Majorization, correlating catalysts, and the one-shot interpretation of entropic quantities

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Michele Pastena[4], and Jakob Scharlau[4]

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[4] Department of Theoretical Physics, University of Heidelberg, Germany

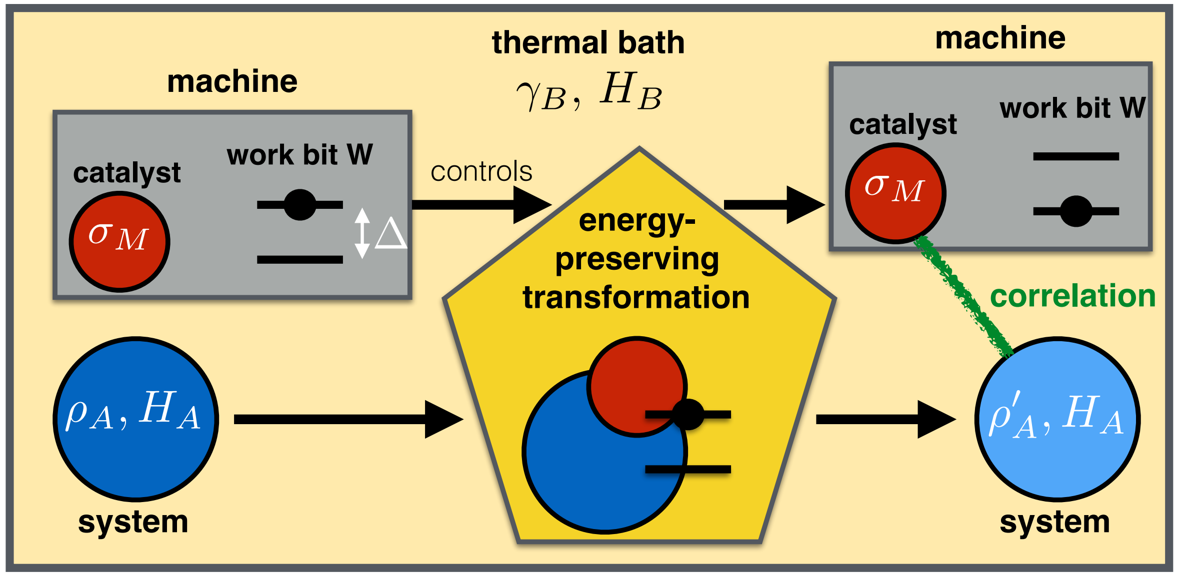


Outline

1. Intro: Majorization in quantum information

2. Main mathematical results

3. Implications for quantum thermodynamics



4. Implications for quantum information (in progress)

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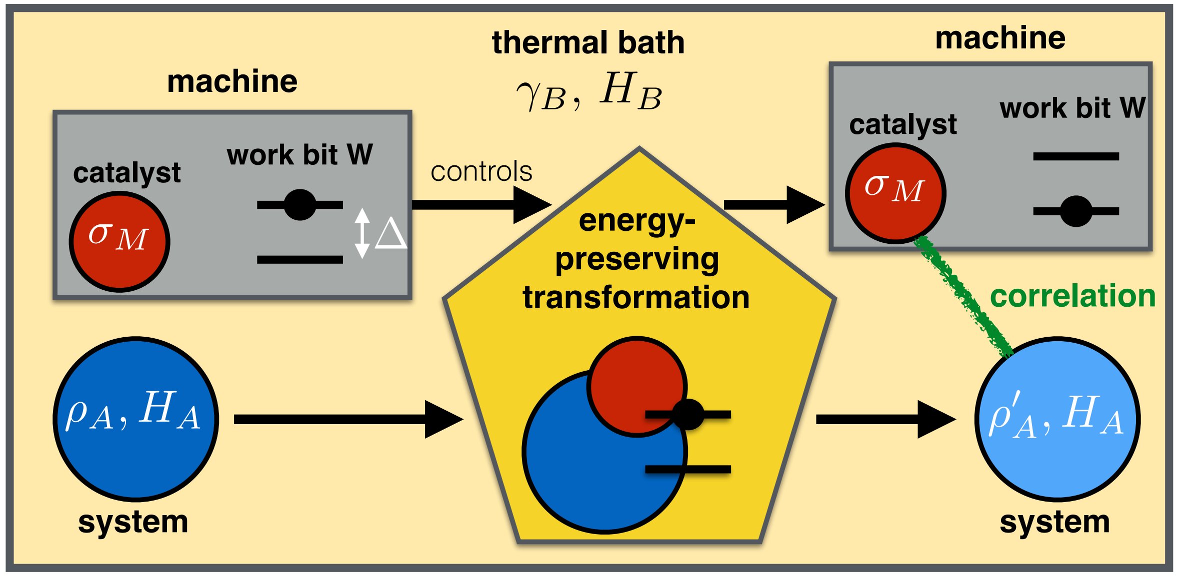

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Intro: Majorization in quantum information

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Nielsen's Theorem: If and only if $\lambda_\varphi \succ \lambda_\psi$.

$$|\psi\rangle = \sum_i \sqrt{\lambda_\psi^{(i)}}|i\rangle \otimes |i\rangle$$

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Majorization: prob. vectors $p = (p_1, \ldots, p_n), \quad q = (q_1, \ldots, q_n)$

$$p \succ q \quad \Leftrightarrow \quad \sum_{i=1}^k p_i^\downarrow \geq \sum_{i=1}^k q_i^\downarrow \qquad (k = 1, \ldots, n).$$

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e.g. $(1,0,0) \succ (.7,.2,.1) \succ \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

(Resource theory of) noisy operations

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M.+P. Horodecki, and J. Oppenheim, *Reversible transformations from pure to mixed states, and the unique measure of information*, Phys. Rev. A **67**, 062104 (2003).

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Write $\rho_A \xrightarrow{\text{noisy}} \rho'_A$ if there is a finite-dim. B, a unitary U_{AB} and maximally mixed state $\mu_B = \mathbf{1}_B/d_B$ such that

$$\rho'_A = \mathrm{Tr}_B\left[U_{AB}\left(\rho_A \otimes \mu_B\right)U^\dagger_{AB}\right].$$

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Blueprint of the resource theory of q. thermodynamics…

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Theorem. For all $\varepsilon > 0$ there is $\rho'_A(\varepsilon)$ and

$$\rho_A \xrightarrow{\text{noisy}} \rho'_A(\varepsilon), \qquad \|\rho'_A - \rho'_A(\varepsilon)\| < \varepsilon$$

if and only if $\mathrm{spec}(\rho_A) \succ \mathrm{spec}(\rho'_A)$.

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Need large d_B for small ε.

Catalysis

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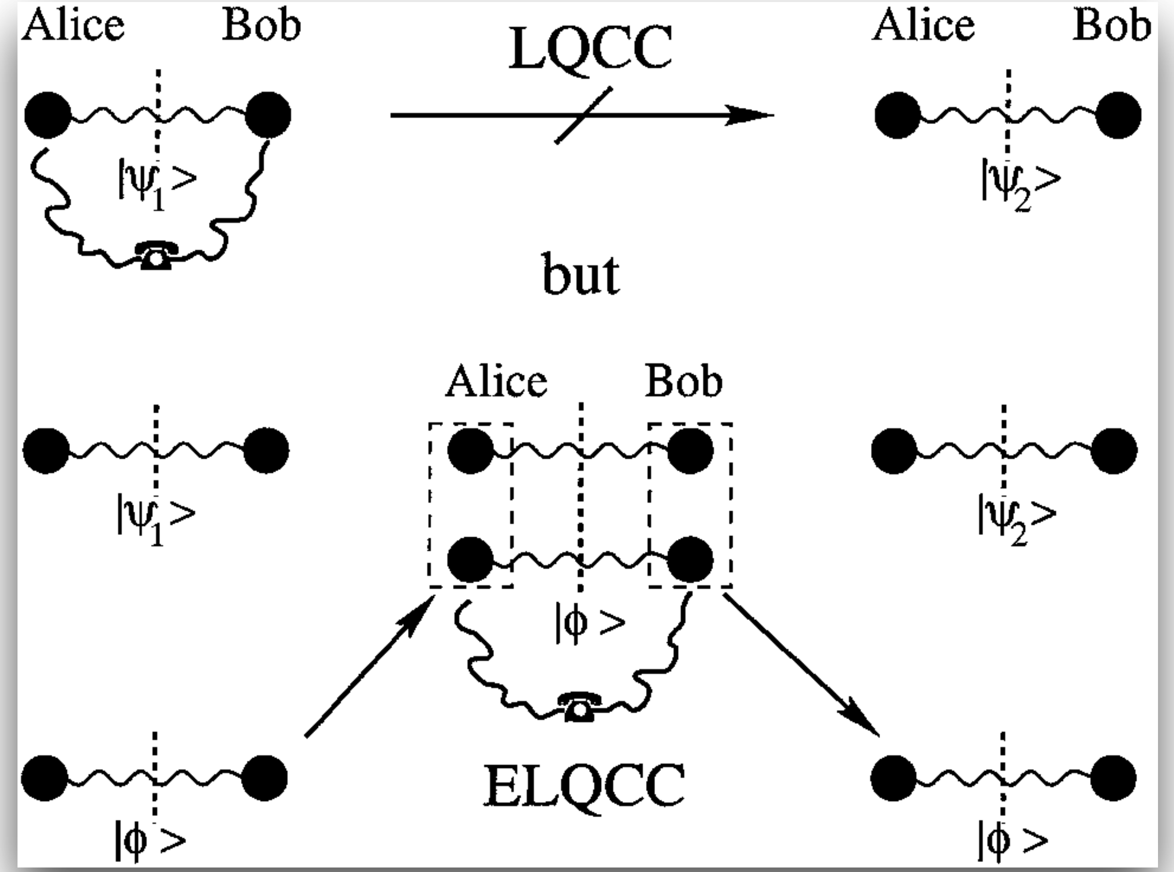

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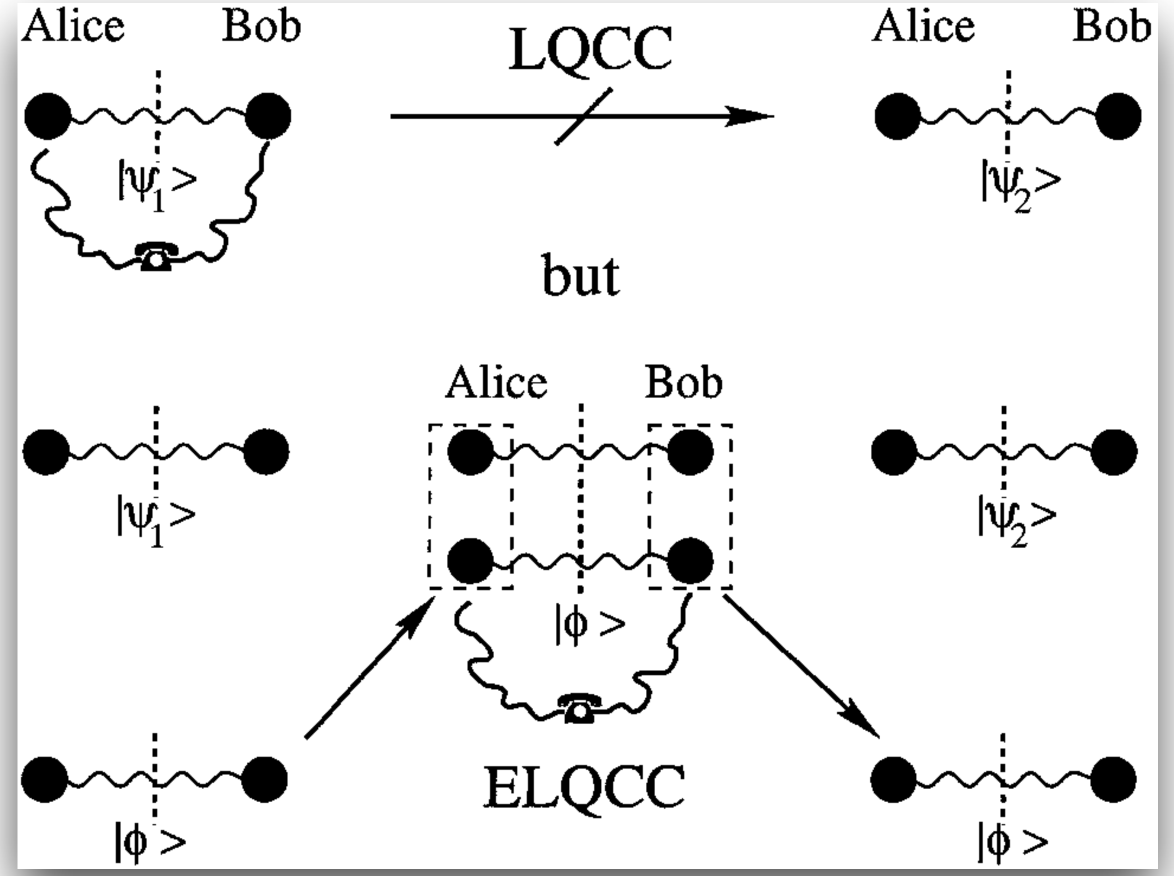

There are states with

$$|\psi_1\rangle \overset{\text{LOCC}}{\not\longrightarrow} |\psi_2\rangle \quad \textbf{but}$$

$$|\psi_1\rangle \otimes |\varphi\rangle \overset{\text{LOCC}}{\longrightarrow} |\psi_2\rangle \otimes |\varphi\rangle$$

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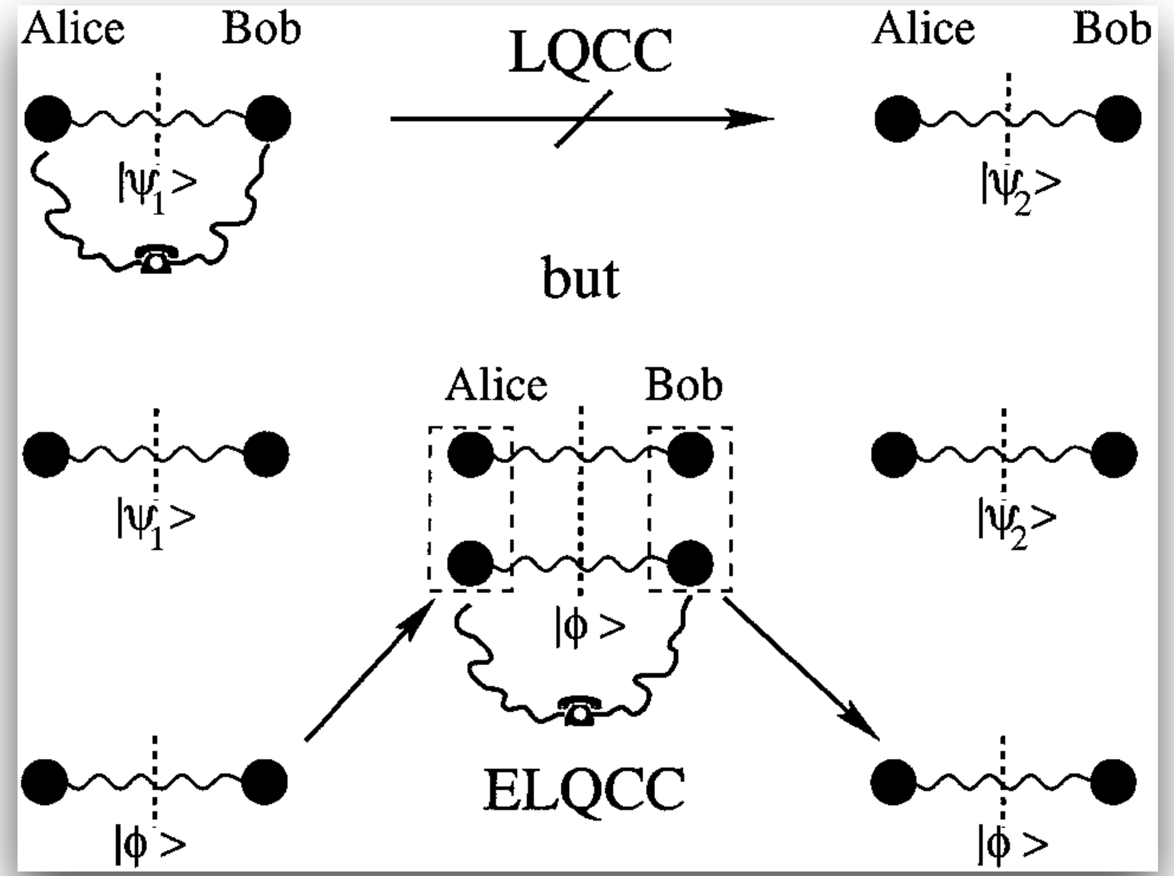

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because there are prob. vectors p, q, c with $p \not\succ q$

but $p \otimes c \succ q \otimes c.$

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Lemma (Klimesh; Turgut 2007):

Assuming $p^{\downarrow} \neq q^{\downarrow}$, there is such a c if and only if

$$H_\alpha(p) < H_\alpha(q) \text{ for all } \alpha \in \mathbb{R} \setminus \{0\} \text{ and}$$
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$$H_\alpha(p) = \frac{\text{sgn}(\alpha)}{1-\alpha} \log \sum_{i=1}^{n} p_i^\alpha, \qquad H_{\text{Burg}}(p) = \frac{1}{n} \sum_{i=1}^{n} \log p_i.$$

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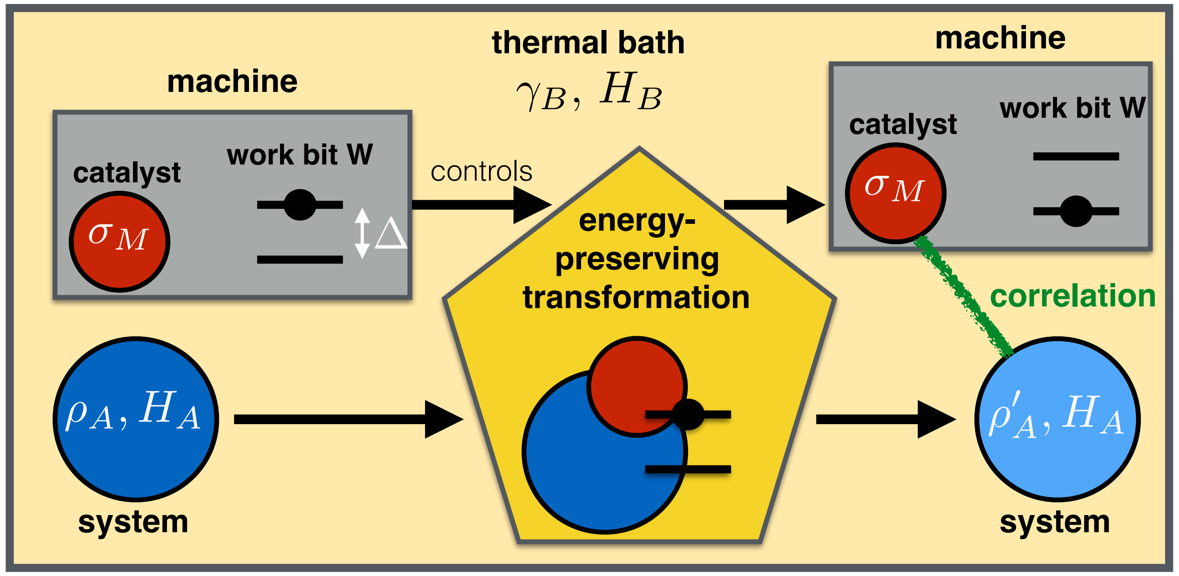

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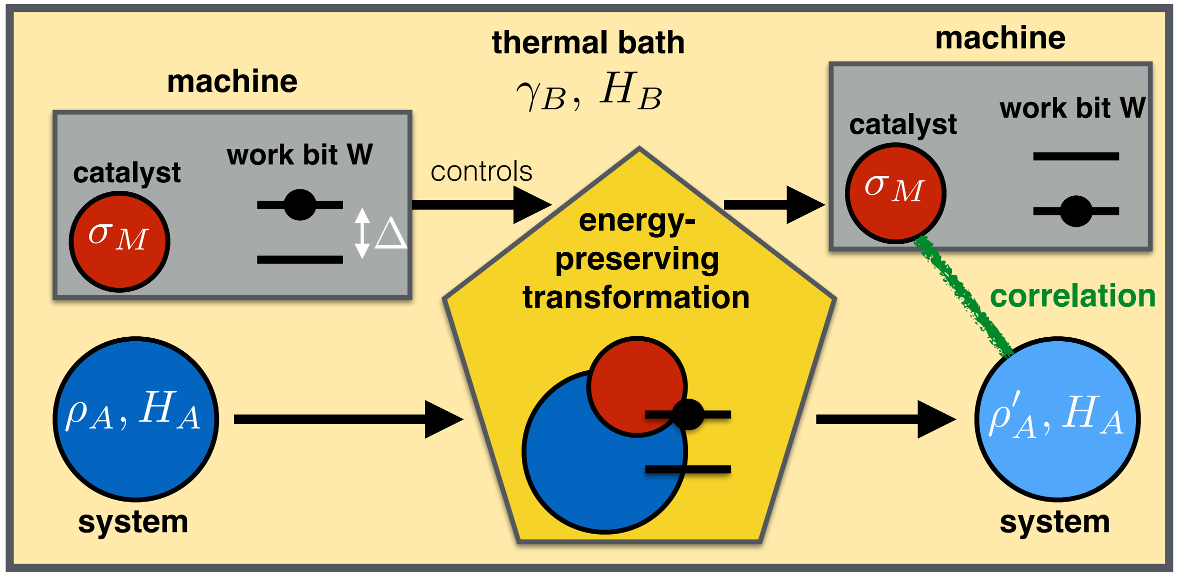

4. Implications for quantum information (in progress)

Main mathematical results

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Theorem 3 (Ref. [2]). *Let ρ and ρ' be quantum states on A such that $\rho \succ \rho'$, and let B be a copy of A. Then there exists a unitary U_{AB} such that*

$$\rho'_A = \mathrm{Tr}_B \left[U_{AB}(\rho_A \otimes \mu_B) U_{AB}^\dagger \right],$$

that is, the noisy transition from ρ to ρ' can be achieved exactly *with an auxiliary system that is of the same size as A. Moreover, U_{AB} can be chosen to leave the maximally mixed state μ_B on B invariant.*

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Answers a question by Bengtsson and Życzkowski.

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$$p_X \otimes p'_Y \succ p'_{XY} \tag{1}$$

if and only if $H_0(p) \leq H_0(p')$ and $H(p) < H(p')$. Moreover, for every $\varepsilon > 0$, we can choose Y and p'_{XY} such that the mutual information is $I(X:Y) \equiv S(p'_{XY} \| p'_X \otimes p'_Y) < \varepsilon$.

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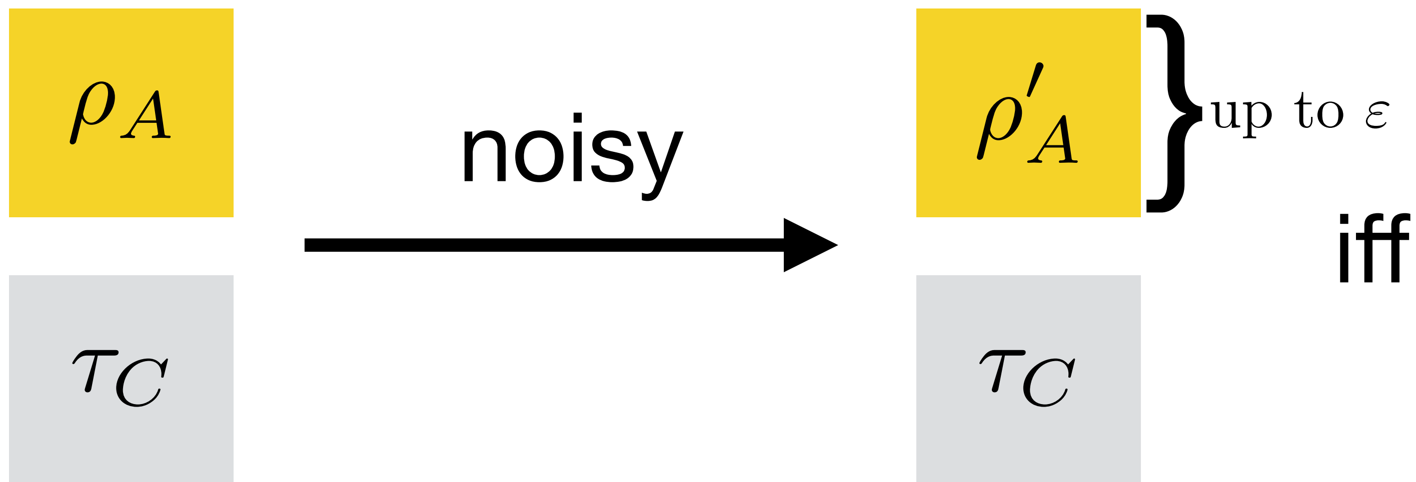
$$H_0(p) = \log \#\{i : p_i \neq 0\}, \qquad H(p) = -\sum_i p_i \log p_i.$$

Klimesh/Turgut's 2007 catalysis result implies:



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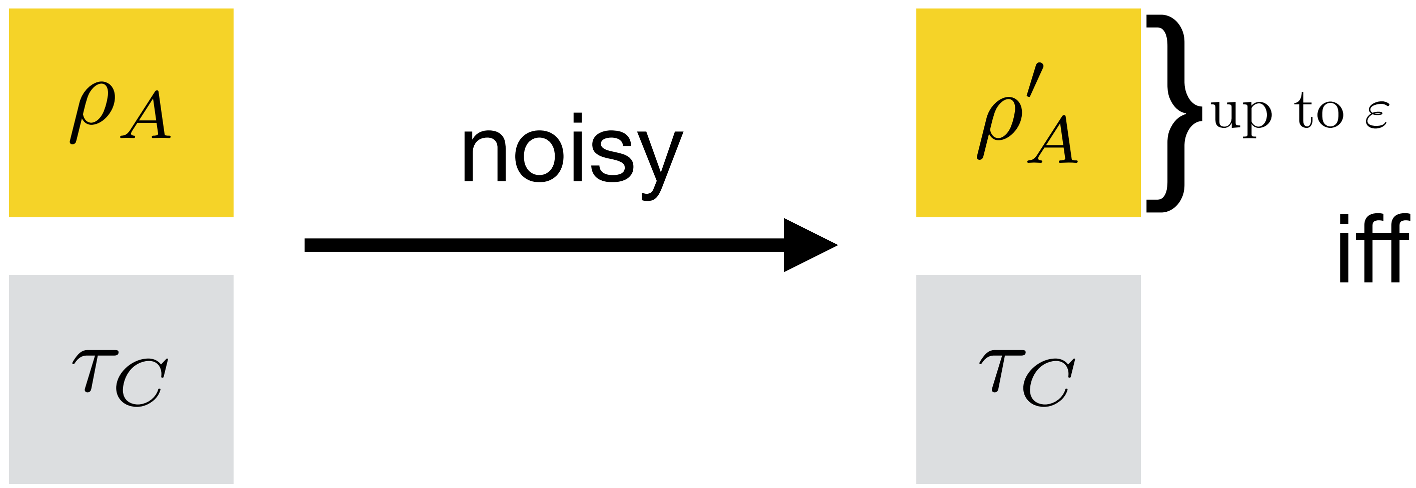

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Catalytic noisy operations

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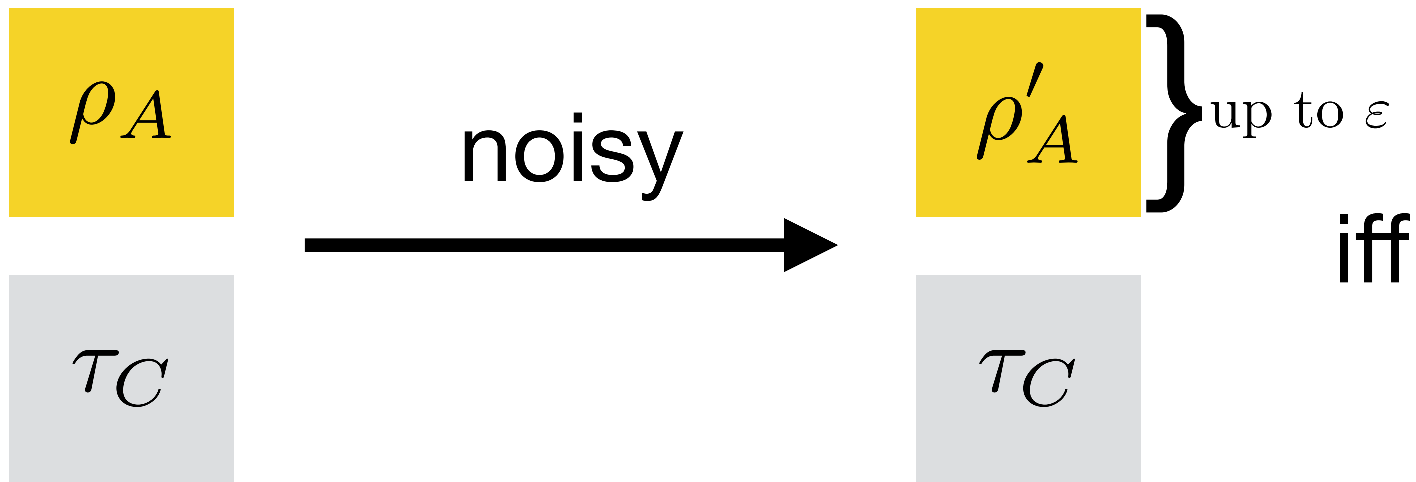

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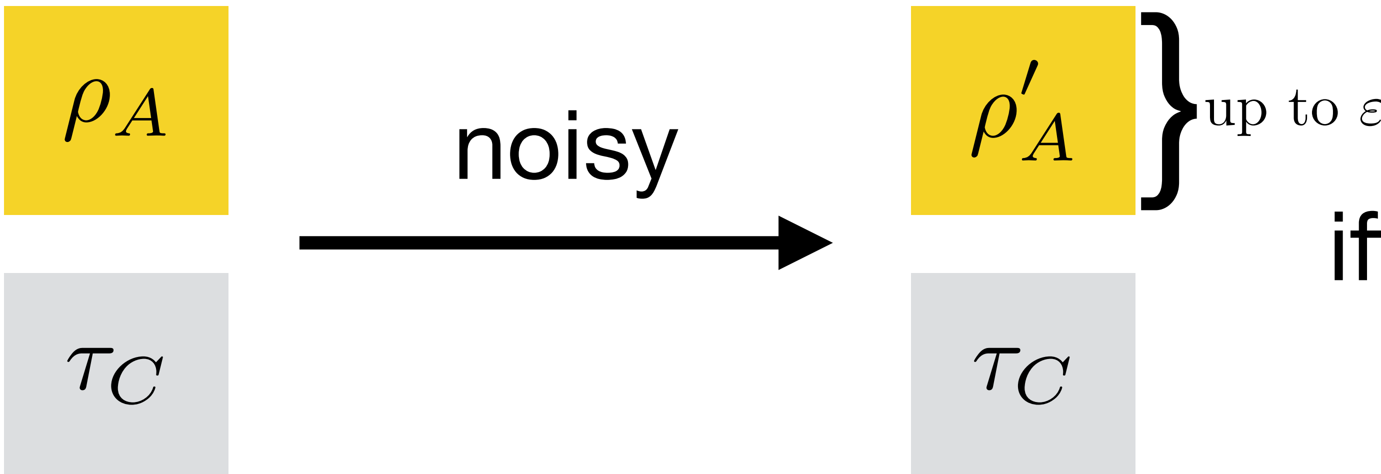

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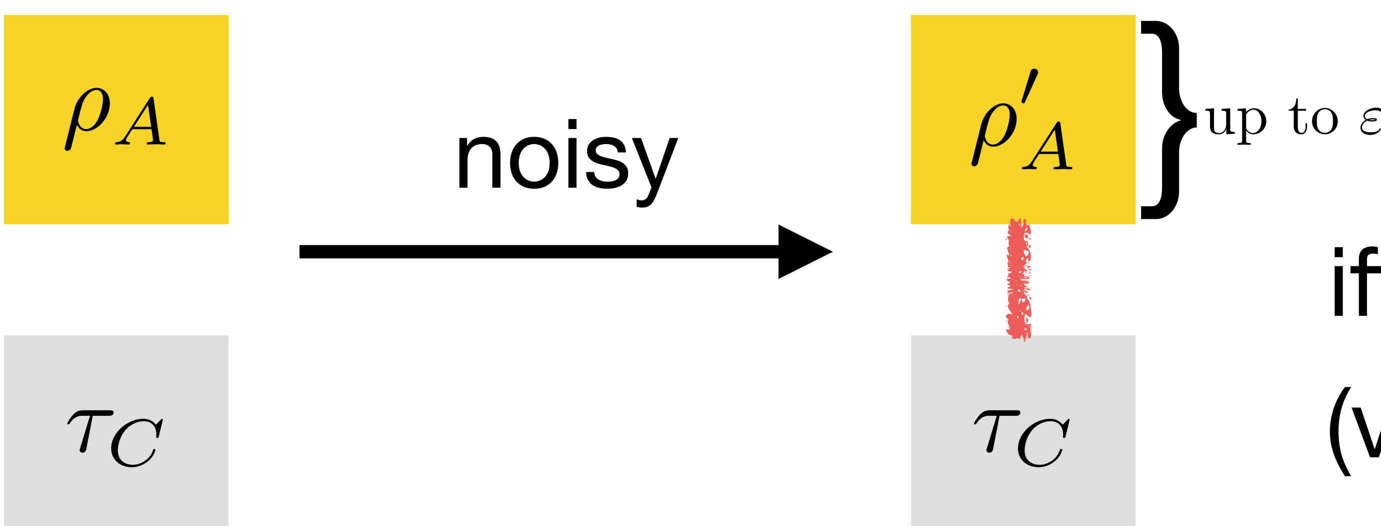
Our result implies:



iff $H(\rho_A) \leq H(\rho'_A)$

(von Neumann entropy!)

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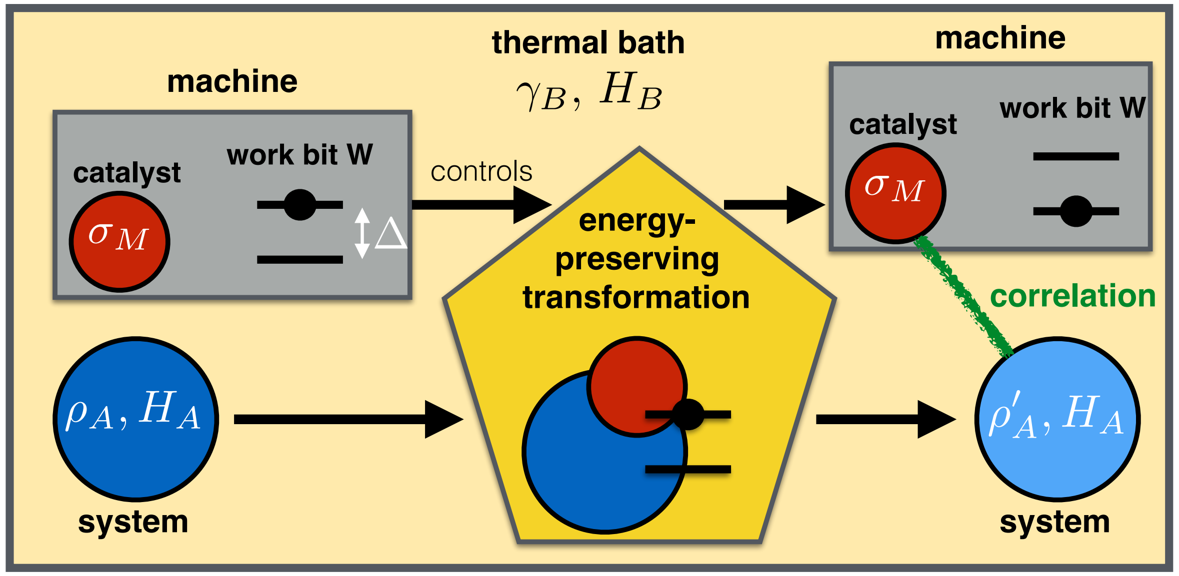

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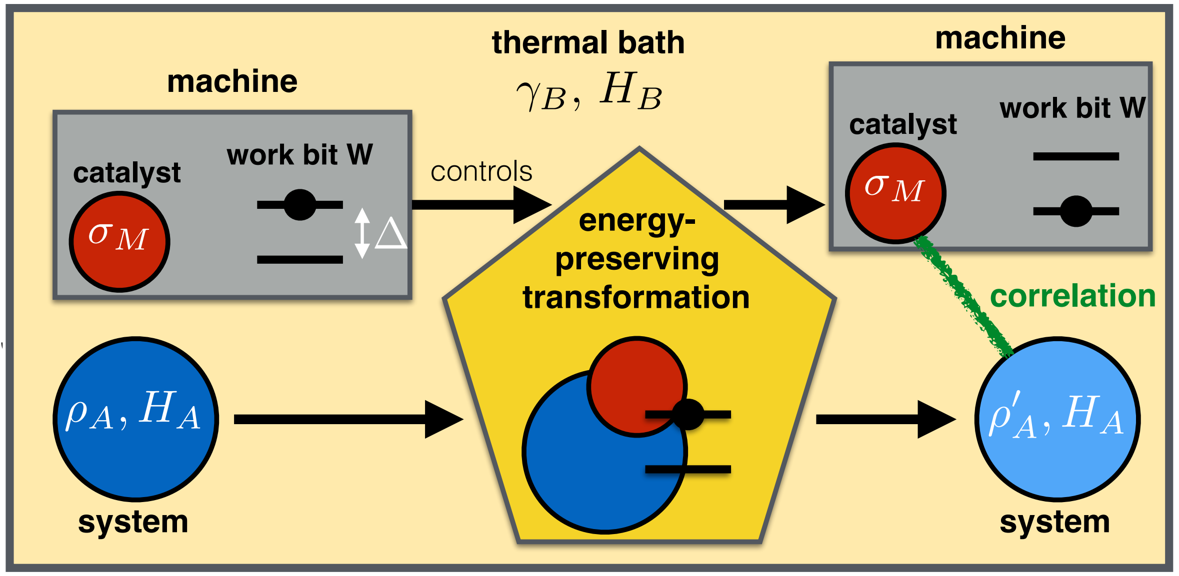

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Implications for quantum thermodynamics

The second laws of quantum thermodynamics

Fernando Brandão[a,1], Michał Horodecki[b], Nelly Ng[c], Jonathan Oppenheim[c,d,2], and Stephanie Wehner[c,e]

[a]Department of Computer Science, University College London, London WC1E 6BT, United Kingdom; [b]Instytut Fizyki Teoretycznej i Astrofizyki, University of Gdansk, 80-952 Gdansk, Poland; [c]Centre for Quantum Technologies, National University of Singapore, 117543 Singapore; [d]Department of Physics & Astronomy, University College London, London WC1E 6BT, United Kingdom; and [e]School of Computing, National University of Singapore, 117417 Singapor

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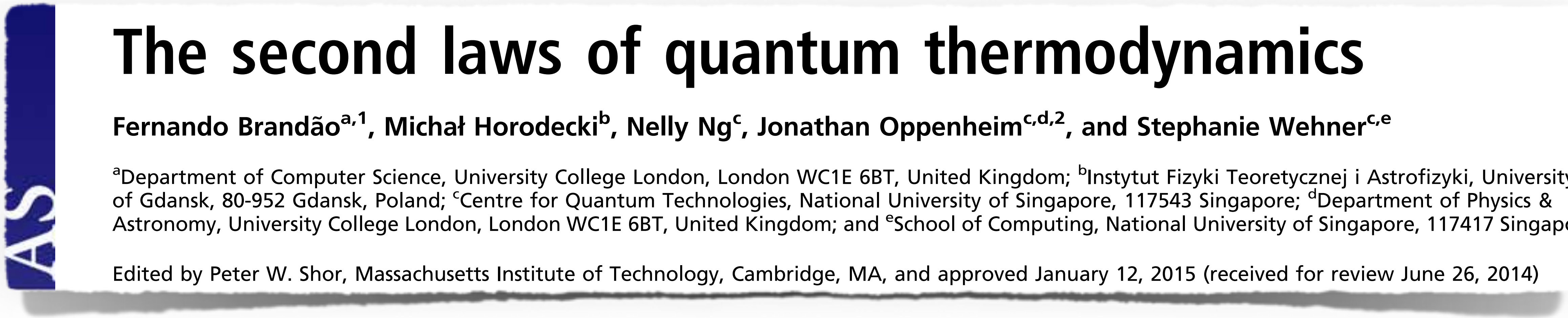
Quantum thermo for small&strongly correlated systems: formulate as a **resource theory**.



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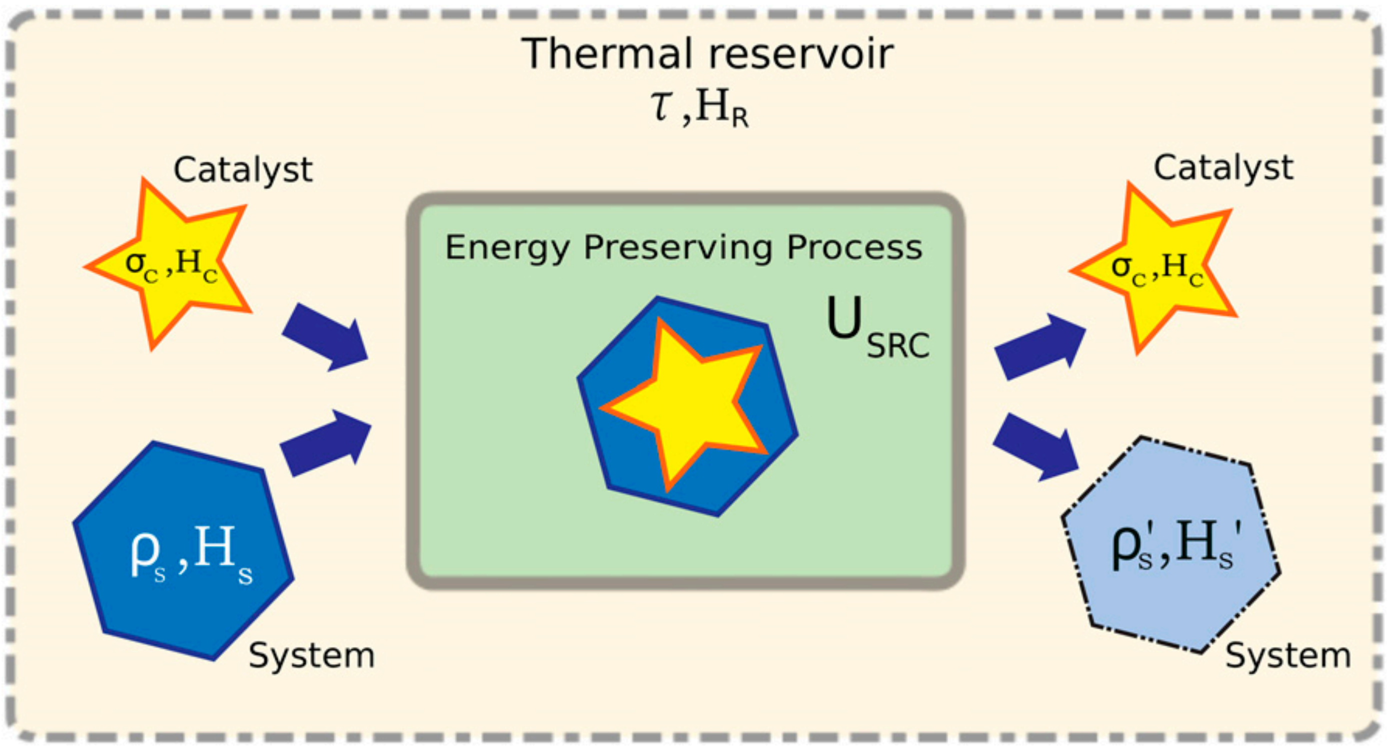

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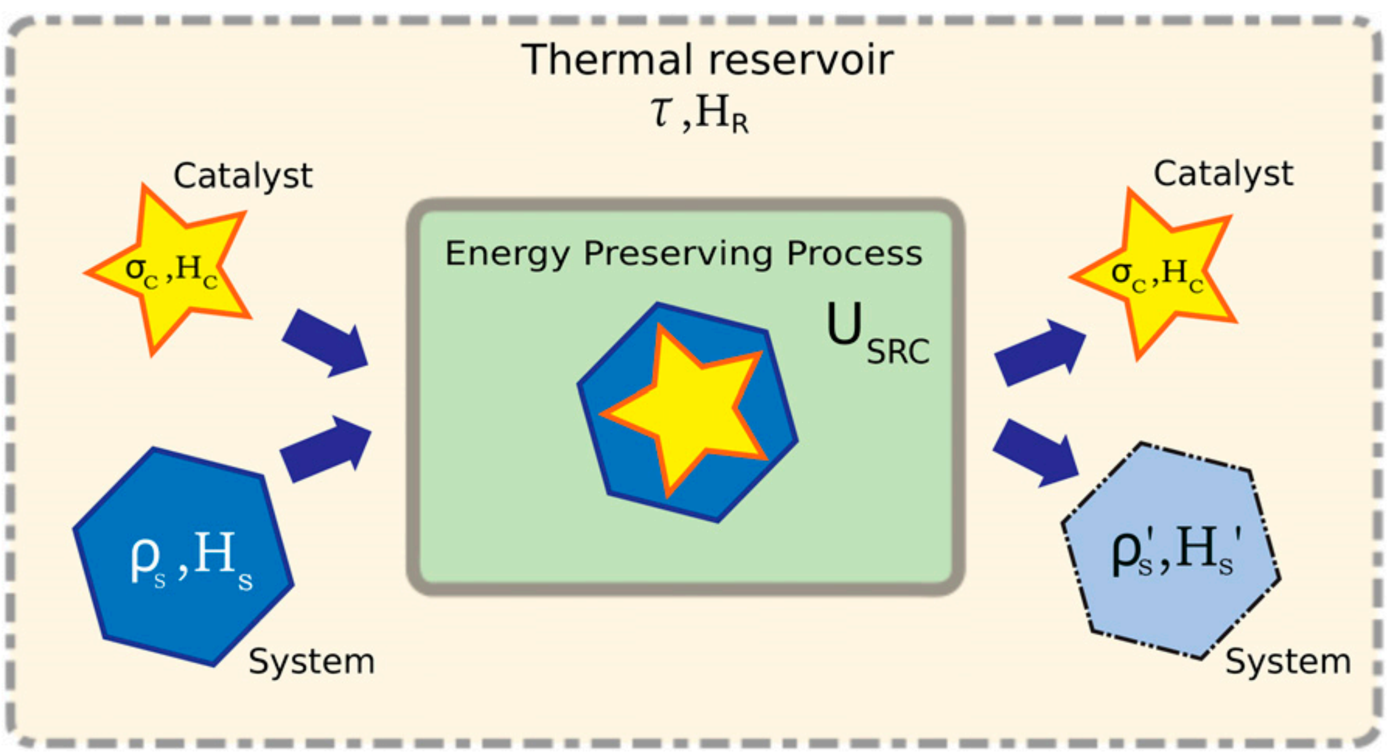

$$\mathrm{Tr}_R\left[U_{SRC}(\rho_S\otimes\sigma_C\otimes\gamma_R)U_{SRC}^\dagger\right]$$
$$=\rho'_S\otimes\sigma_C$$

The second laws of quantum thermodynamics

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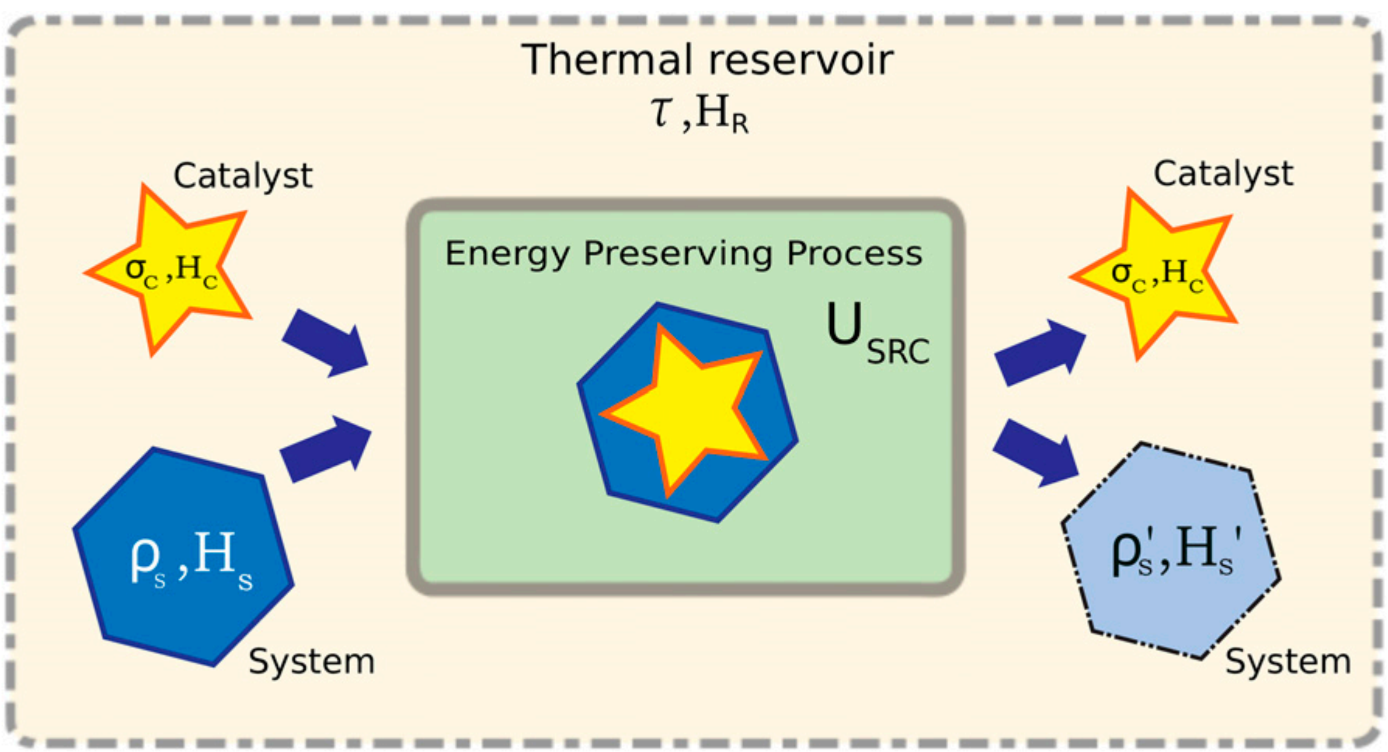

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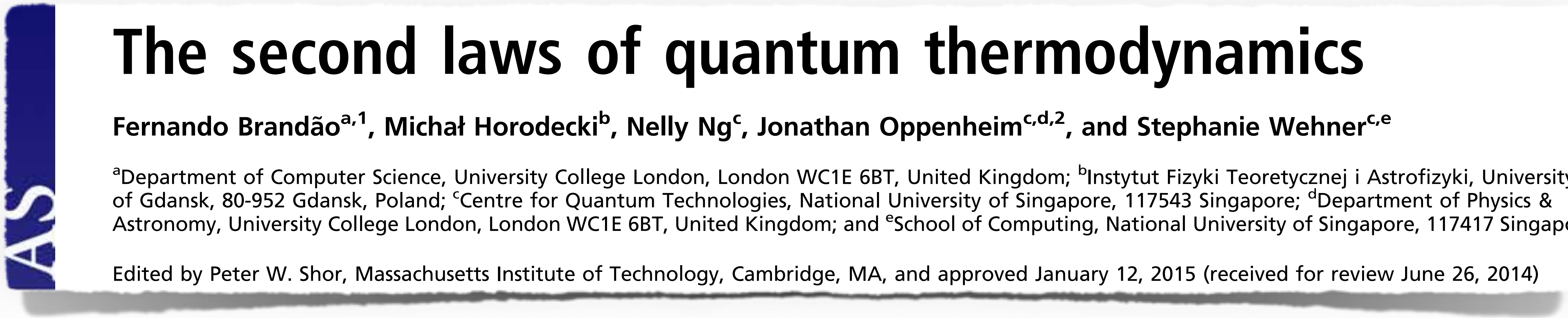
thermal reservoir

Implications for quantum thermodynamics



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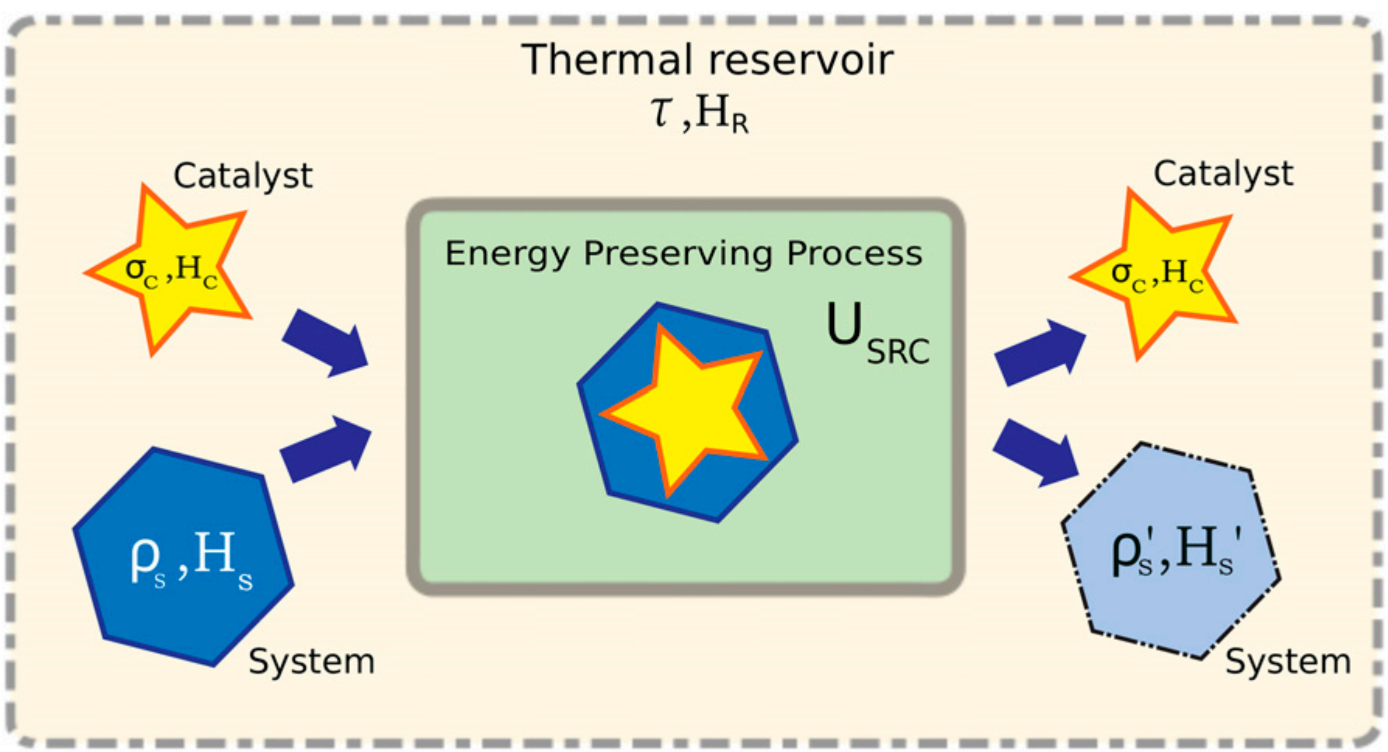

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thermal reservoir

$$[U_{SRC},H_S+H_R+H_C]=0$$

(energy strictly preserved)

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[a]Department of Computer Science, University College London, London WC1E 6BT, United Kingdom; [b]Instytut Fizyki Teoretycznej i Astrofizyki, University of Gdansk, 80-952 Gdansk, Poland; [c]Centre for Quantum Technologies, National University of Singapore, 117543 Singapore; [d]Department of Physics & Astronomy, University College London, London WC1E 6BT, United Kingdom; and [e]School of Computing, National University of Singapore, 117417 Singapor

Edited by Peter W. Shor, Massachusetts Institute of Technology, Cambridge, MA, and approved January 12, 2015 (received for review June 26, 2014)

Theorem: $\rho \to \rho'$ is possible (for block-diagonal states) iff

$$F_\alpha(\rho) \geq F_\alpha(\rho') \ \ \forall \alpha$$ ("free energies").

Implications for quantum thermodynamics

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$$F(\rho_A) \equiv F_1(\rho) = \mathrm{tr}(\rho_A H_A) - k_B T S(\rho_A),$$

$$F_\alpha(\rho) = k_B T S_\alpha(\rho \| \gamma) + F_\alpha(\gamma).$$

Rényi divergence

Implications for quantum thermodynamics

MM, *Correlating thermal machines and the second law at the nanoscale*, arXiv:1707.03451

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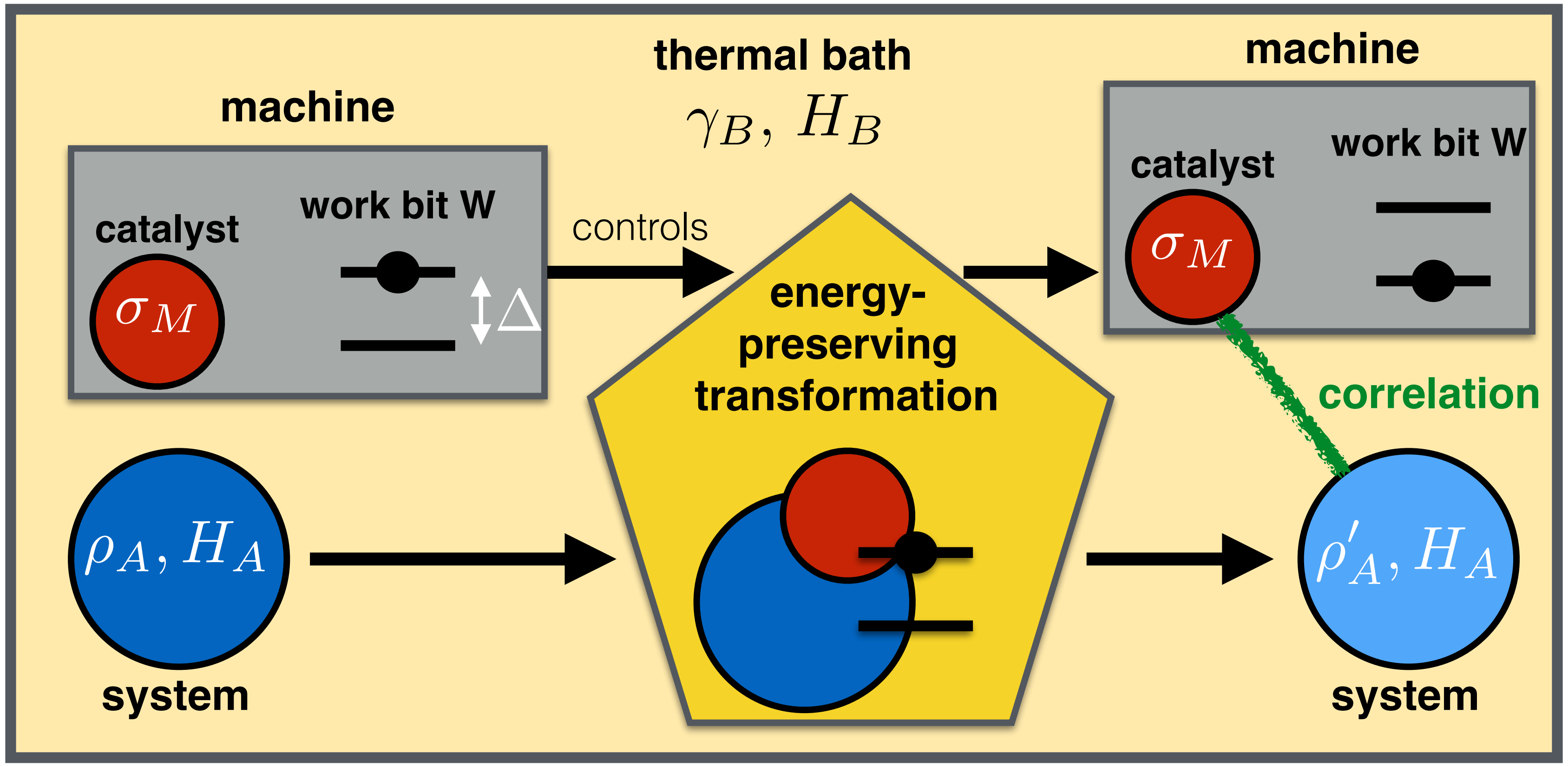

Preserve catalyst exactly, but allow **correlations** to build up.

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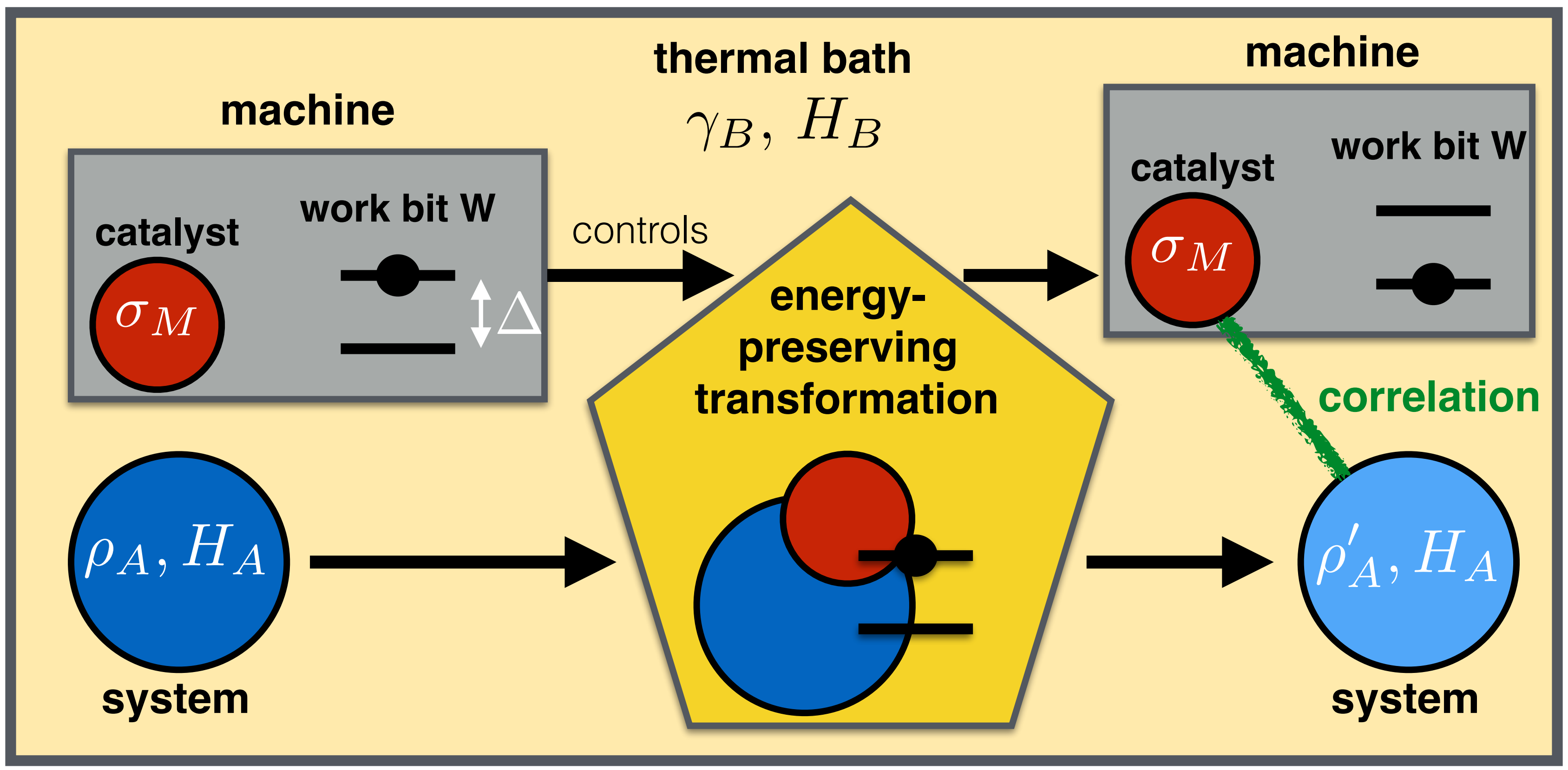

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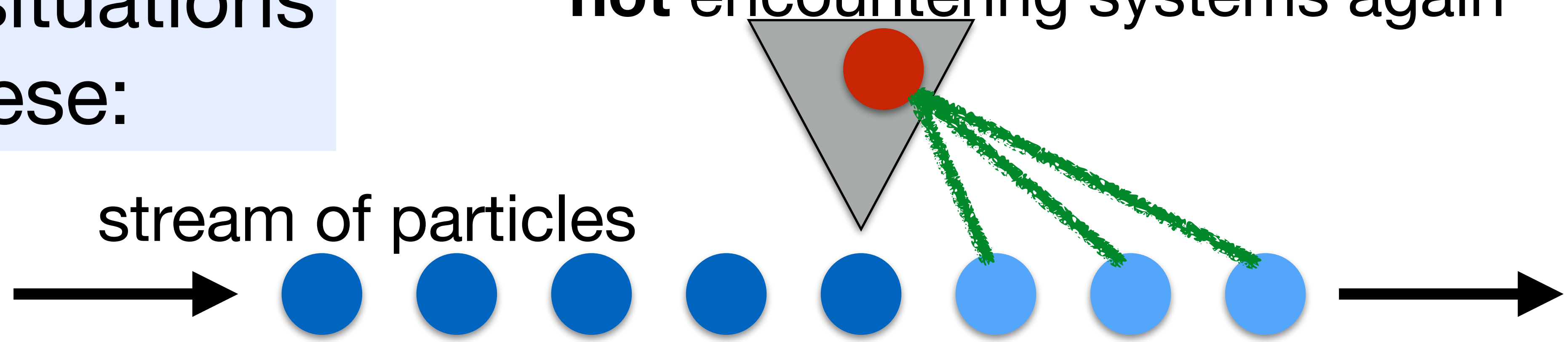
Applies to situations like these:

thermal machine,
acting single-shot,
not encountering systems again



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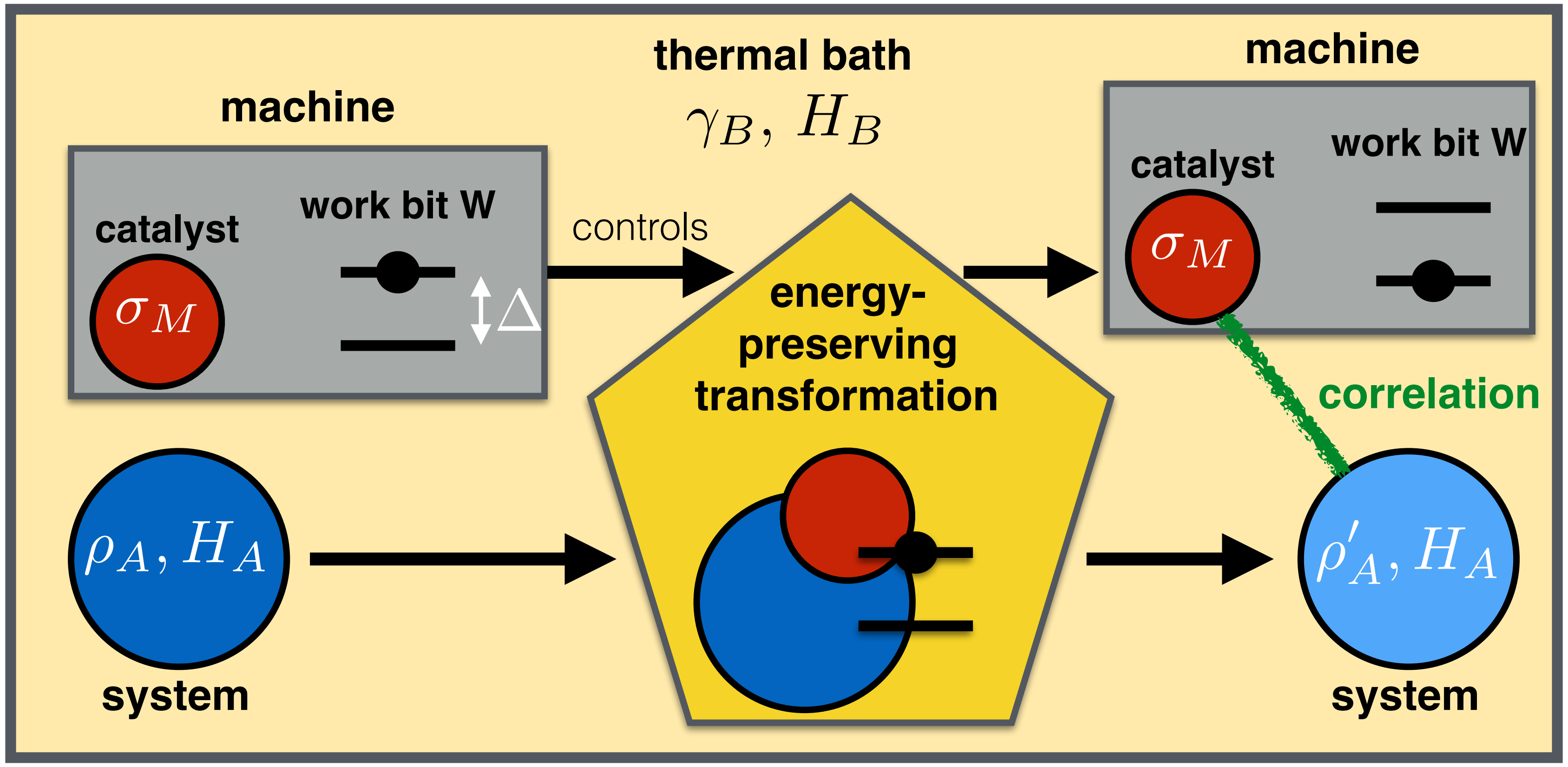

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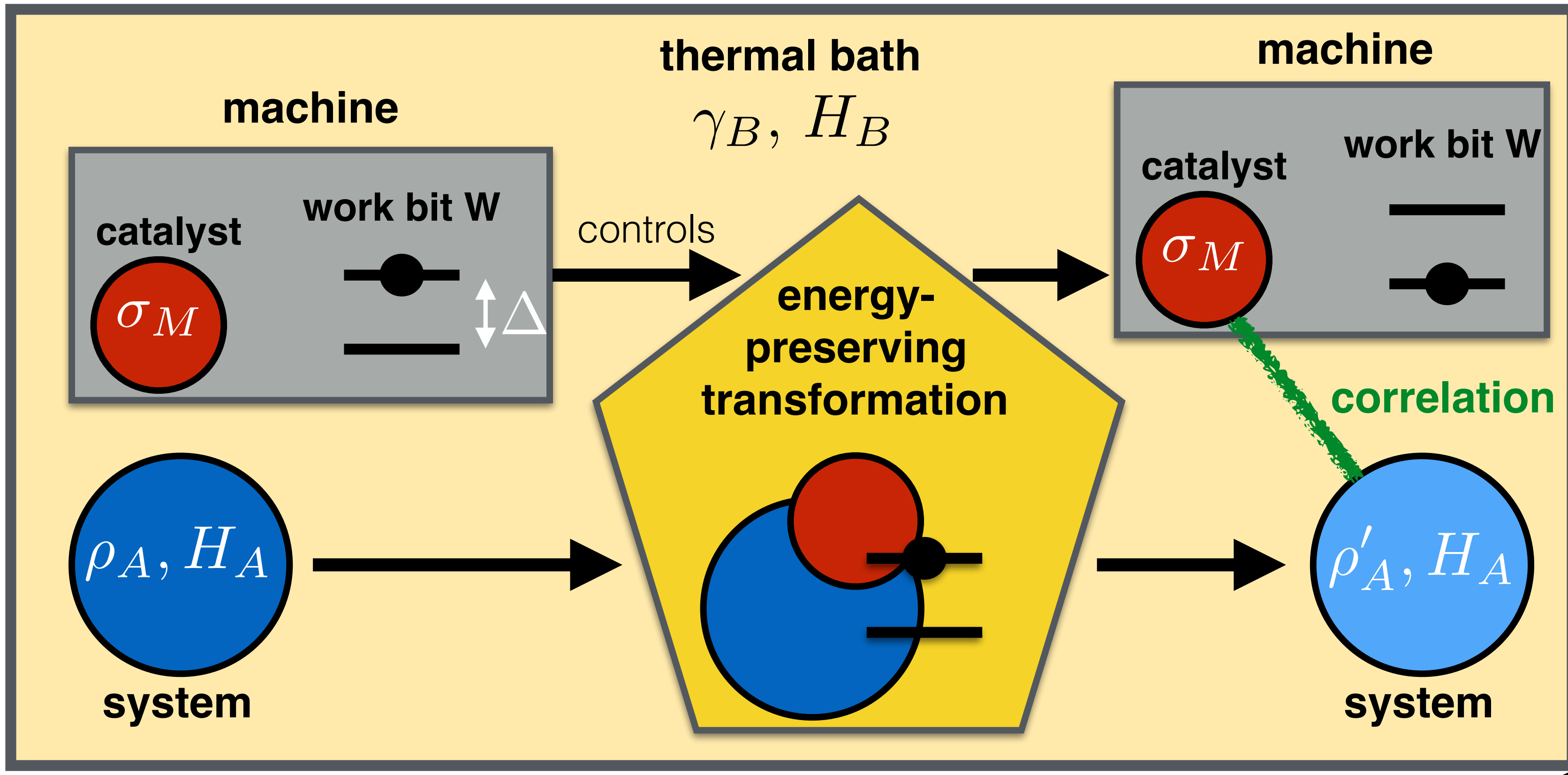

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to arbitrary accuracy $\varepsilon > 0$.

Theorem: $\rho \to \rho'$ is possible (for block-diagonal states) iff $F(\rho) \geq F(\rho')$ (if work bit not involved).

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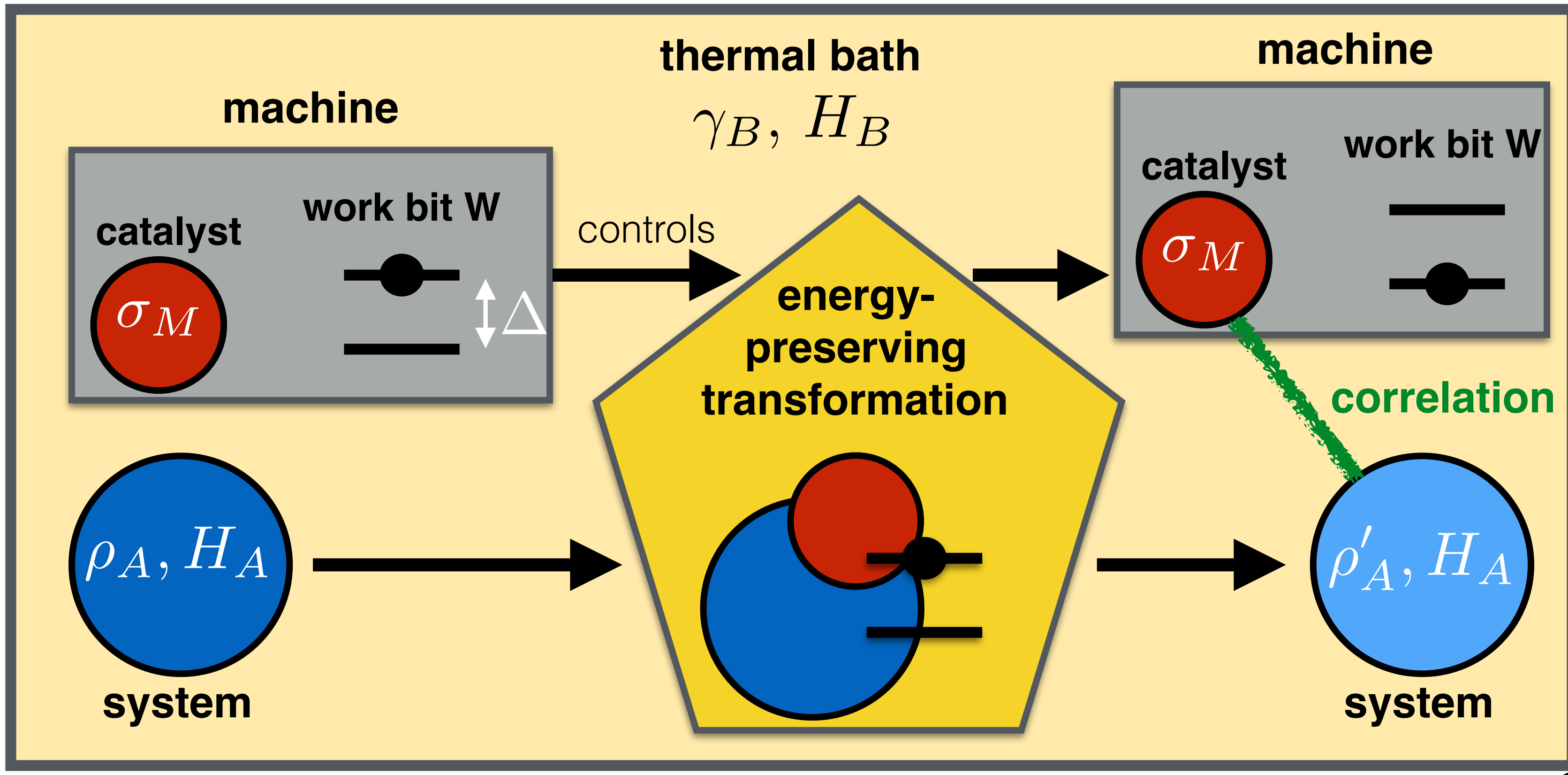

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One-shot operational interpretation of Helmholtz free energy!

Second laws ⟶ second law!

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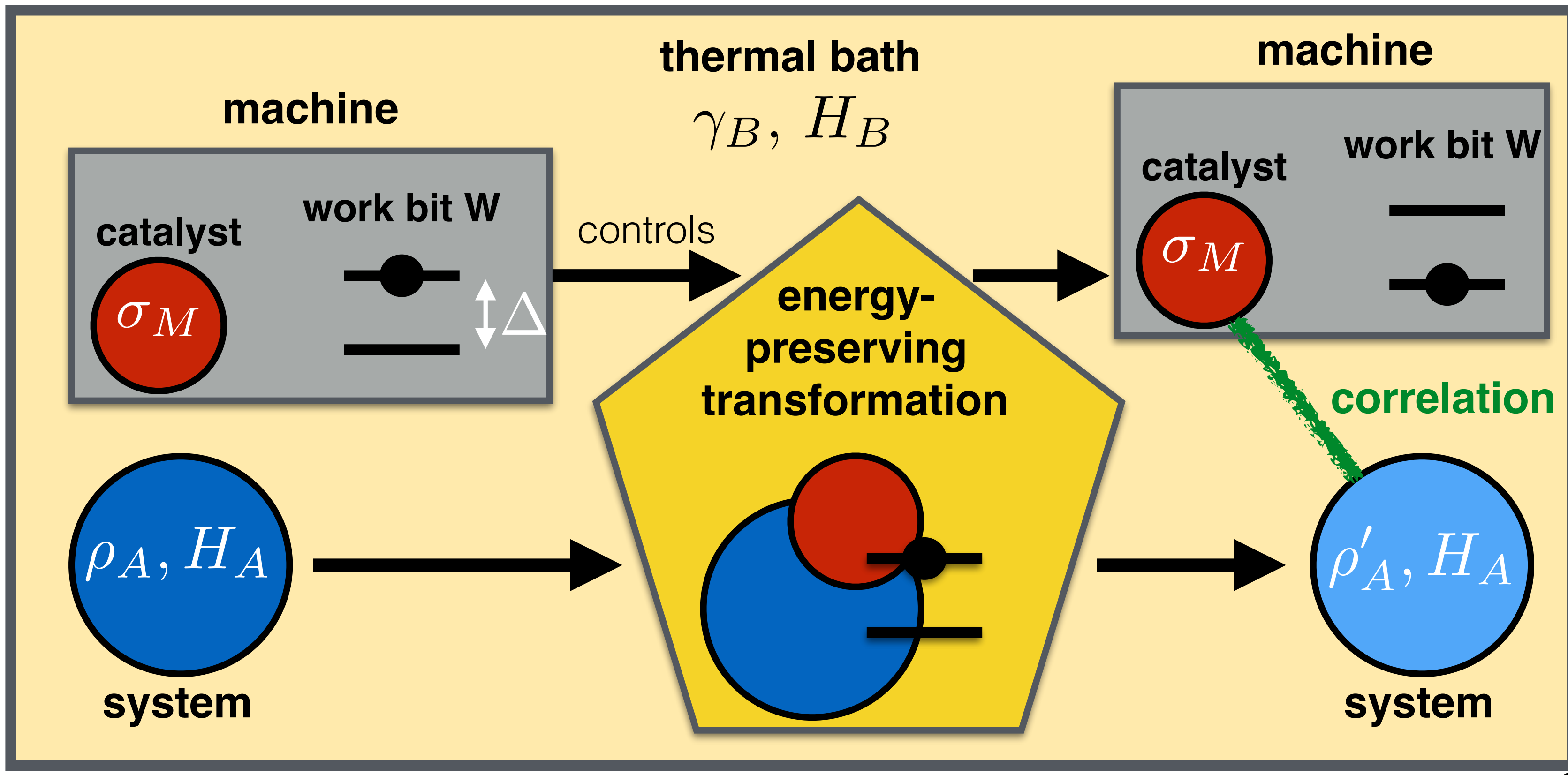

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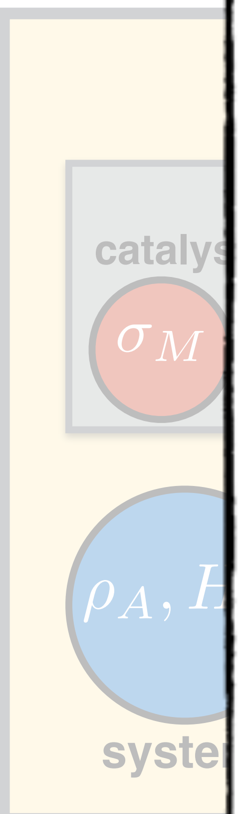
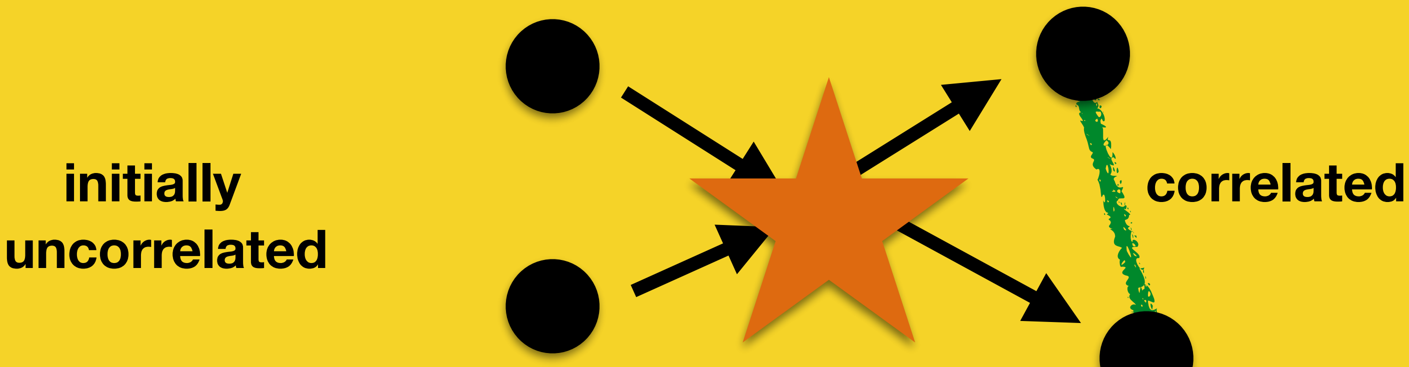


In spirit somewhat close to Boltzmann's original **Stoßzahlansatz** (1896):



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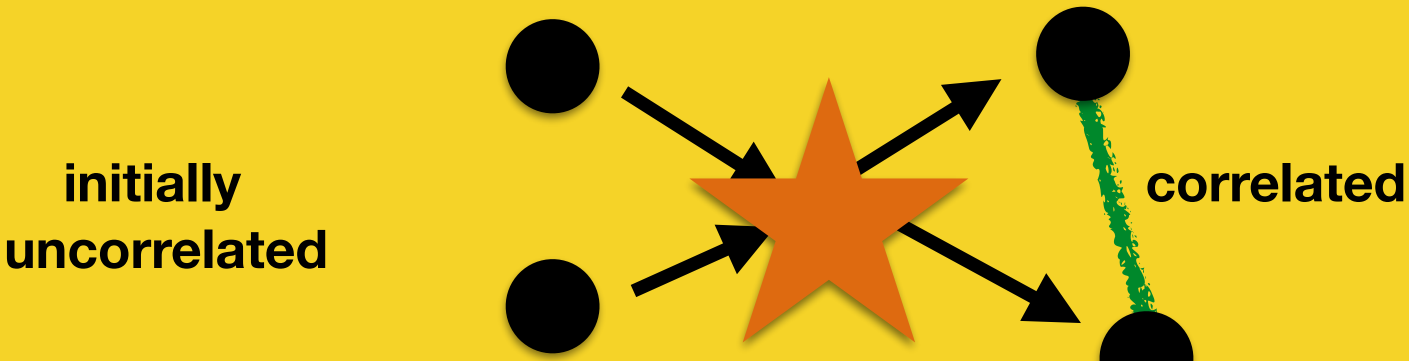

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Theore... ...onal states) iff ... involved).

Conjecture. Result also true in the presence of quantum coherence.

One-sh... ...ltz free energy!

Extractable work / work of formation…

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… are both in general random variables.

ARTICLE

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Michał Horodecki[1,*] & Jonathan Oppenheim[2,3,*]

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If we want reliability (success prob. $\geq 1-\varepsilon$), then

- extractable work: $F_0^\varepsilon(\rho) - F(\gamma)$
- work of formation: $F_\infty^\varepsilon(\rho) - F(\gamma)$

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… are both in general random variables.

$$\lim_{n\to\infty}\frac{1}{n}F^{\varepsilon}_{0/\infty}(\rho^{\otimes n})=F(\rho).$$

Work characterized by *F* only in the thermodynamic limit.

If we want reliability (success prob. $\geq 1-\varepsilon$), then

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- work of formation: $F_\infty^{\varepsilon}(\rho)-F(\gamma)$

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Work of formation (when allowing correlations)…

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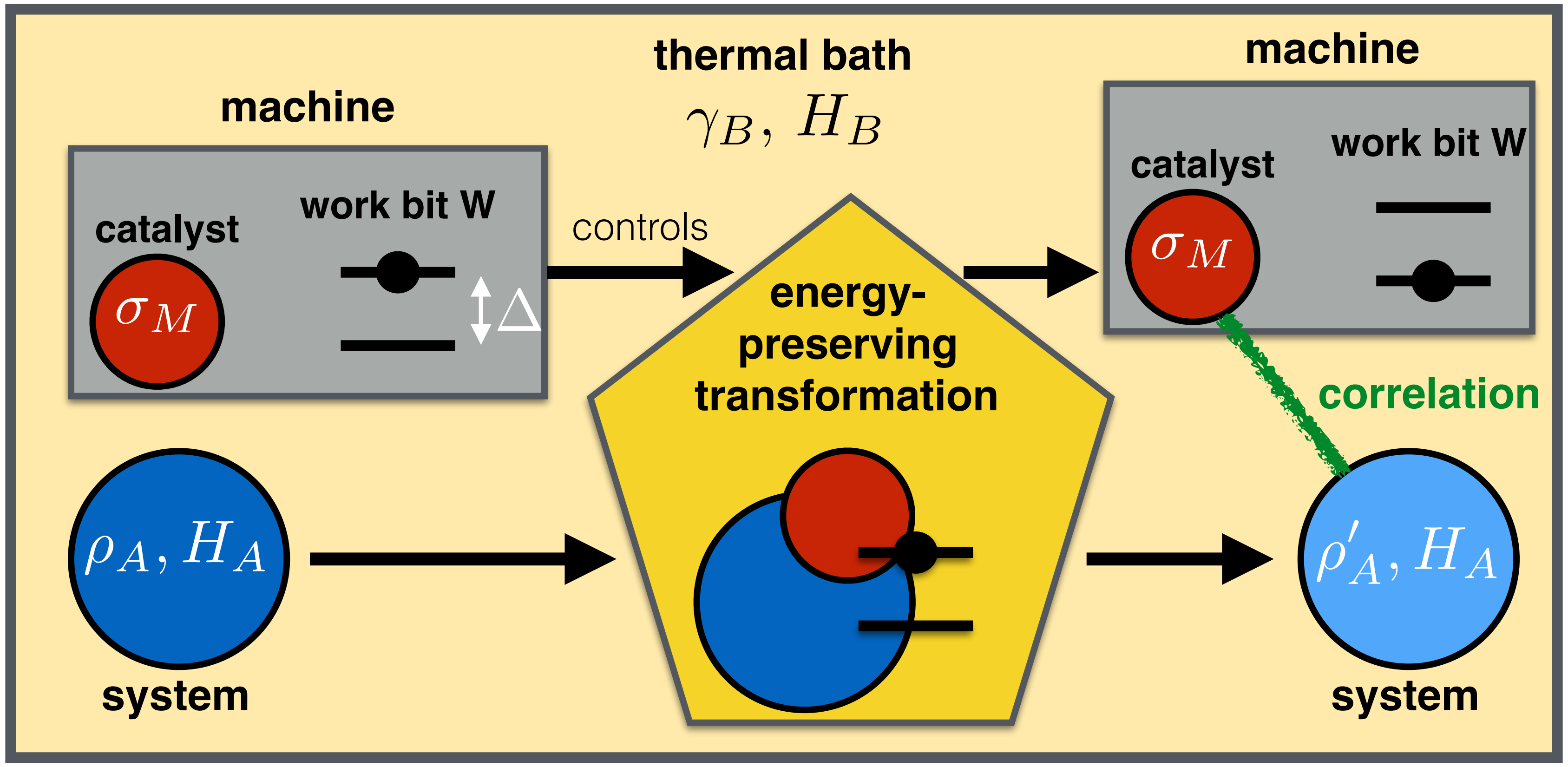

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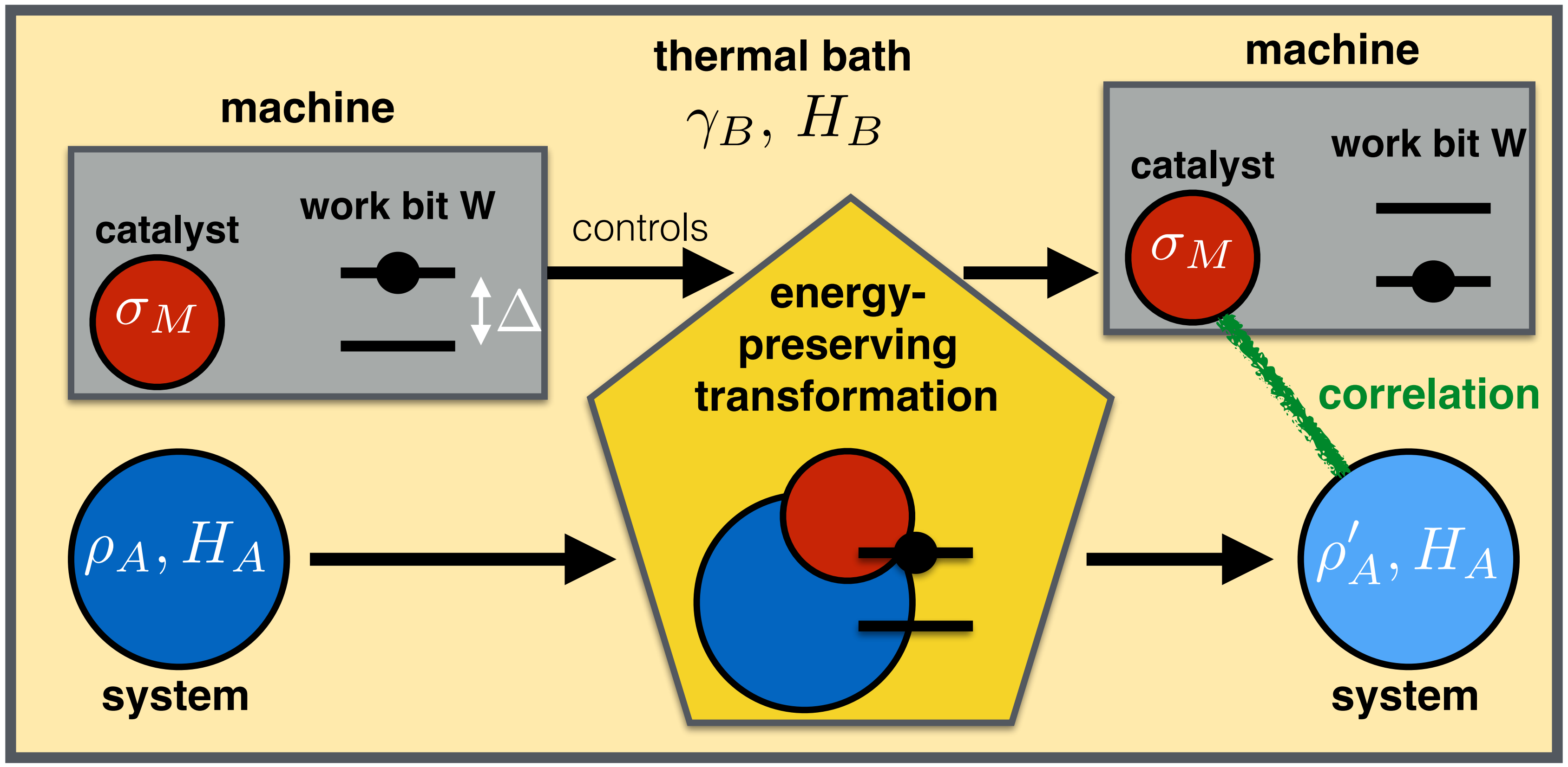

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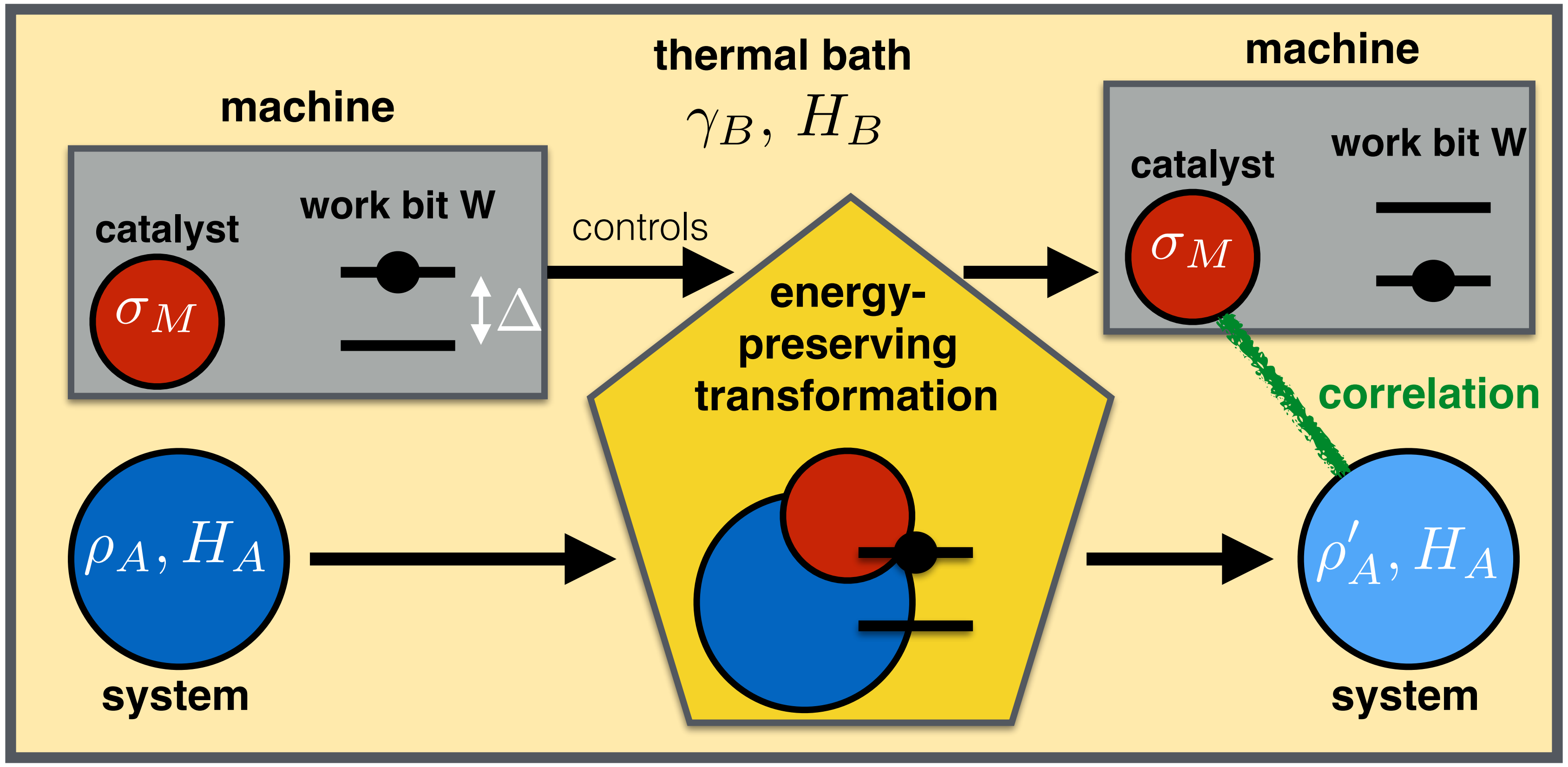

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$$\rho_A \otimes \sigma_M \otimes |e\rangle\langle e|_W$$
$$\mapsto$$
$$\sigma_{AM} \otimes |g\rangle\langle g|_W$$

Theorem. *Fix any initial state ρ_A and target state ρ'_A, both block-diagonal, such that $F(\rho'_A) \geq F(\rho_A)$. Using a work bit W with some energy gap Δ larger than, but arbitrarily close to $F(\rho'_A) - F(\rho_A)$, the transition*

$$\rho_A \otimes \sigma_M \otimes |e\rangle\langle e|_W \mapsto \sigma_{AM} \otimes |g\rangle\langle g|_W$$

can be achieved by a thermal operation, where $\sigma_A := \mathrm{Tr}_M \sigma_{AM}$ is arbitrarily close to ρ'_A.

The state σ_M is exactly identical before and after the transformation, M is finite-dimensional, and the resulting correlations between A and M can be made arbitrarily small.

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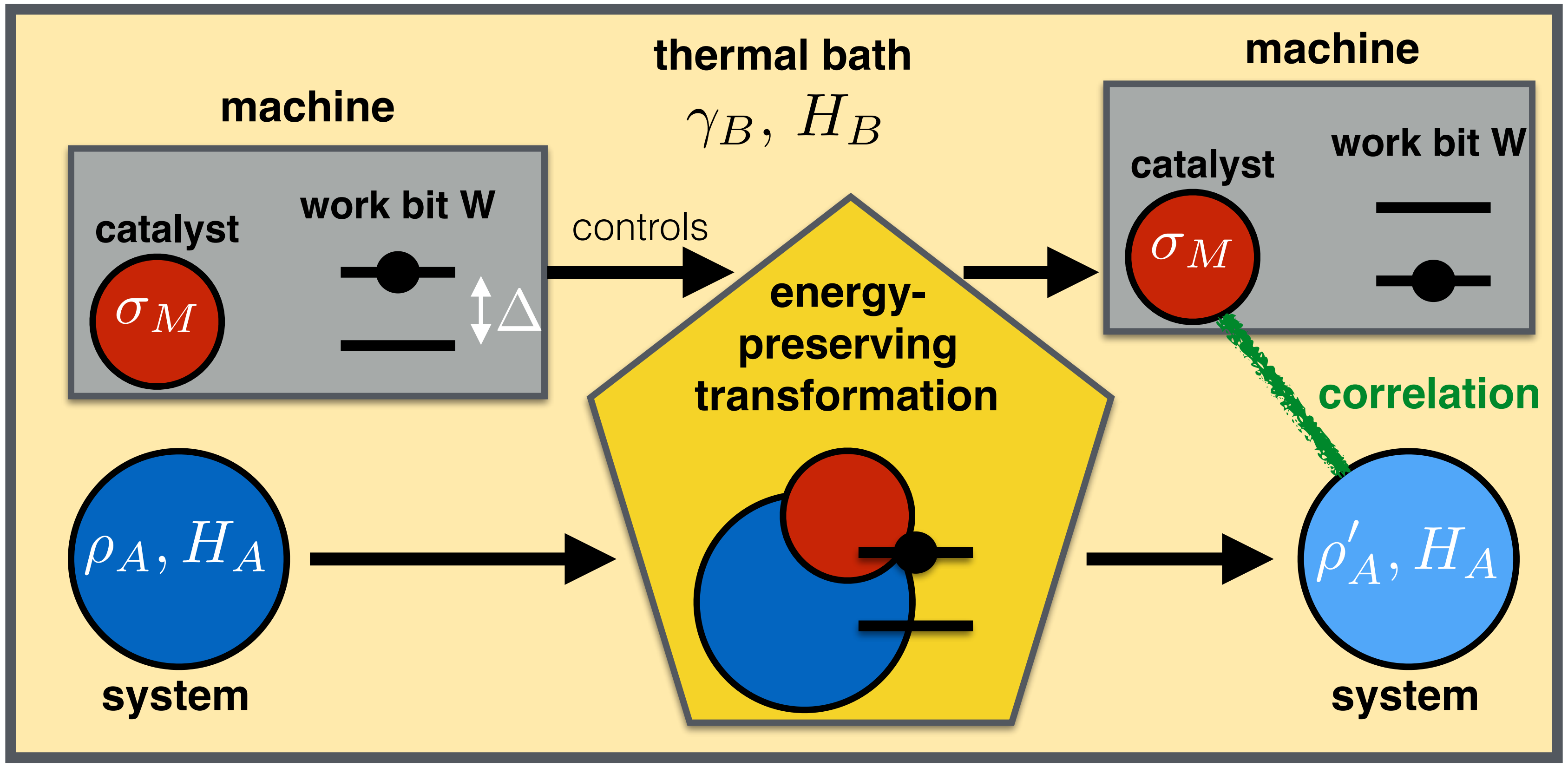

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can be achie… σ_A as close to ρ'_A as you like, *…rily close to* ρ'_A.

The sta… … finite-dimensional, and the resulting correlatio… σ_M exactly preserved, $\dim M < \infty$.

Extractable work (when allowing correlations)…

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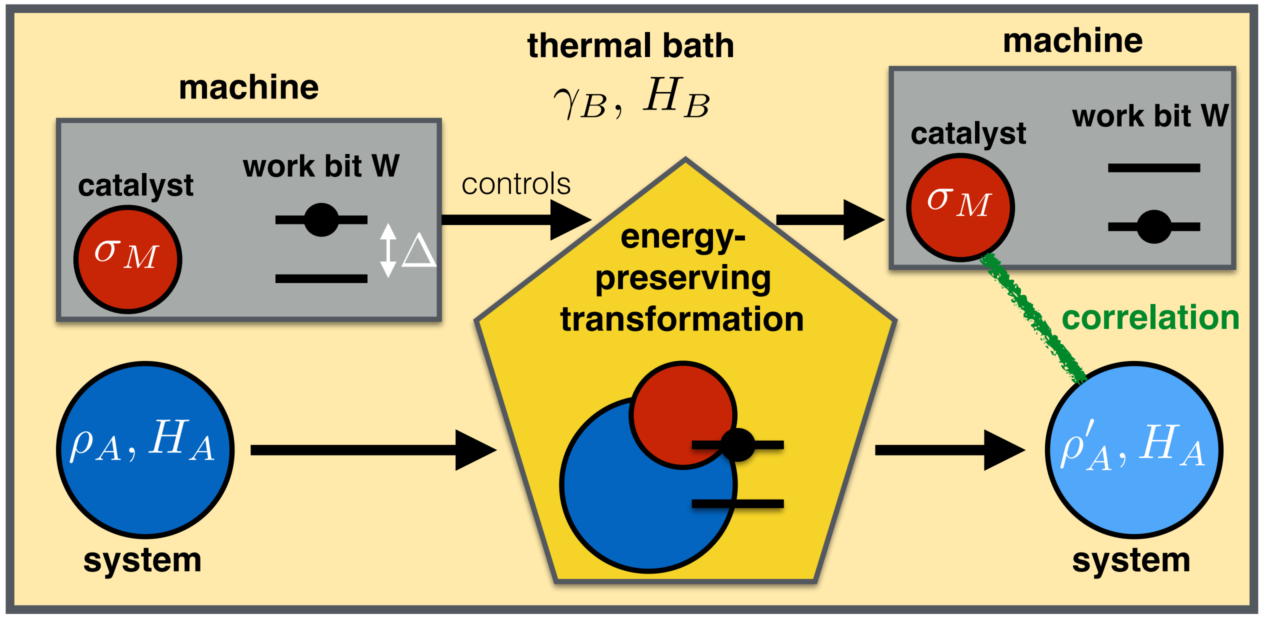

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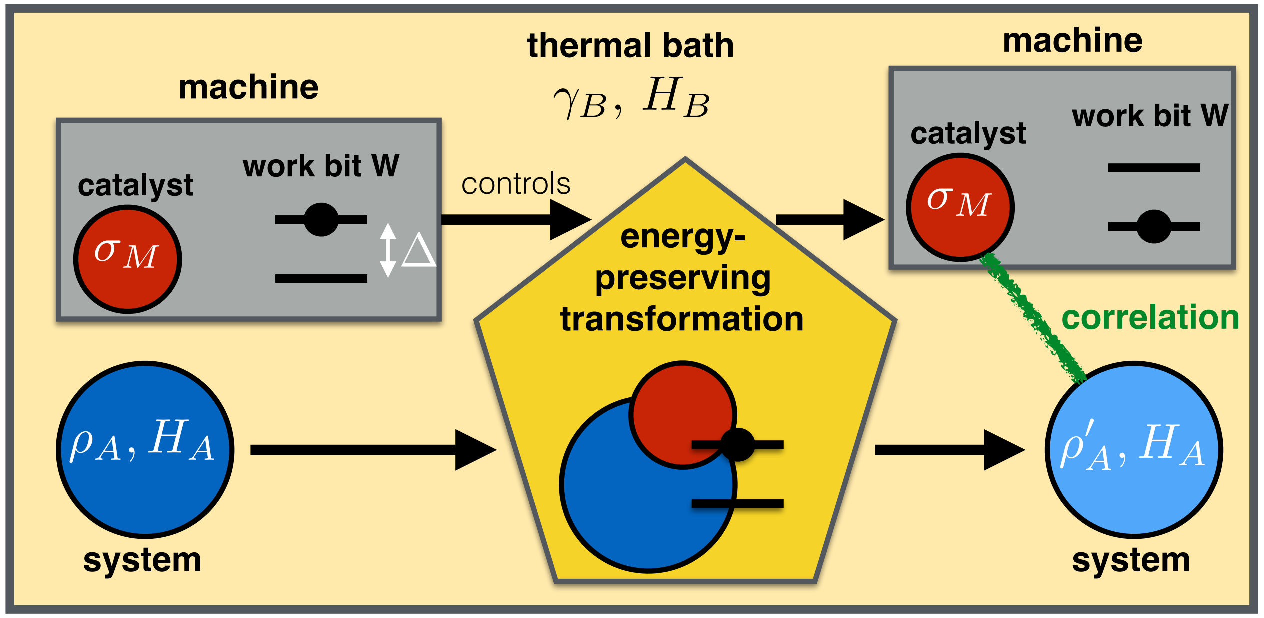

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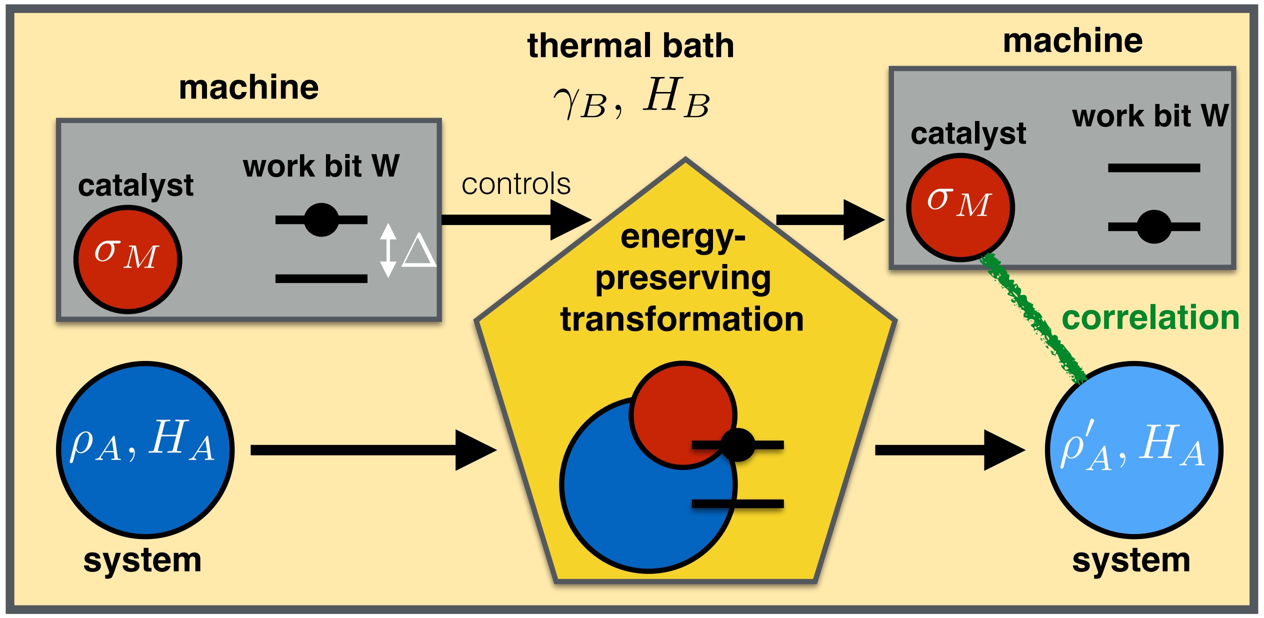

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$$\rho_A \otimes \sigma_M \otimes (1, 0, \ldots, 0) \otimes |g\rangle\langle g|_W$$

$$\big\downarrow$$

$$\sigma_{AM} \leftrightarrow (1-\varepsilon, \varepsilon/n, \ldots, \varepsilon/n) \otimes |e\rangle\langle e|_W$$

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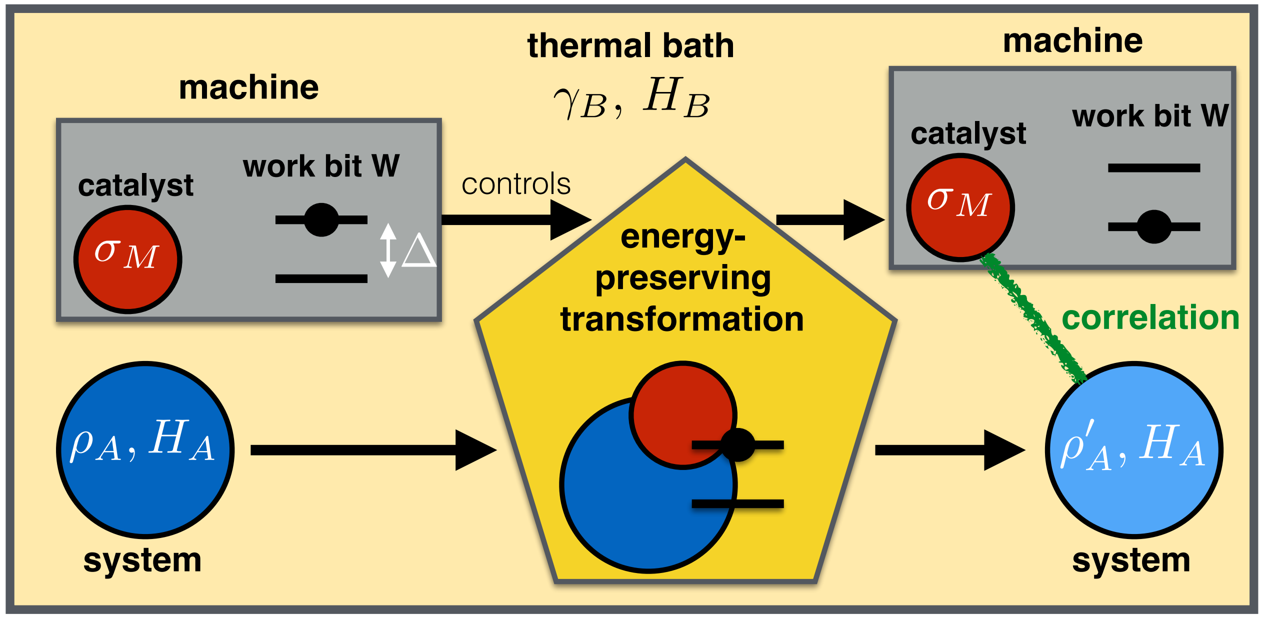

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$$\rho_A \otimes \sigma_M \otimes \tau_S^{(m,n)} \otimes |g\rangle\langle g|_W \mapsto \sigma_{AMS} \otimes |e\rangle\langle e|_W.$$

Here $\sigma_M = \mathrm{Tr}_{AS}\sigma_{AMS}$ *remains identical during the transformation,* $\sigma_S = \tau_S^{(m,n,\varepsilon)}$*, and* σ_A *is as close to* ρ'_A *as we like. This can be achieved for any choice of* $\varepsilon > 0$*, as long as* n/m *is large enough.*

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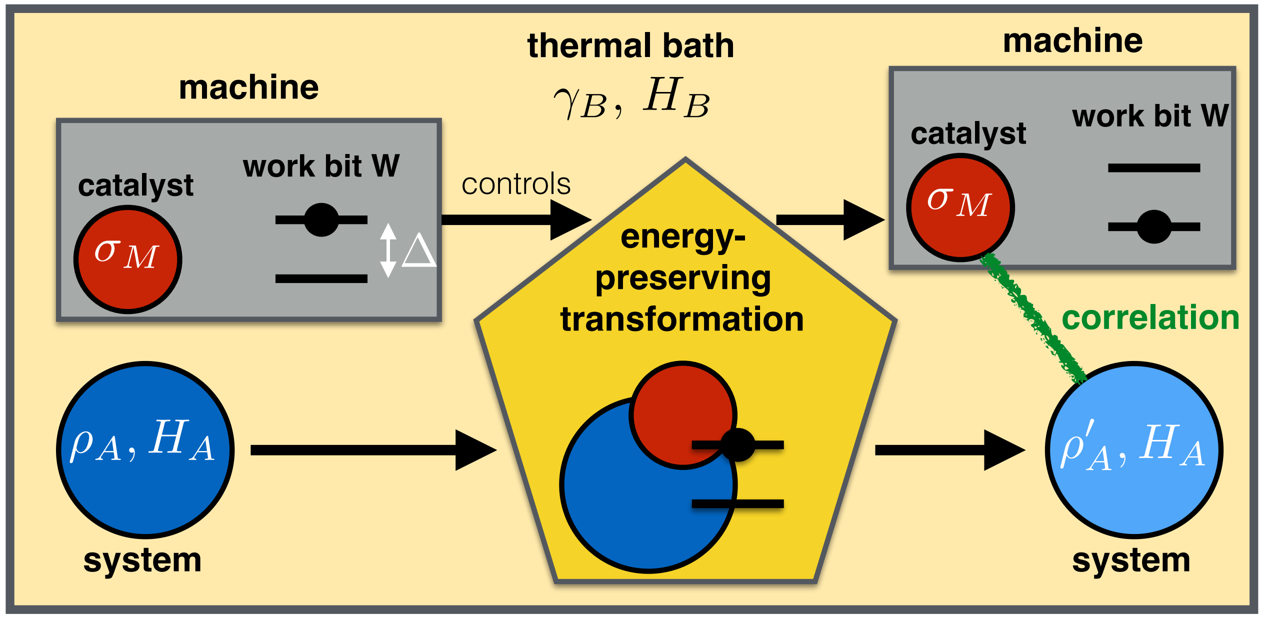

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I.e. can make fluctuations arbitrarily small (but not zero).

Stochastic independence as a resource

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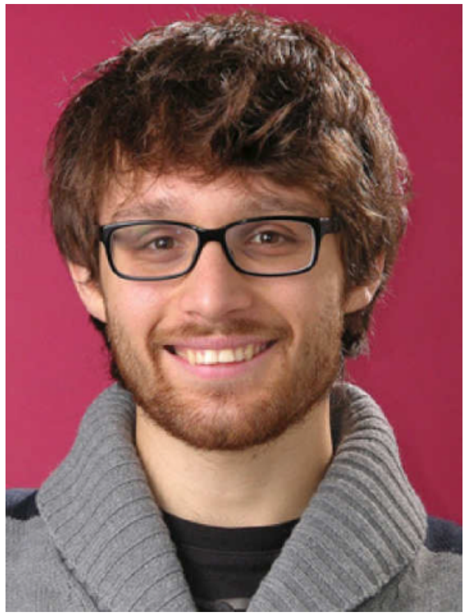
M. Lostaglio, MM, and M. Pastena, *Stochastic independence as a resource in small-scale thermodynamics*, Phys. Rev. Lett. **115**, 150402 (2015).



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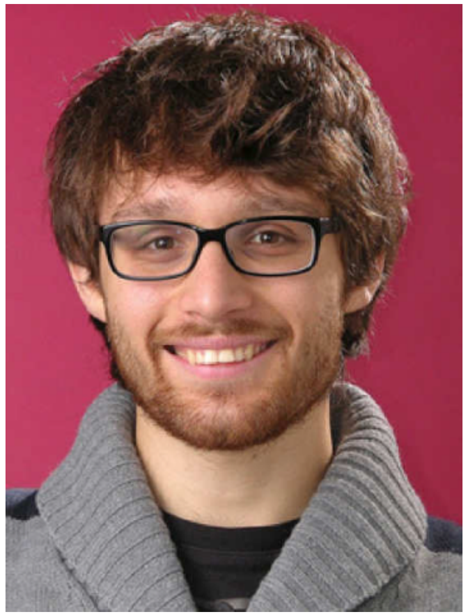

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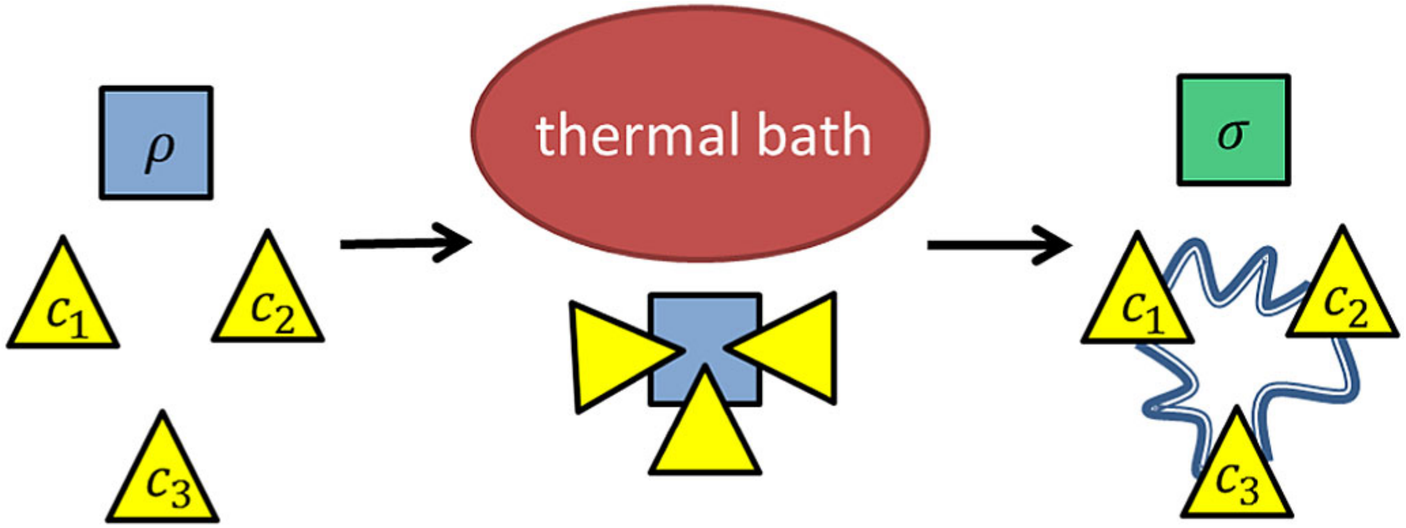

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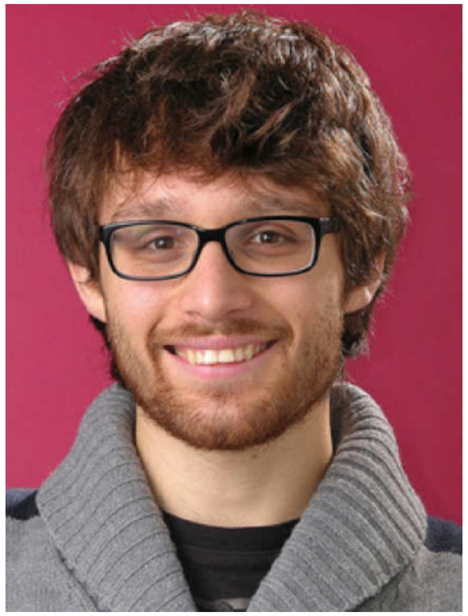

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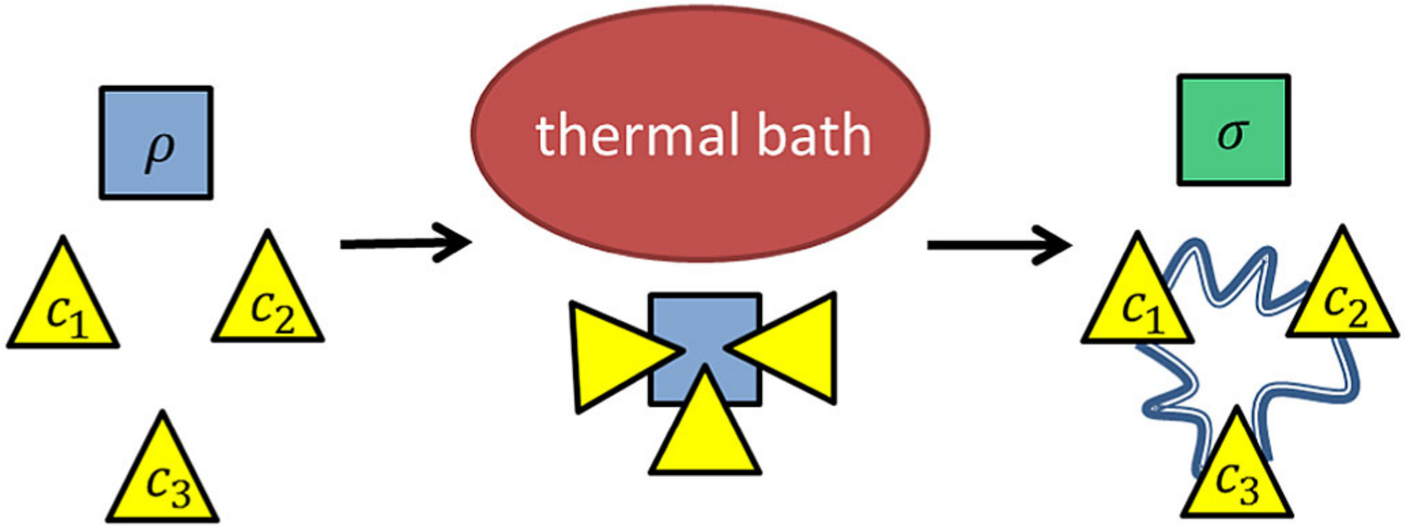

Correlating external systems can allow otherwise impossible state transitions.

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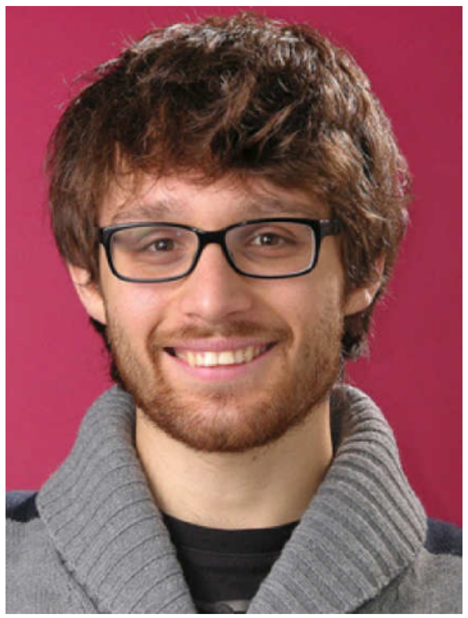

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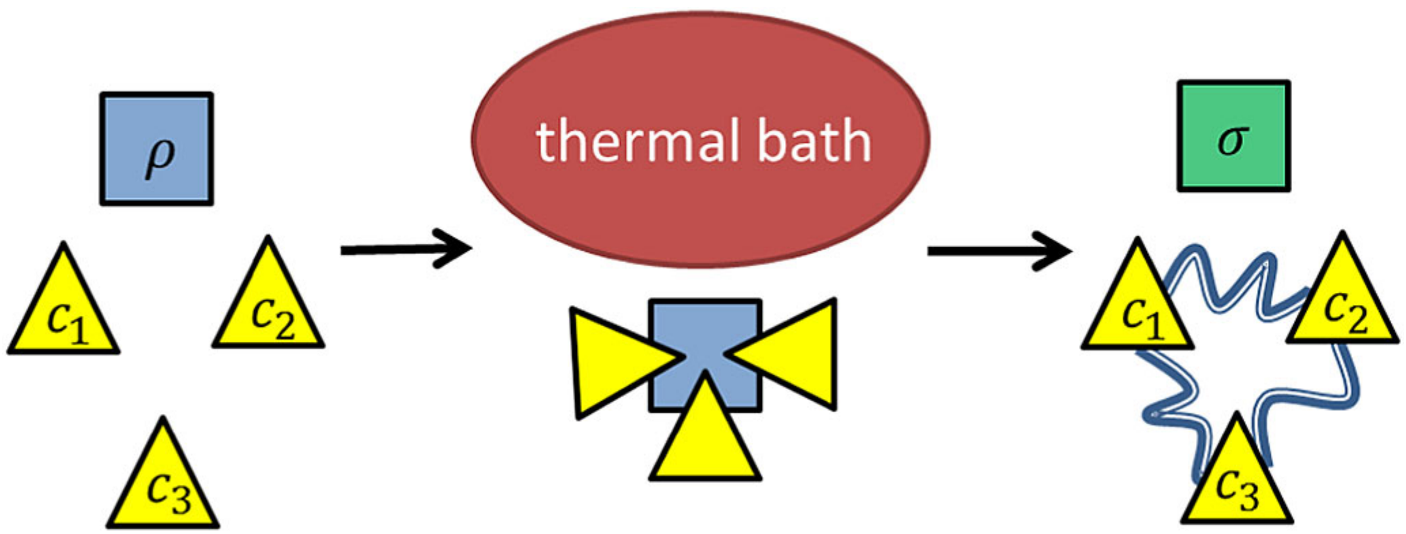

Correlating external systems can allow otherwise impossible state transitions.

The exact opposite of what one would expect from standard thermodynamics!

$$F(\rho_{AB}) \geq F(\rho_A \otimes \rho_B).$$

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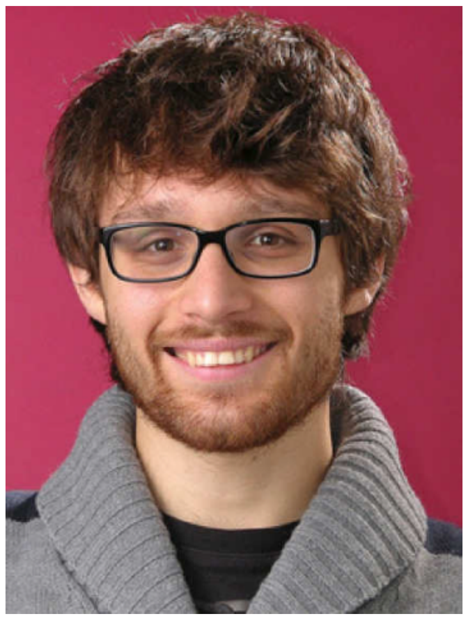

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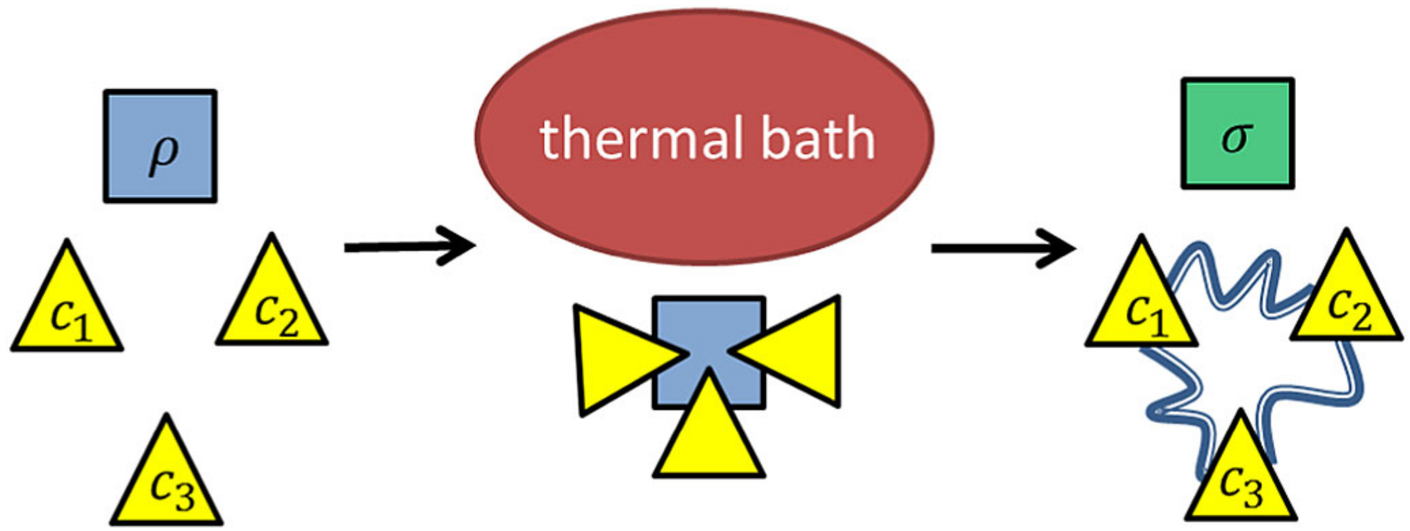

Correlating external systems can allow otherwise impossible state transitions. **“Trade fluctuations for correlations.”**

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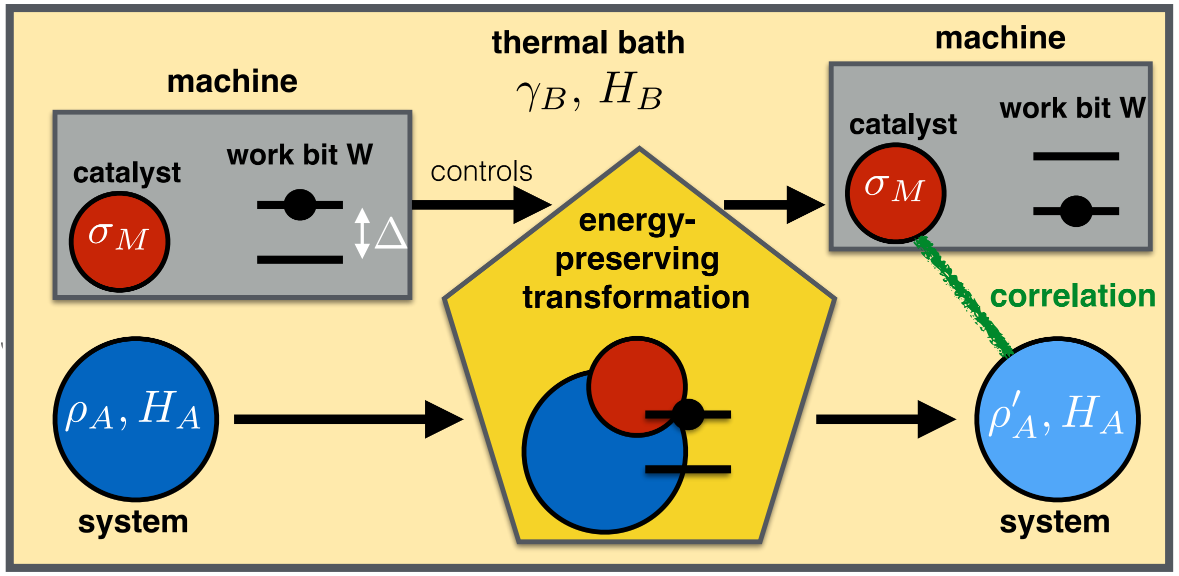
Outline

1. Intro: Majorization in quantum information

2. Main mathematical results

3. **Implications for quantum thermodynamic**



4. Implications for quantum information (in progress)

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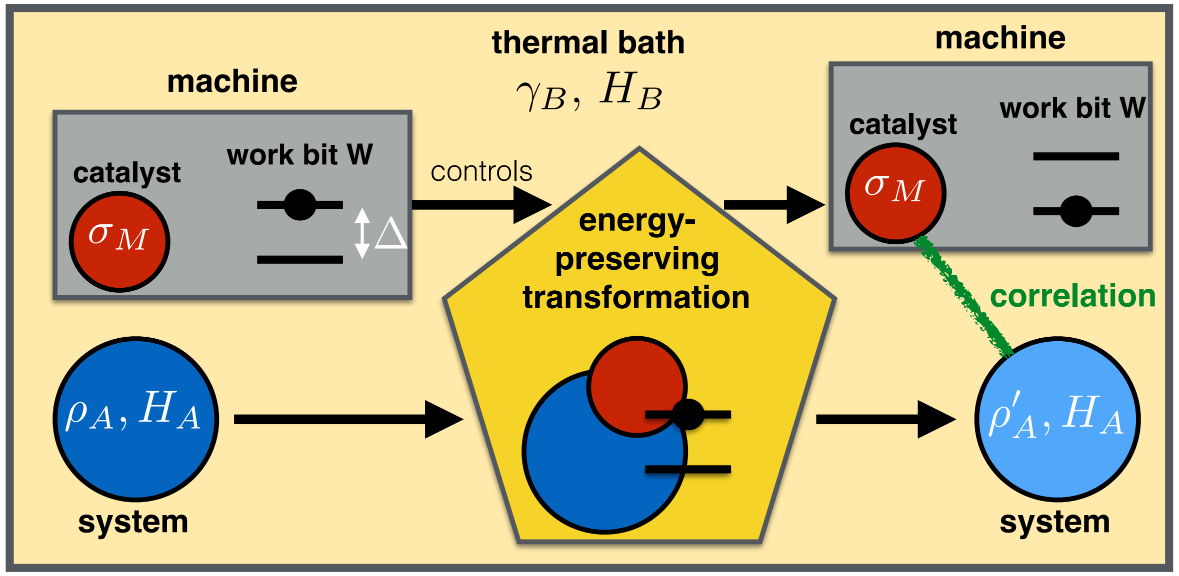
2. Main mathematical results

3. Implications for quantum thermodynamics



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Implications for quantum information

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One-shot operational tasks are typically characterized by one-shot entropies. E.g.:

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PRL **118,** 080503 (2017) PHYSICAL REVIEW LETTERS week ending 24 FEBRUARY 2017

Catalytic Decoupling of Quantum Information

Christian Majenz,[1,*] Mario Berta,[2] Frédéric Dupuis,[3] Renato Renner,[4] and Matthias Christandl[1]
[1]*Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø*
[2]*Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, California 91125, USA*
[3]*Faculty of Informatics, Masaryk University, Brno, Czech Republic*
[4]*Institute for Theoretical Physics, ETH Zurich, 8093 Zürich, Switzerland*
(Received 24 May 2016; published 23 February 2017)

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[3]*Faculty of Informatics, Masaryk University, Brno, Czech Republic*

[4]*Institute for Theoretical Physics, ETH Zurich, 8093 Zürich, Switzerland*

(Received 24 May 2016; published 23 February 2017)

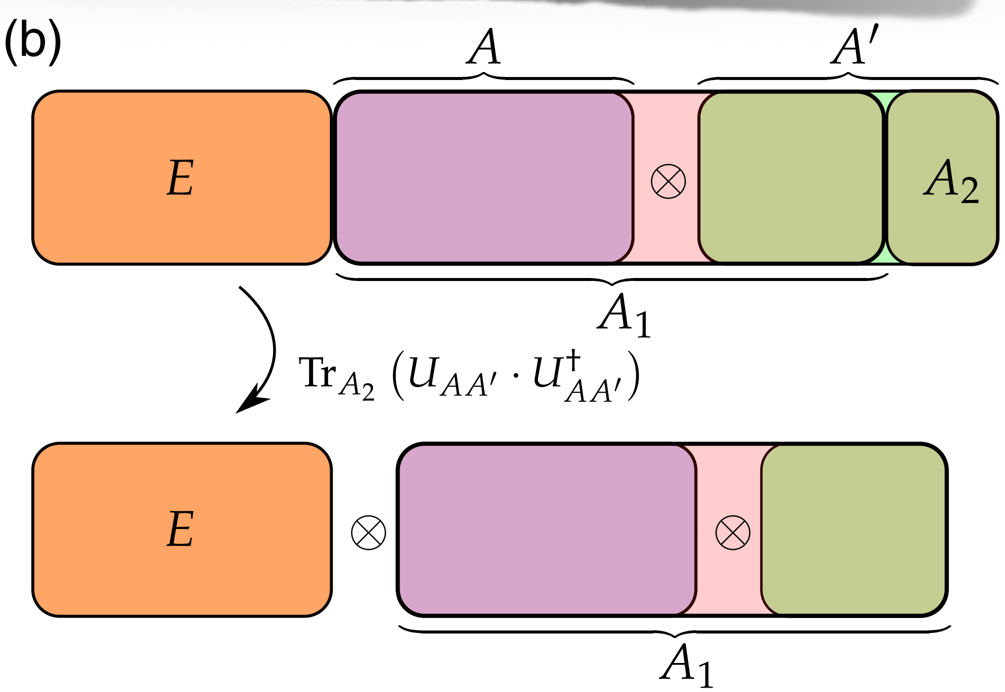
Theorem 1: (Catalytic decoupling) For any bipartite quantum state ϱ_{AE} and $0 < \delta \leq \varepsilon \leq 1$, we have:

$$R_c^\varepsilon(A;E)_\varrho \lesssim \frac{1}{2} I_{\max}^{\varepsilon-\delta}(E;A)_\varrho, \tag{11}$$

where $\lesssim$ stands for smaller or equal up to terms $\mathcal{O}(\log\log|A| + \log(1/\delta))$. We also have the converse

$$R_c^\varepsilon(A;E)_\varrho \geq \frac{1}{2} I_{\max}^{\varepsilon}(E:A)_\varrho. \tag{12}$$

(b)



Implications for quantum information

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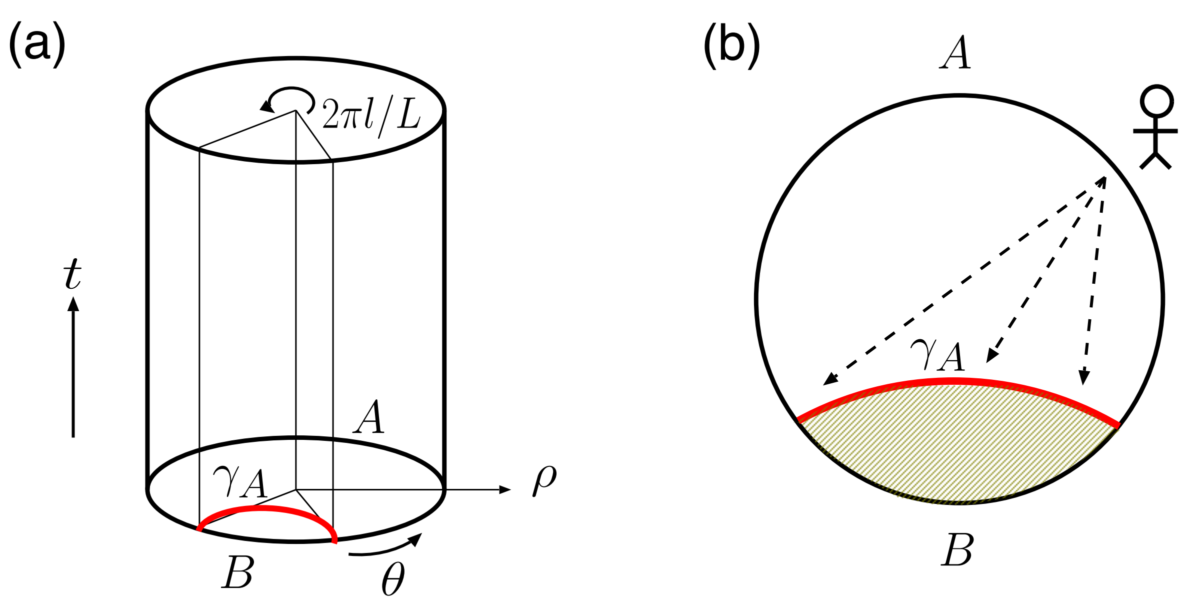
Interesting, for example, because standard entropies have **dual spacetime interpretations**:

S. Ryu and T. Takayanagi, *Holographic Derivation of Entanglement Entropy from AdS/CFT*, Phys. Rev. Lett. **96**, 181602 (2006).

$$S_A = \frac{\text{Area of } \gamma_A}{4G_N^{(d+2)}}$$



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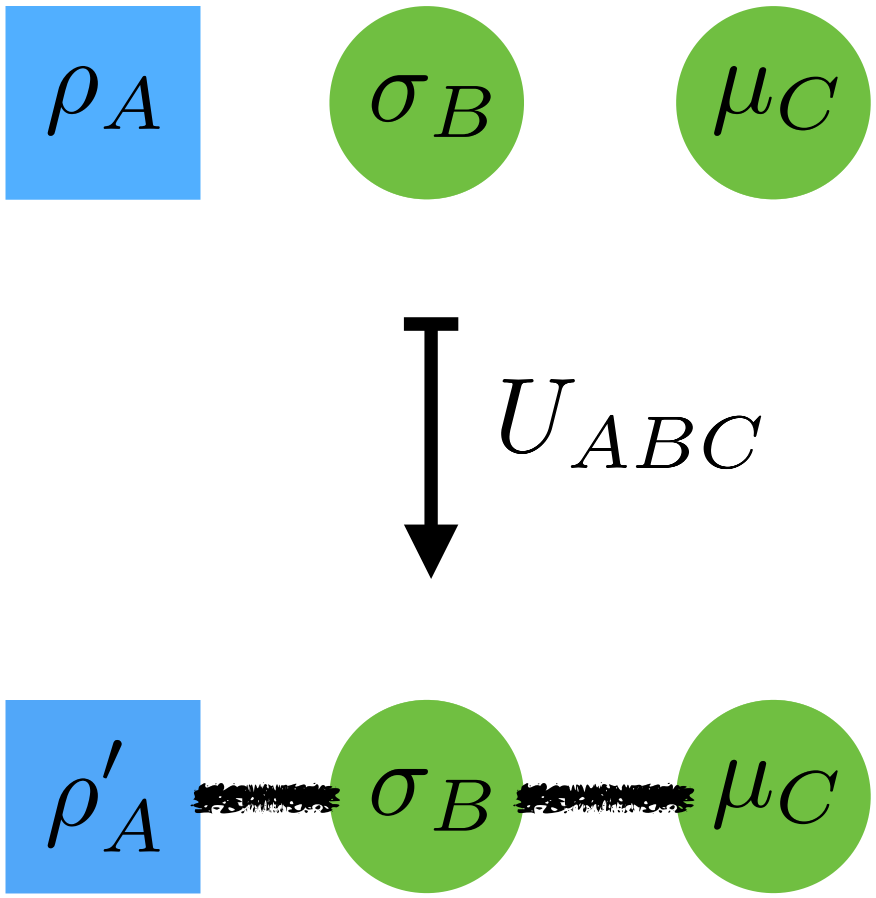

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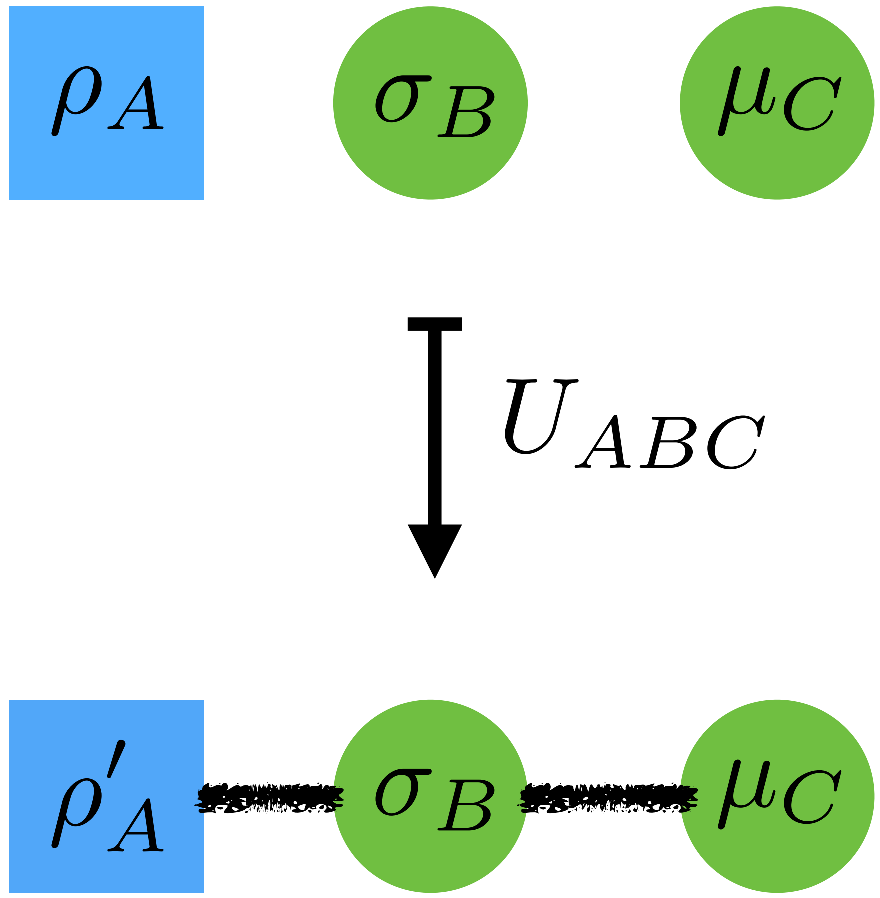

$$S(\rho_A) + S(\sigma_B) + S(\mu_C) = S(\rho'_{ABC}) \leq S(\rho'_A) + S(\sigma_B) + S(\mu_C).$$

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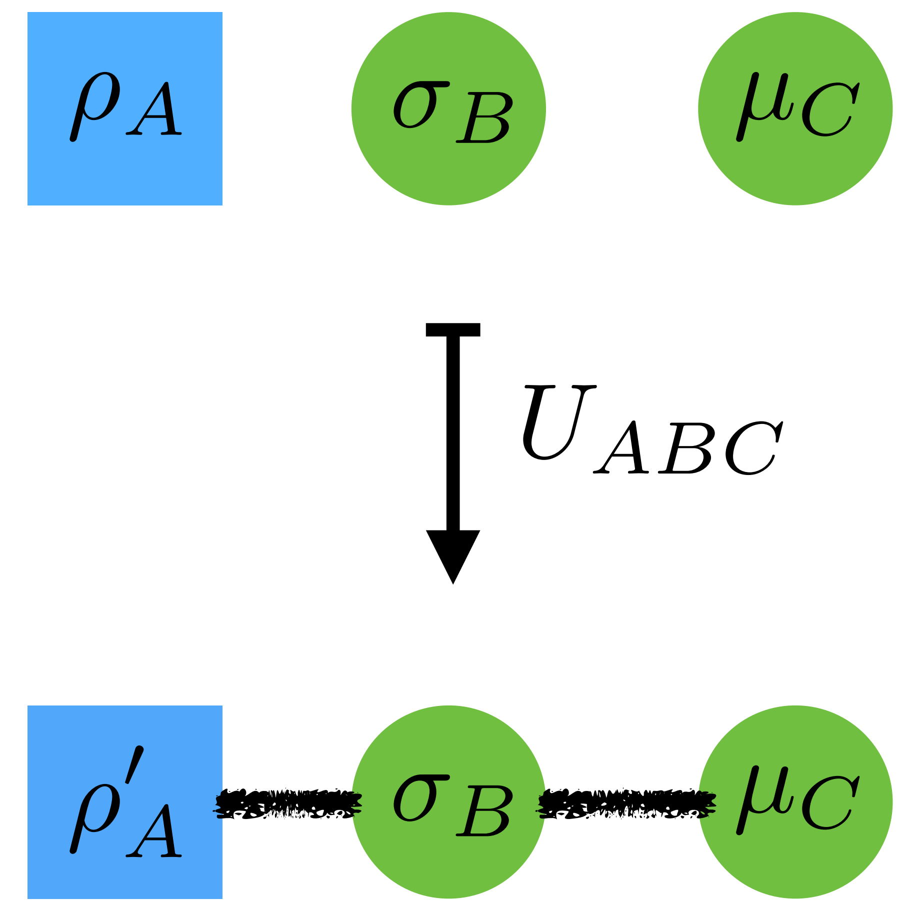
But here's an example, following from the above:

Theorem 5. *Let ρ_A and ρ'_A be quantum states with full rank which are not unitarily equivalent, i.e. do not have the exact same set of eigenvalues. Then there exists a finite auxiliary system B, a quantum state σ_B, and a copy C of AB with maximally mixed state μ_C as well a unitary U_{ABC} such that*

$$U_{ABC}(\rho_A \otimes \sigma_B \otimes \mu_C)U^\dagger_{ABC} = \rho'_{ABC}$$

with marginals ρ'_A on A, $\rho'_B = \sigma_B$ and $\rho'_C = \mu_C$ if and only if $S(\rho_A) < S(\rho'_A)$ for the von Neumann entropy S.



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Open Questions. Can we do without the C system? Or recycle BC? And do the same if A is correlated with some other system (decoupling)?

Relation to versions of the quantum marginal problem.

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$$S(\rho_A) + S(\sigma_B) + S(\mu_C) = S(\rho'_{ABC}) \leq S(\rho'_A) + S(\sigma_B) + S(\mu_C).$$

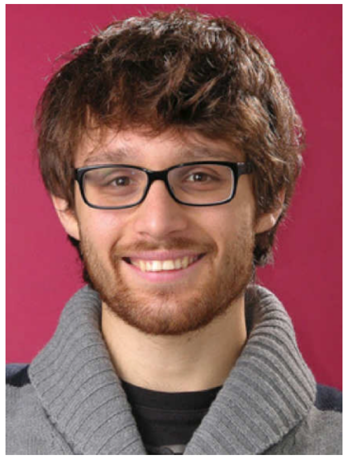
Conclusions

- **Majorization**: some new results
  In particular $p_X \otimes p'_Y \succ p'_{XY} \Leftrightarrow H(X) \leq H(X')$

MM, **arXiv:1707.03451** (+refs)
Further with M. Lostaglio, M. Pastena, J. Scharlau, see http://mpmueller.net

- **Quantum thermodynamics**: standard 2nd law;
  natural one-shot interpretation of free energy *F*

- **Quantum info**: one-shot int. of standard entropies (?)



Thank you!