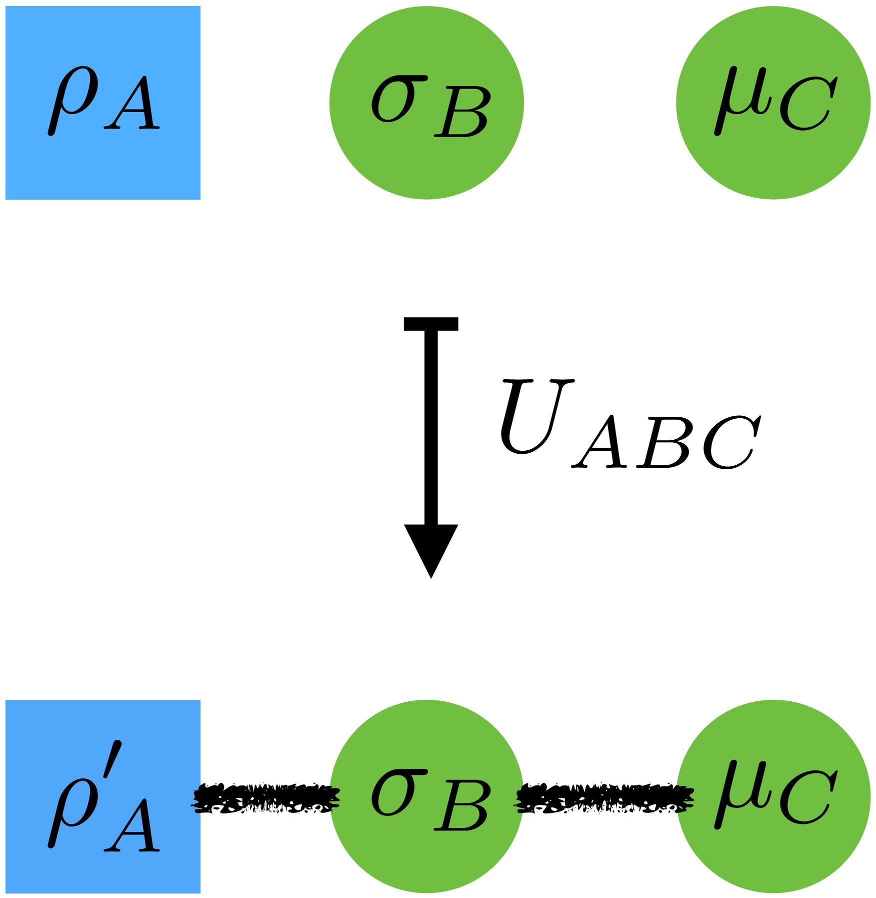

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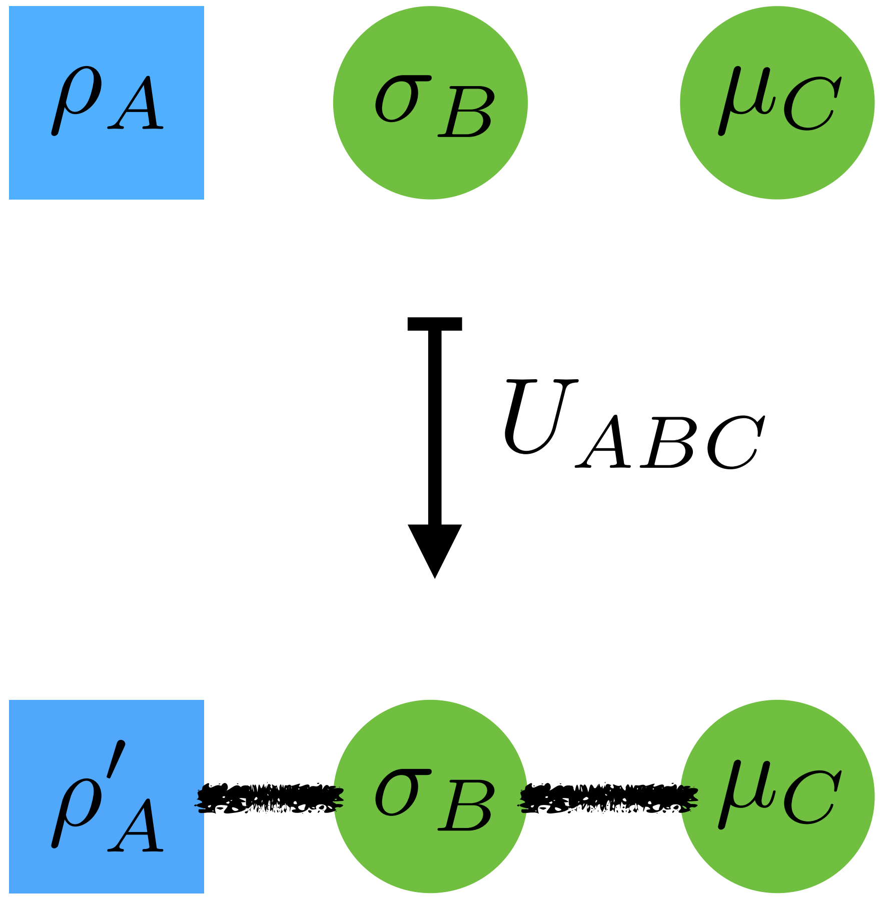


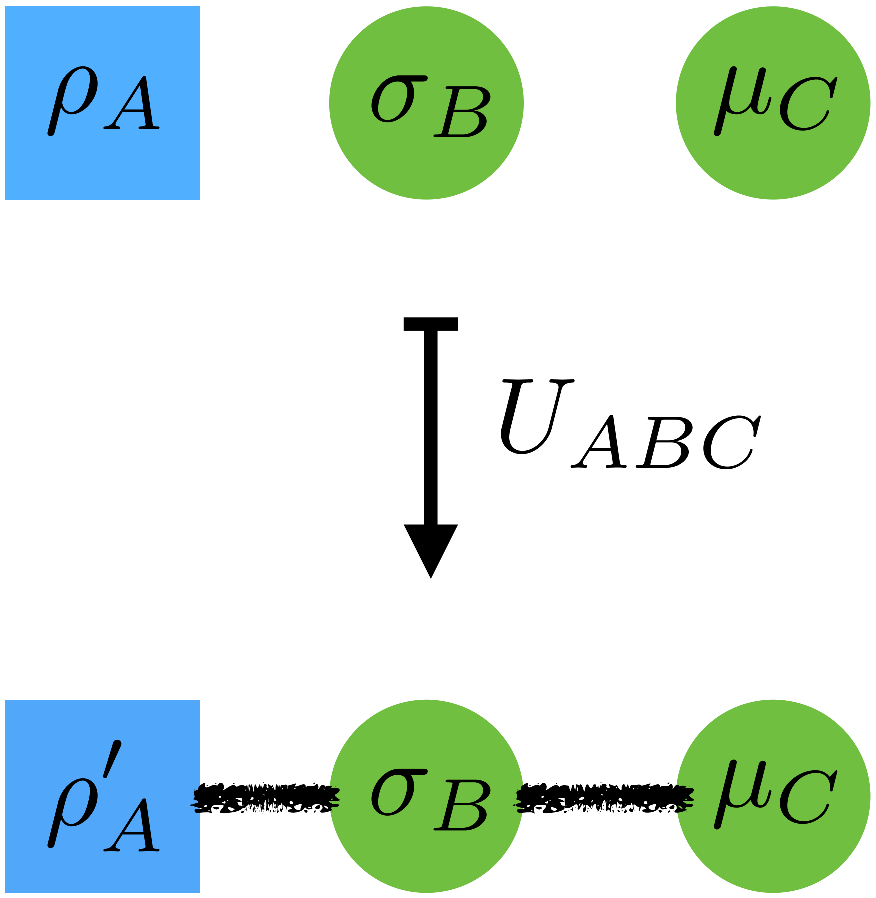
$$S(\rho_A) + S(\sigma_B) + S(\mu_C) = S(\rho'_{ABC}) \leq S(\rho'_A) + S(\sigma_B) + S(\mu_C).$$

What's possible here? Don't know (yet).
But here's an example, following from the above:

Theorem 5. *Let ρ_A and ρ'_A be quantum states with full rank which are not unitarily equivalent, i.e. do not have the exact same set of eigenvalues. Then there exists a finite auxiliary system B, a quantum state σ_B, and a copy C of AB with maximally mixed state μ_C as well a unitary U_{ABC} such that*

$$U_{ABC}(\rho_A \otimes \sigma_B \otimes \mu_C)U^\dagger_{ABC} = \rho'_{ABC}$$

with marginals ρ'_A on A, $\rho'_B = \sigma_B$ and $\rho'_C = \mu_C$ if and only if $S(\rho_A) < S(\rho'_A)$ for the von Neumann entropy S.



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Open Questions. Can we do without the C system? Or recycle BC? And do the same if A is correlated with some other system (decoupling)?

Relation to versions of the quantum marginal problem.

with marginals ρ'_A on A, $\rho'_B = \sigma_B$ and $\rho'_C = \mu_C$ if and only if $S(\rho_A) < S(\rho'_A)$ for the von Neumann entropy S.

$$S(\rho_A) + S(\sigma_B) + S(\mu_C) = S(\rho'_{ABC}) \leq S(\rho'_A) + S(\sigma_B) + S(\mu_C).$$

Conclusions

- Standard interpretation of free energy *F* is only relevant/meaningful in the thermodynamic limit.
- Resource theory approach: generalizes thermo to “small” / strongly correlated systems. “Second laws”: $\Delta F_\alpha \leq 0.$
- But: allowing correlations restores the second law. Operational meaning of *F* for single particles. Conjecture: similar results in quantum information.

MM, *Correlating thermal machines and the second law at the nanoscale*, arXiv:1707.03451

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Thank you!