Reversible computing and the resource-theoretic approach to thermodynamics

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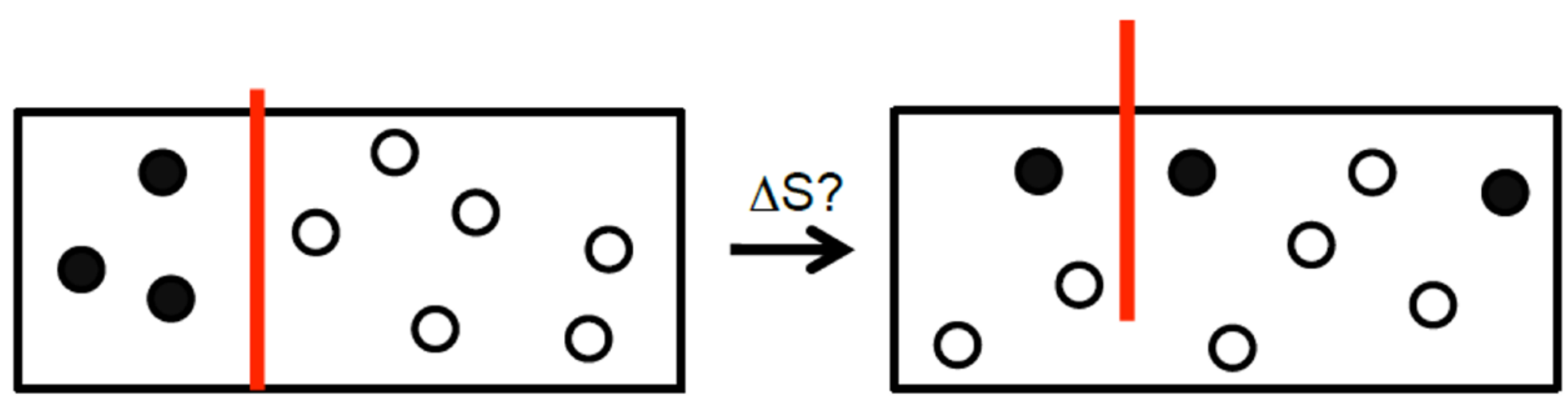
IQOQI Vienna

Standard view: thermodynamic limit



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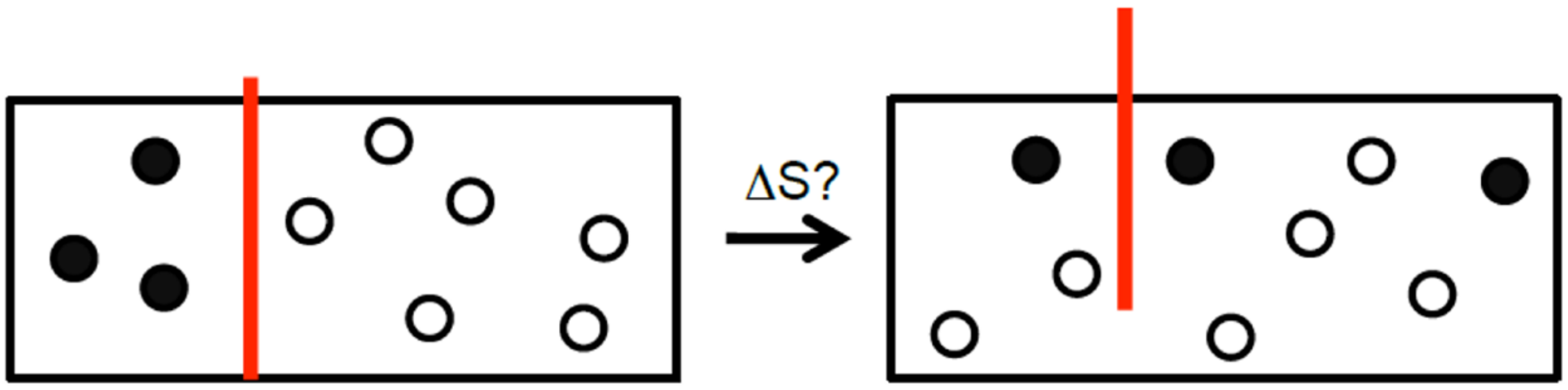

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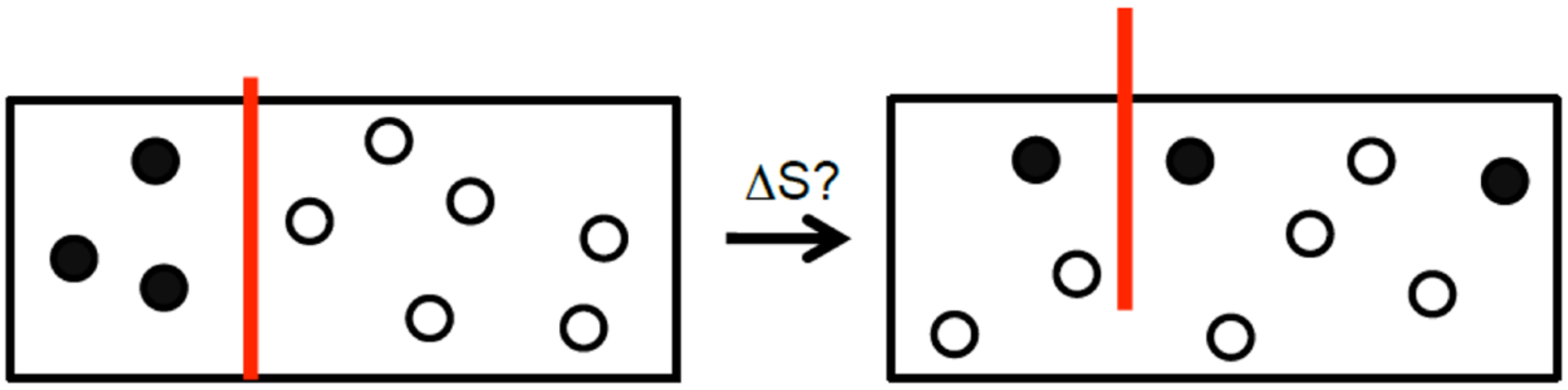
$$\Delta F \leq 0 \qquad \text{(2nd law)},$$

where $F = U - TS$.

If this is negative, then we can extract $|\Delta F|$ of work from the system.



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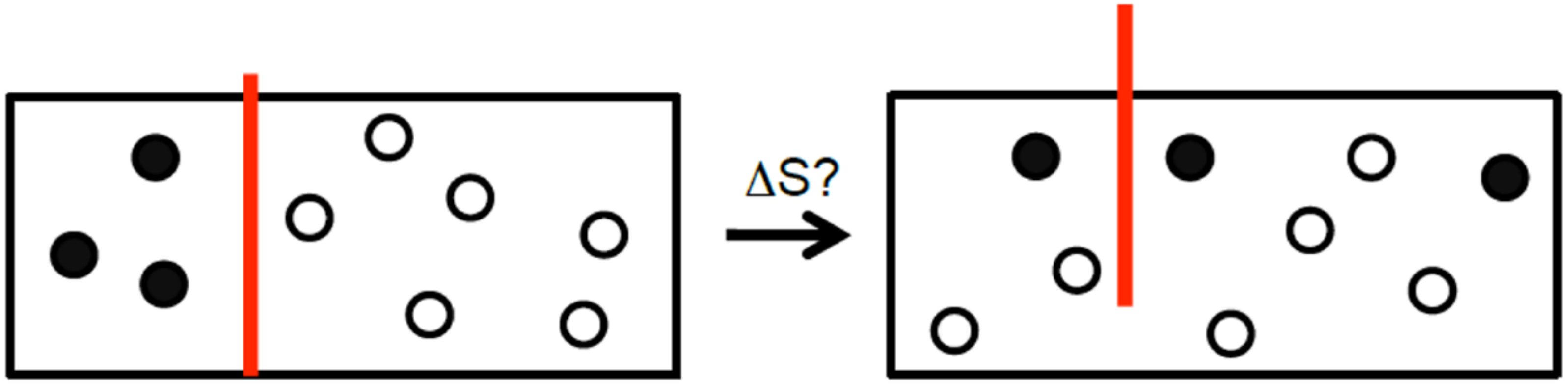
But this is a statement **on average**, since "work" is a random variable.



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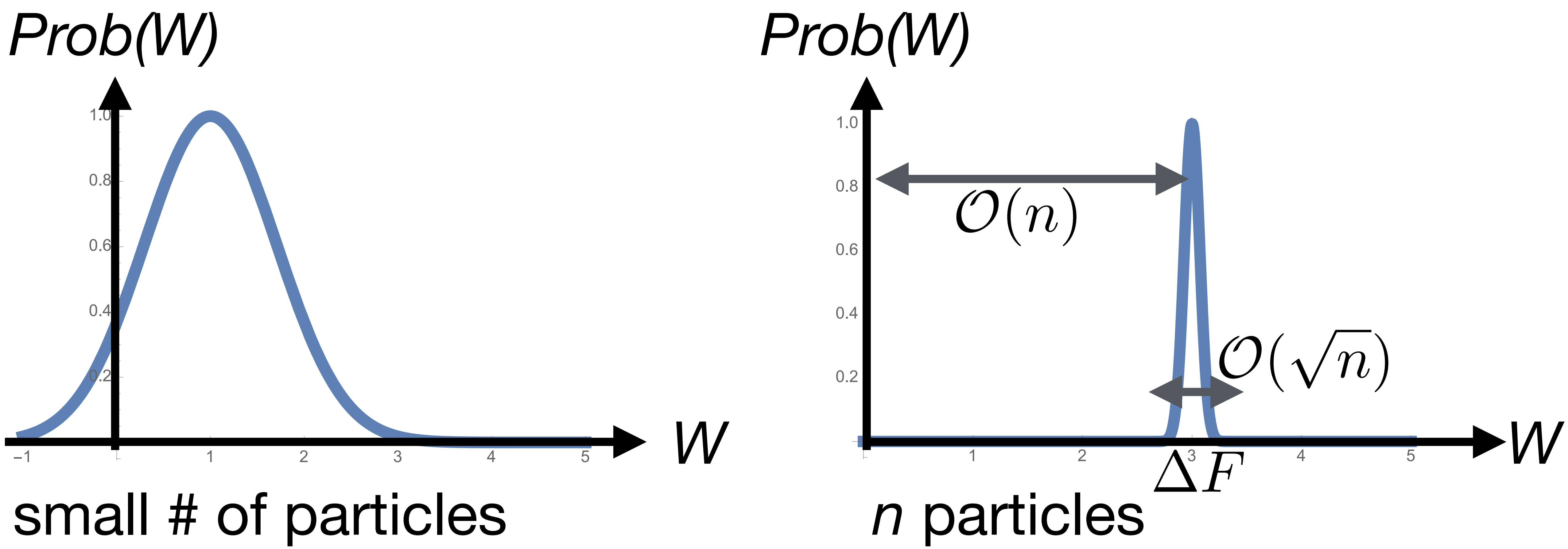

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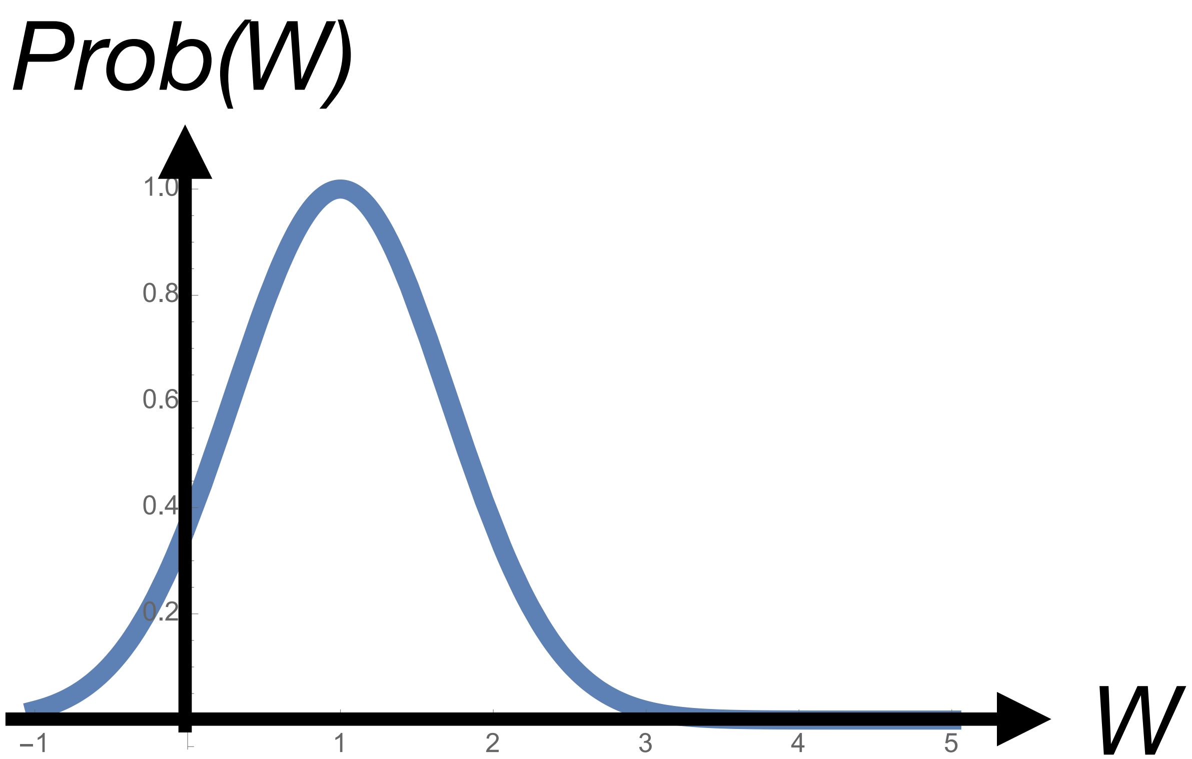
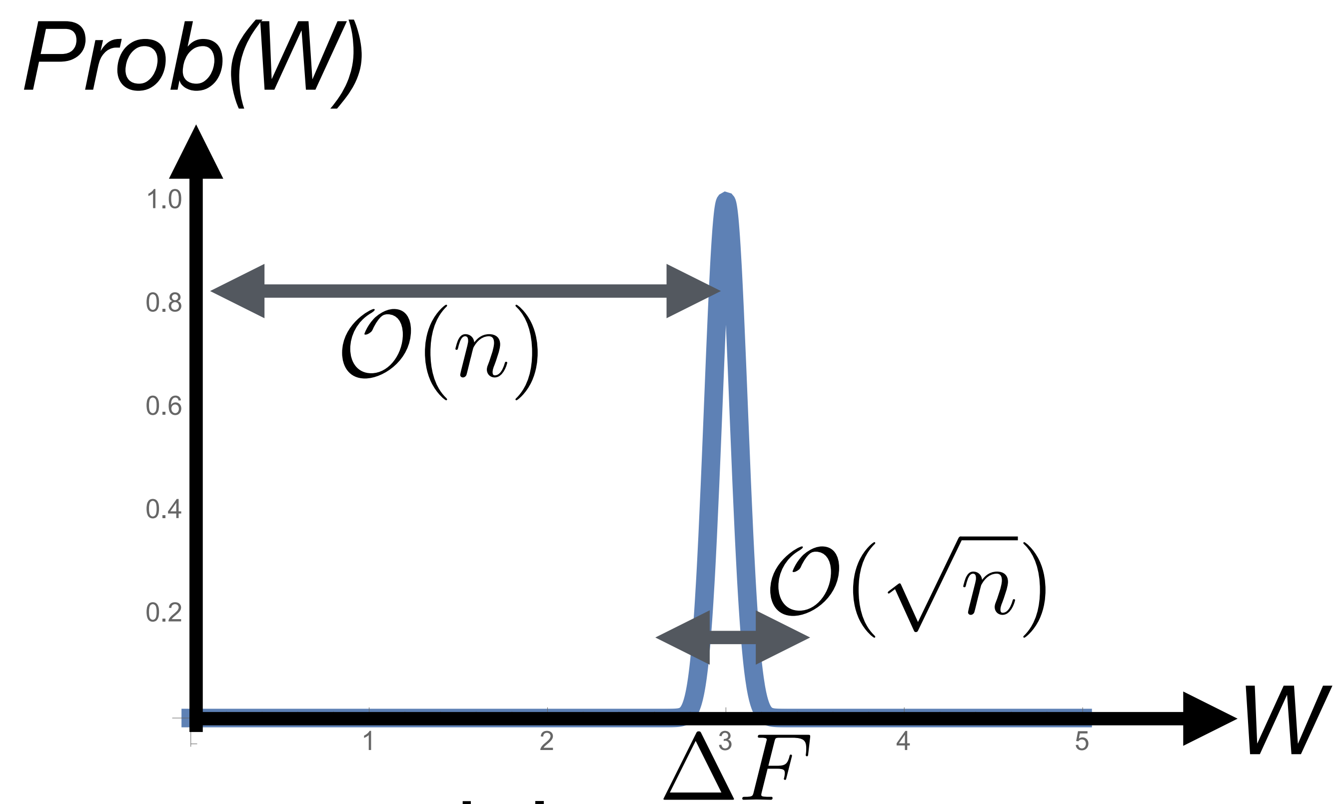


small # of particles



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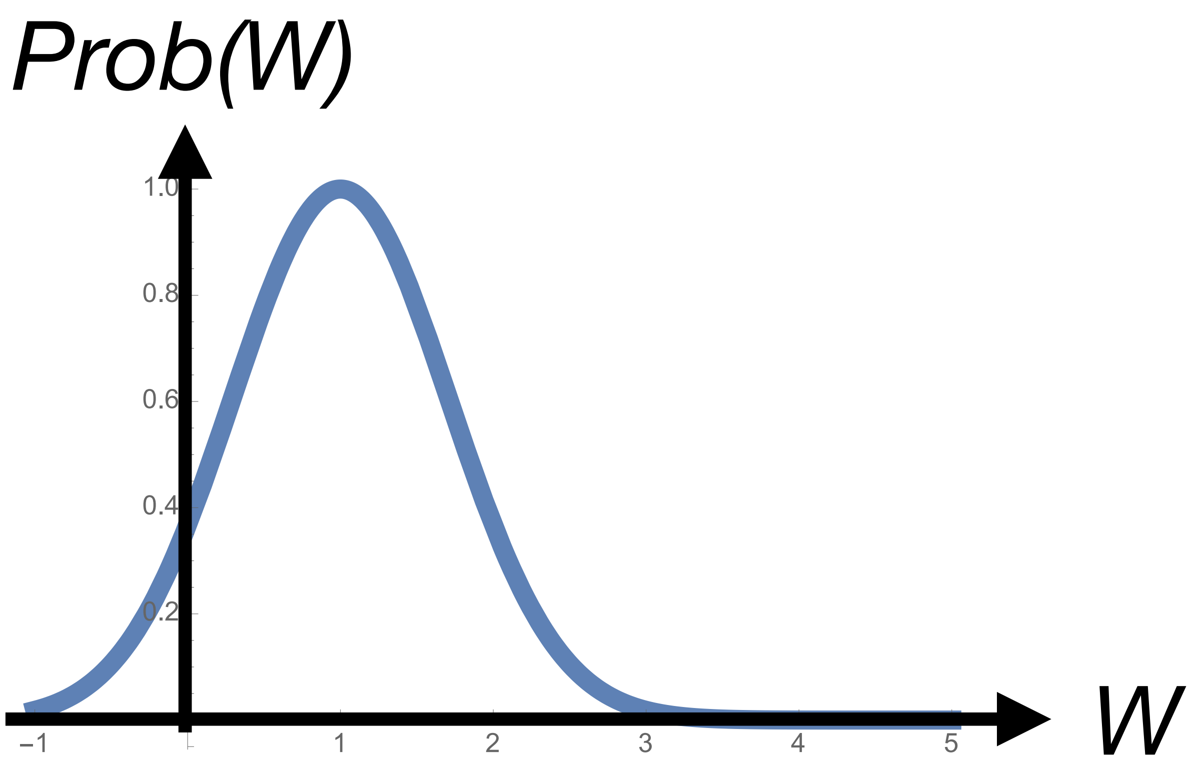

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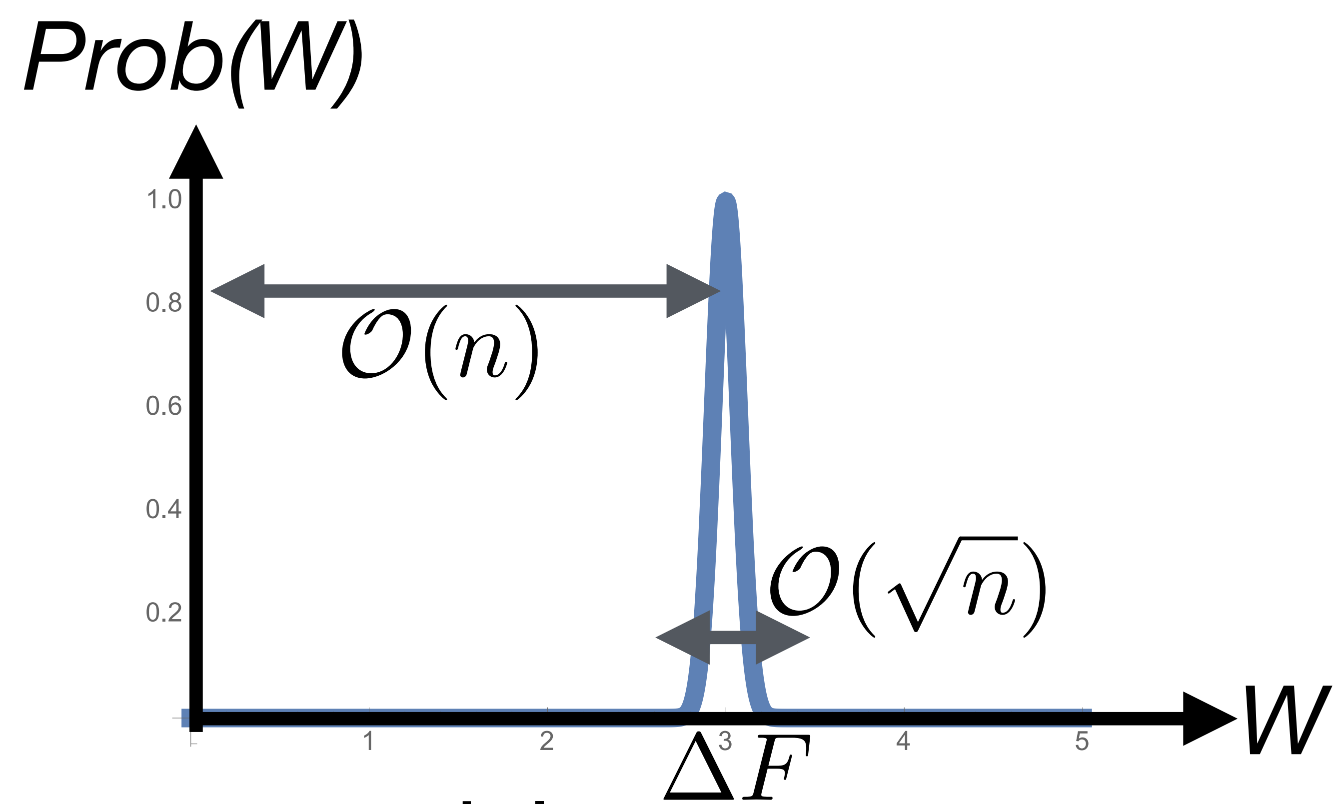

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What can we say for "small" or strongly correlated systems?
Work $\approx$ its fluctuations → reliability?

Single-shot: Thermodynamics as a resource theory

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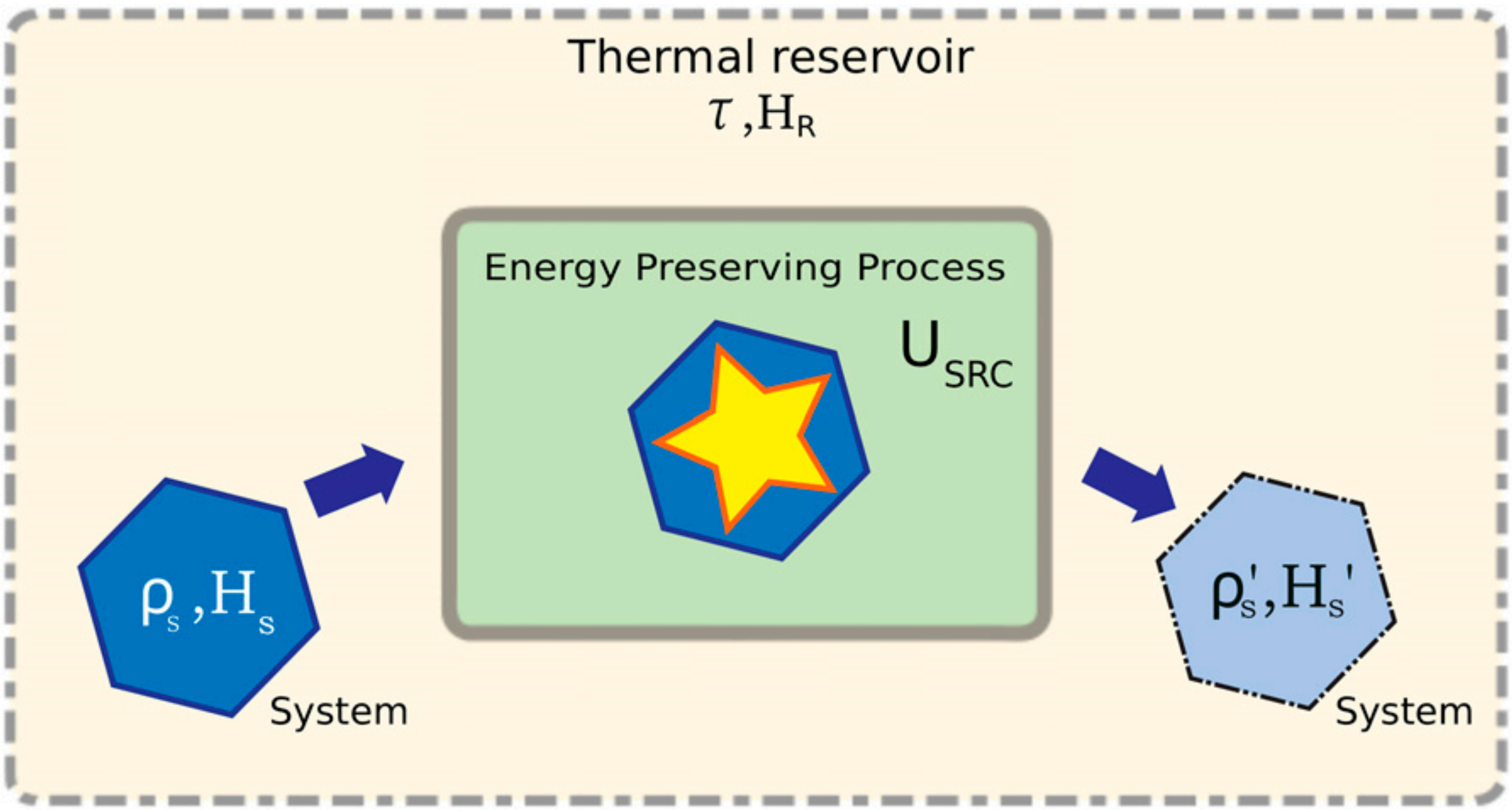
The rules of the game:

- It is "free" to bring in any "bath" *B* in its thermal state $\gamma_B = \exp(-H_B/(k_B T))$,
- strictly energy-preserving unitaries are free,
- and it is free to trace over (ignore) systems.



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Def.: A *thermal operation* $\mathcal{T}$ is a map of the form

$$\mathcal{T}(\rho_A) = \mathrm{Tr}_B \left[U_{AB} \left(\rho_A \otimes \gamma_B \right) U_{AB}^\dagger \right]$$

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Question: Which transitions (work extraction etc.) are possible via thermal operations?

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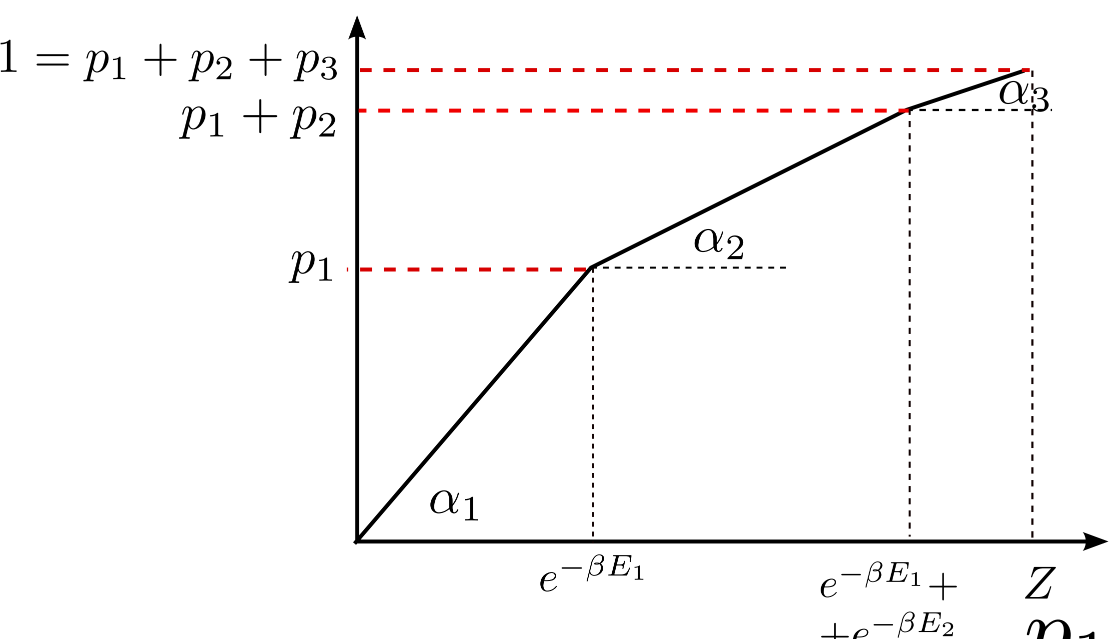
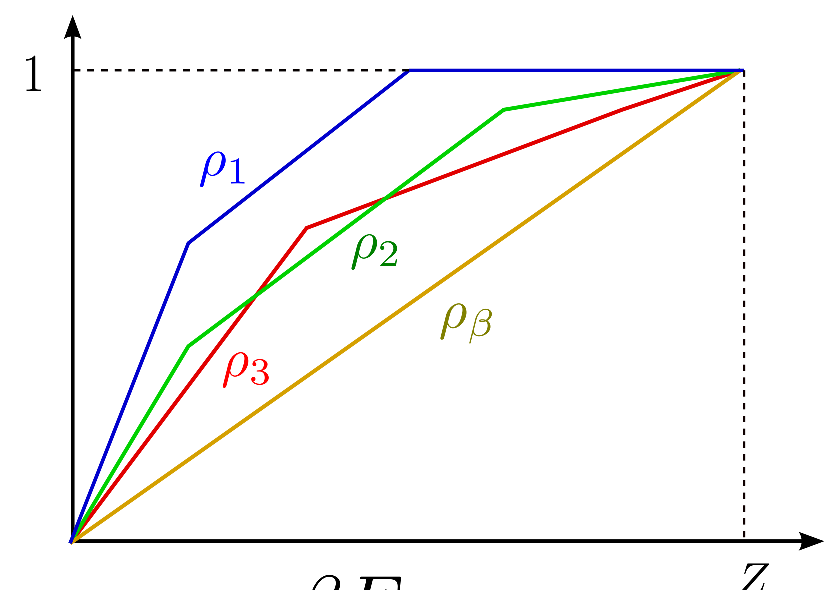
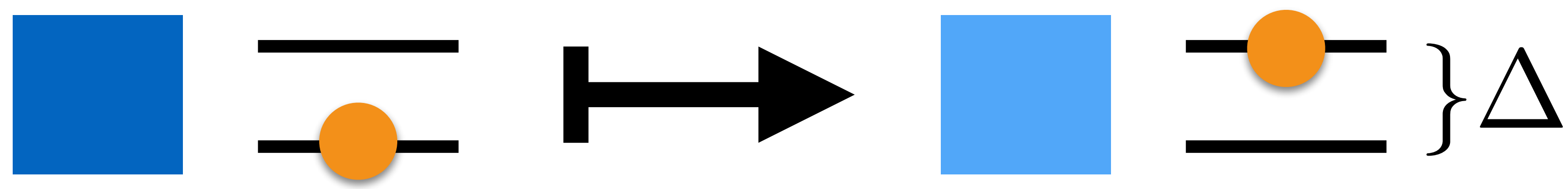


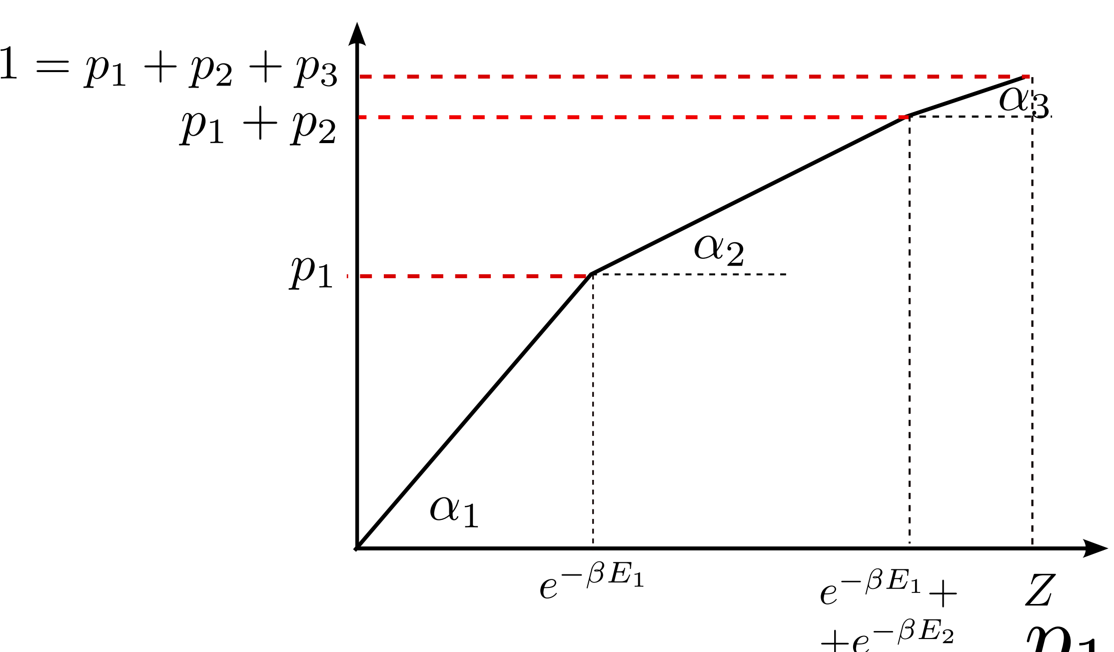


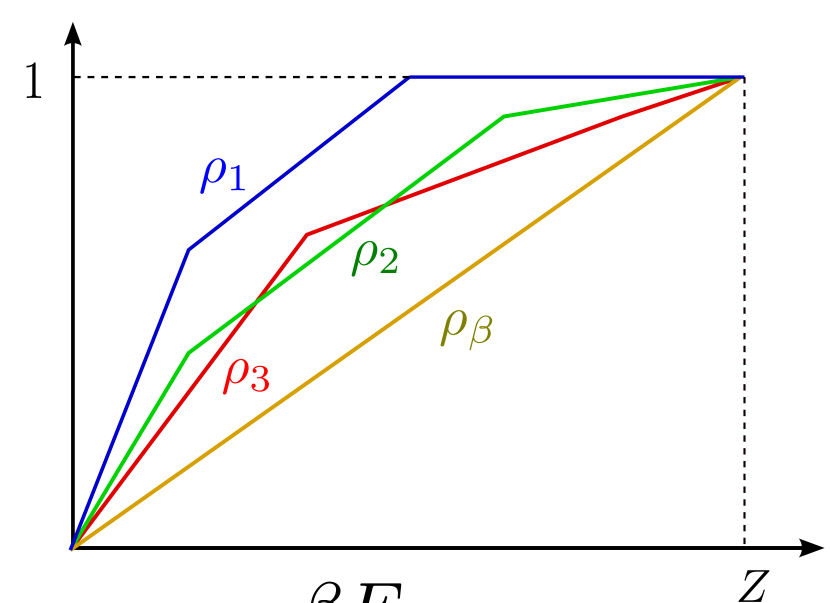
$$p_1 e^{\beta E_1} \geq p_2 e^{\beta E_2} \geq \ldots$$

Work extraction and work of formation

Work extraction: $\sigma_A \otimes |g\rangle\langle g|_W \mapsto \sigma'_A \otimes |e\rangle\langle e|_W$







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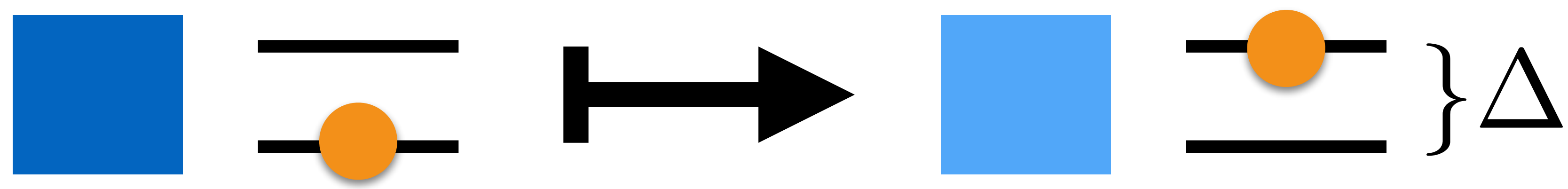

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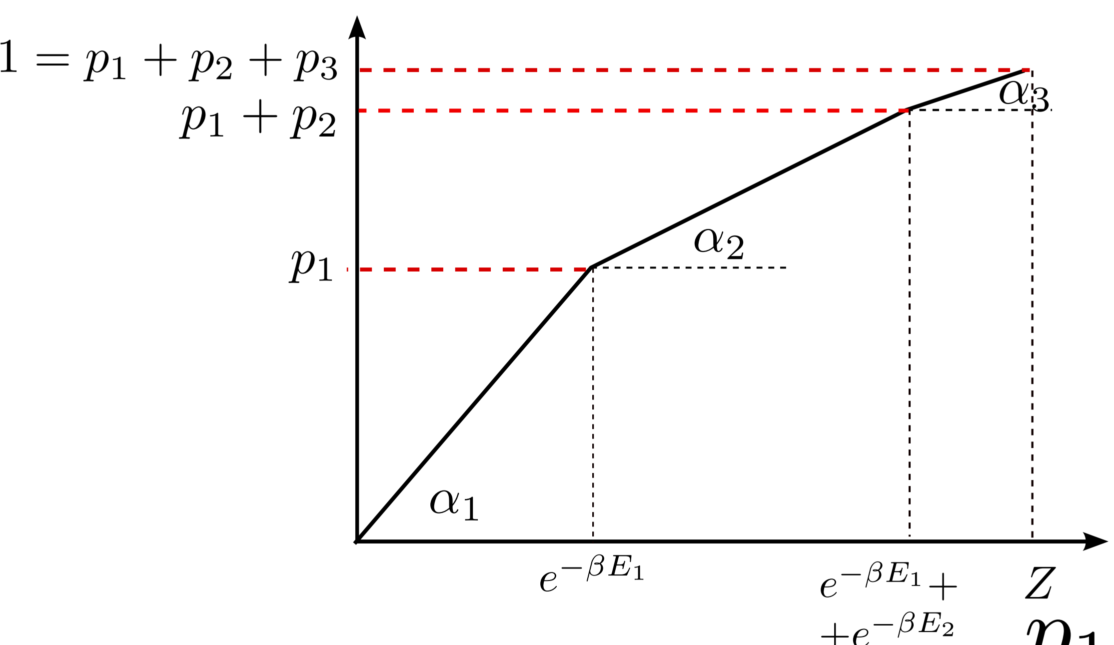

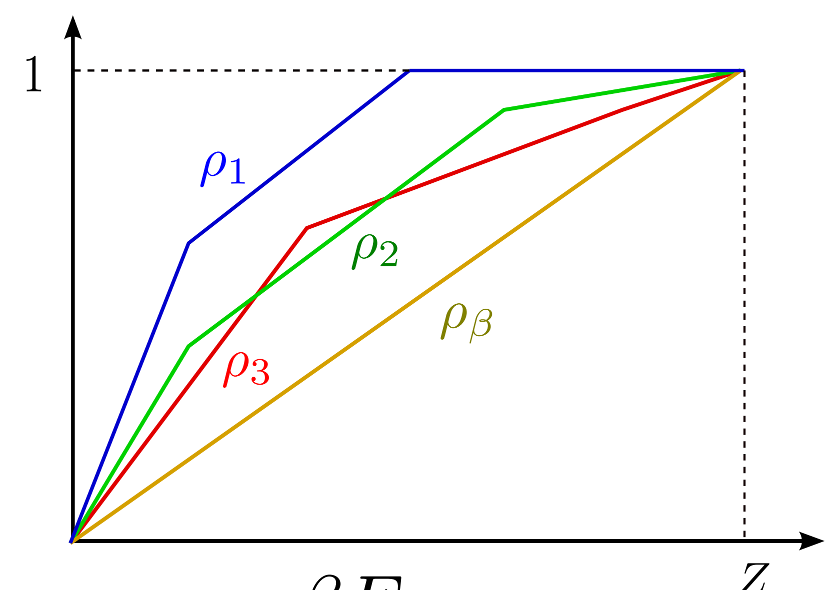

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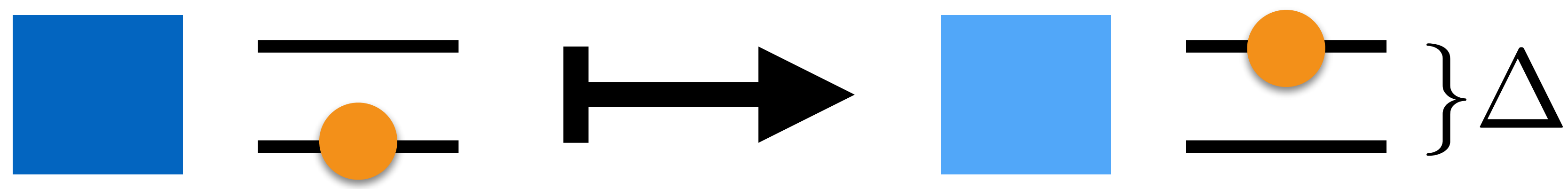

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where $F_0(\sigma) = k_B T \log \sum_{p_i \neq 0} e^{-\beta E_i}$.

Similarly, *work cost* is $F_\infty(\sigma_A) - F(\gamma_A)$

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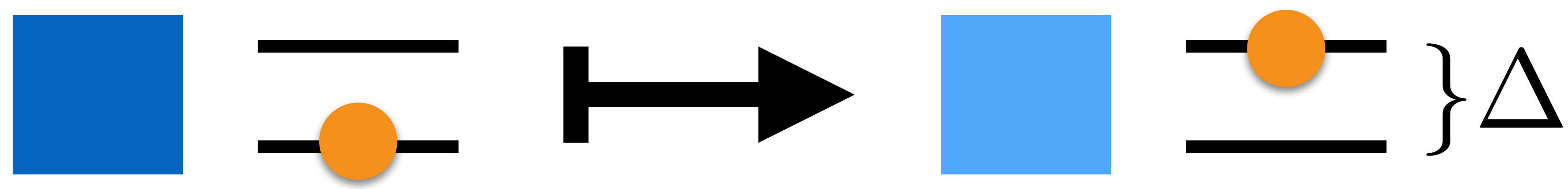

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Fundamental irreversibility: $F_0 \ll F \ll F_\infty$.

Thermodynamic limit

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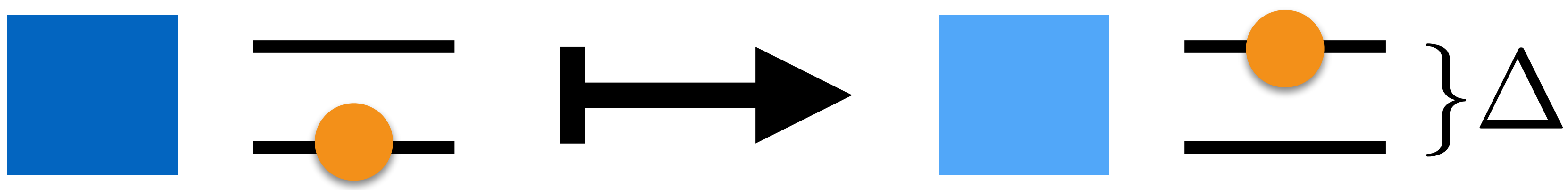

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Allowing small errors ε, we have

$$\frac{1}{n} F^{(\varepsilon)}_{0/\infty}(\rho^{\otimes n}) \xrightarrow{n\to\infty} F(\rho).$$

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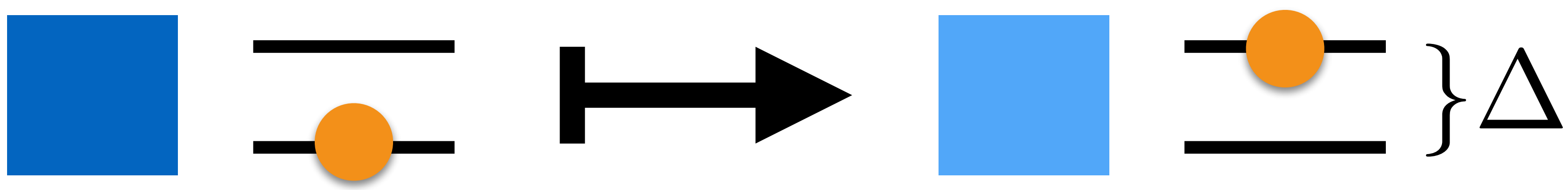

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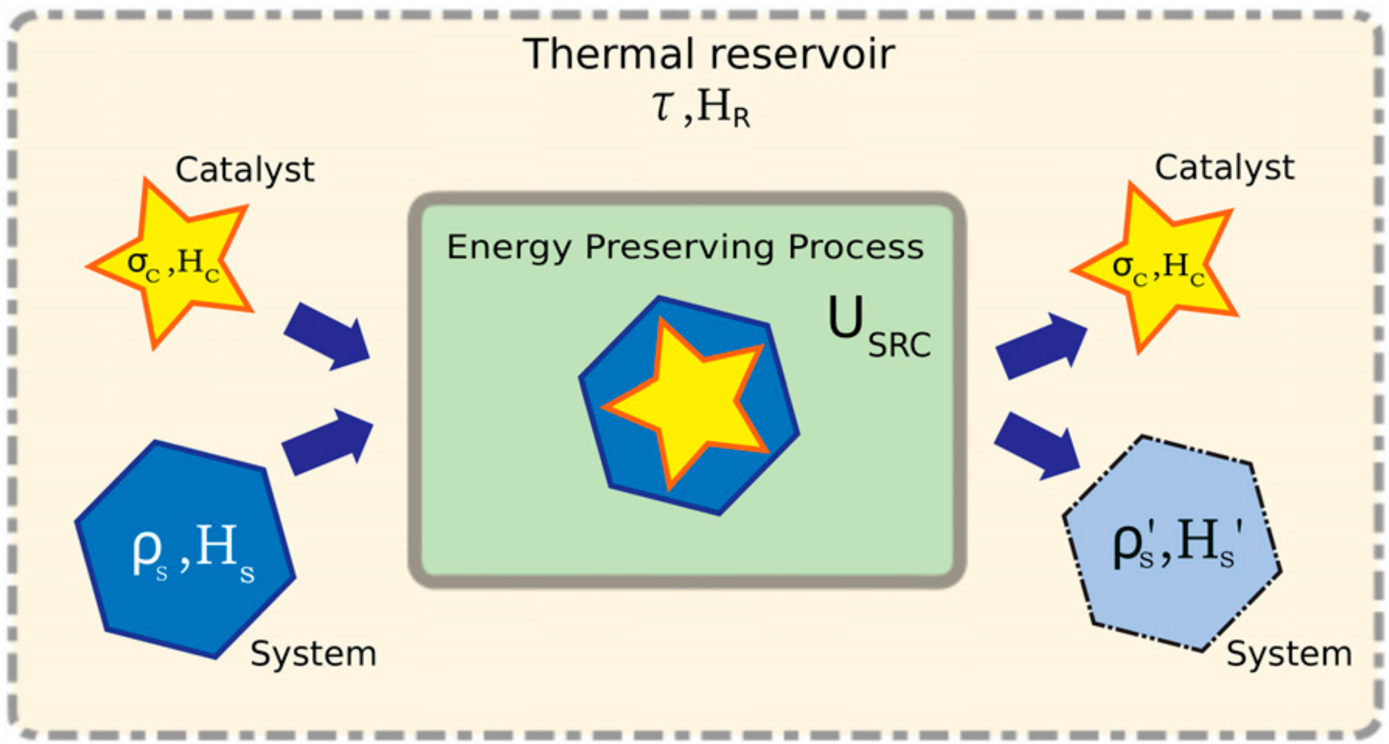
(Rates of) work cost and extractable work become *F*.
Reversibility is restored in the thermodynamic limit!

General state transitions: **catalysts**

Allow for additional system *C* that is involved but doesn't change.

Brandão et al., *The second laws of quantum thermodynamics*, PNAS **112**, 3275 (2015).



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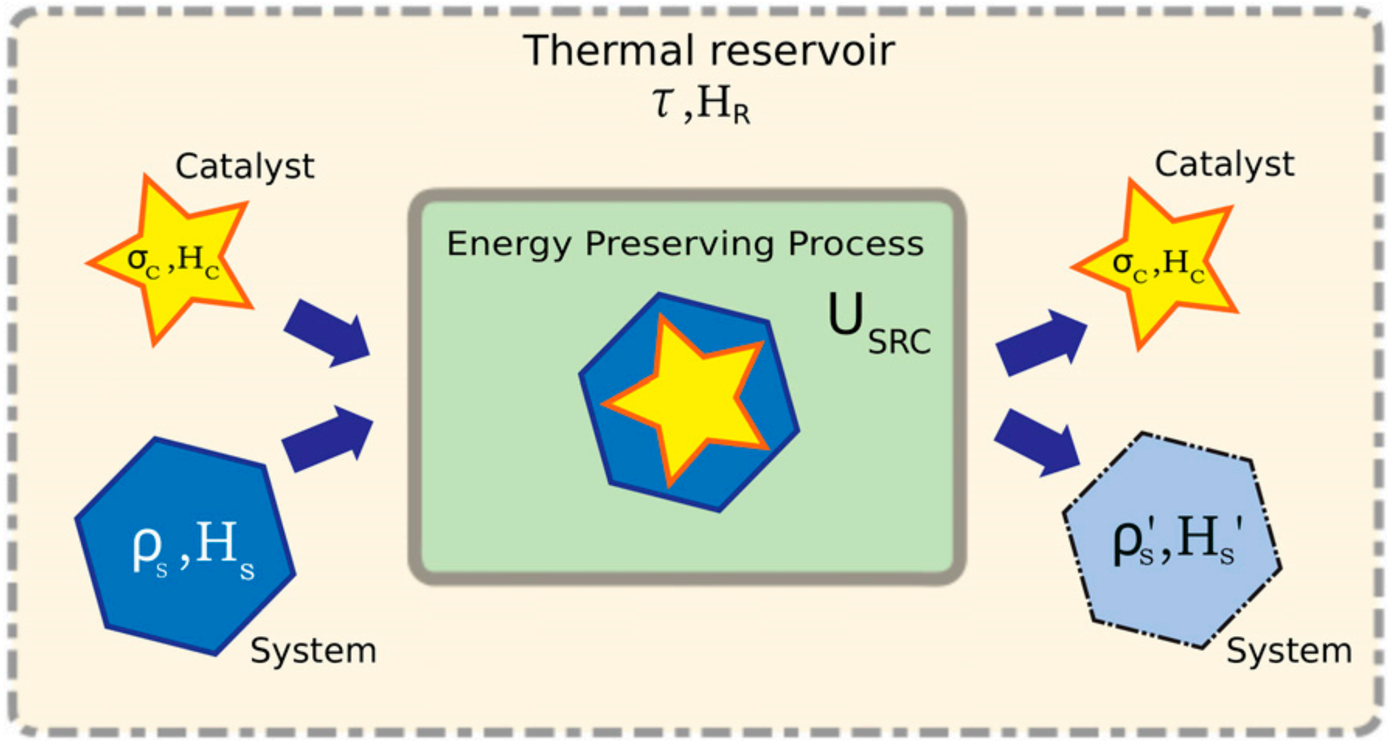

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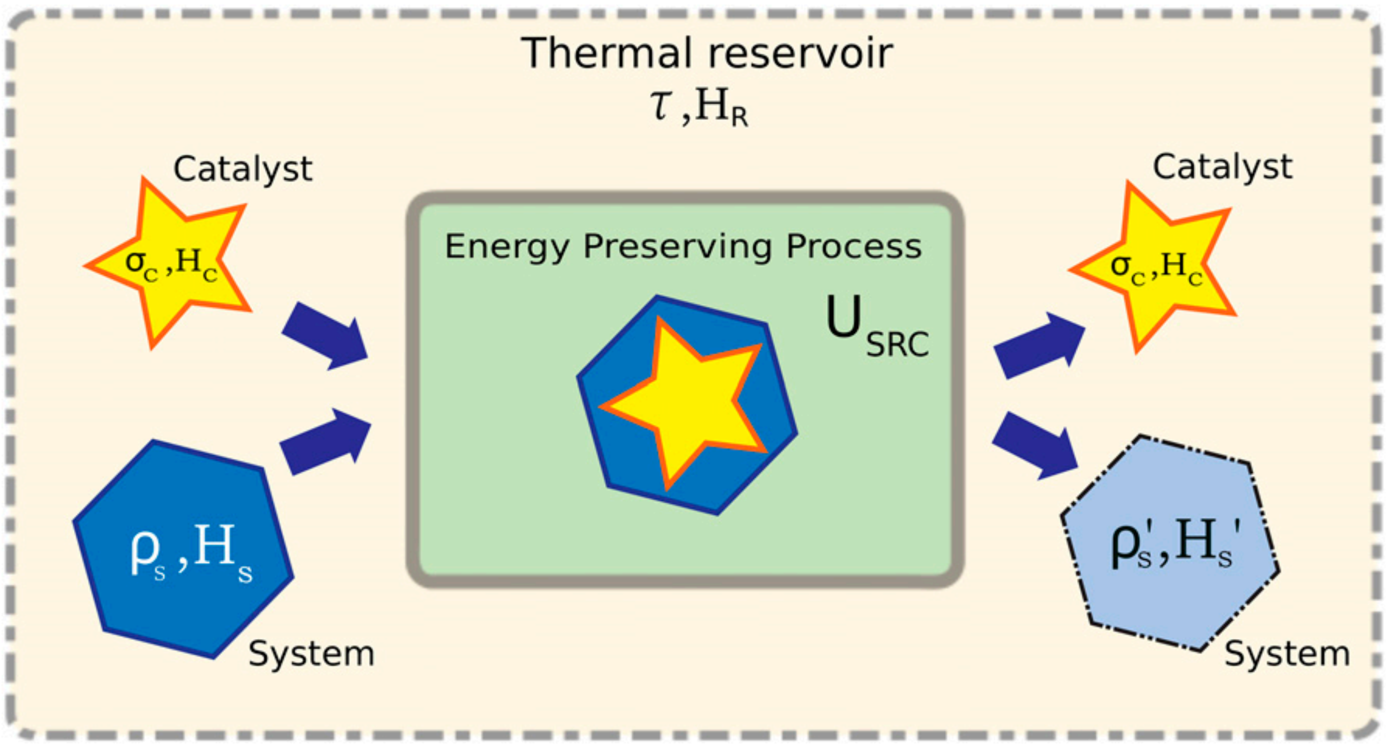

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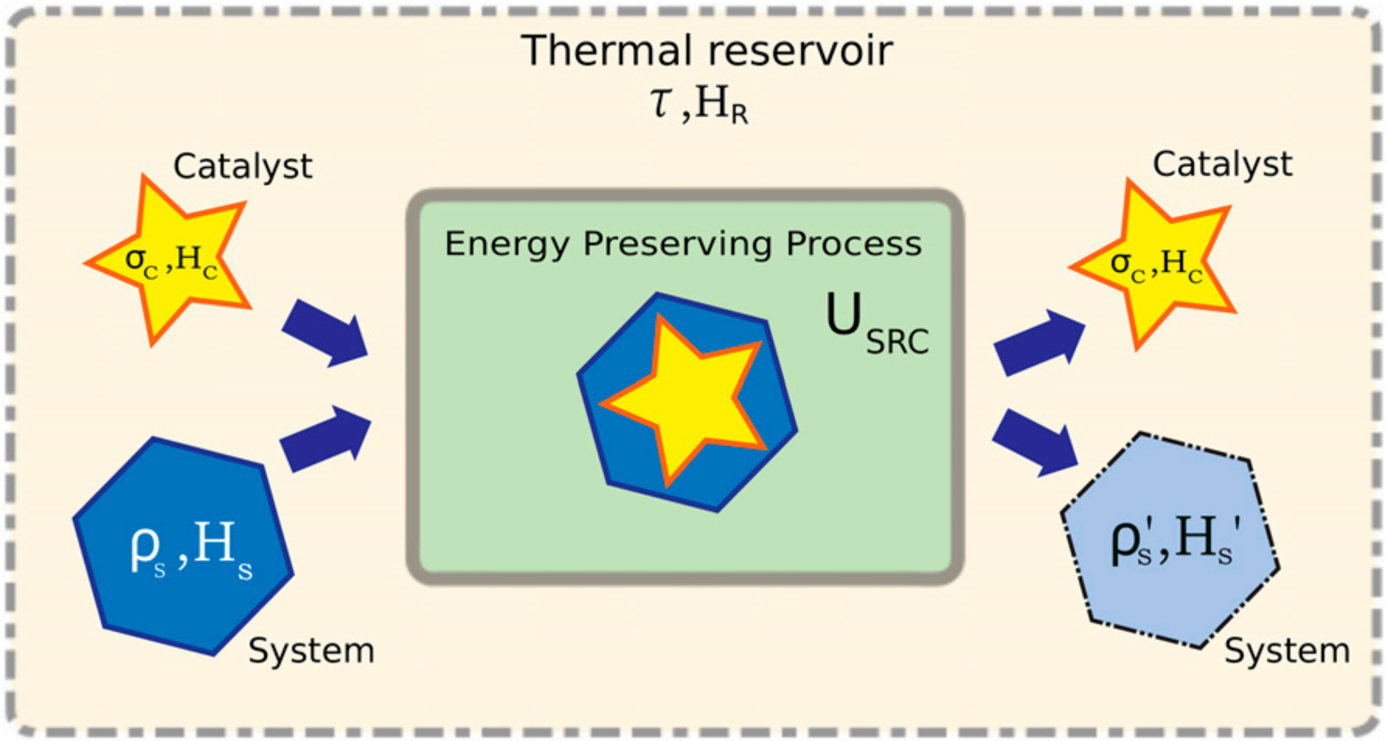

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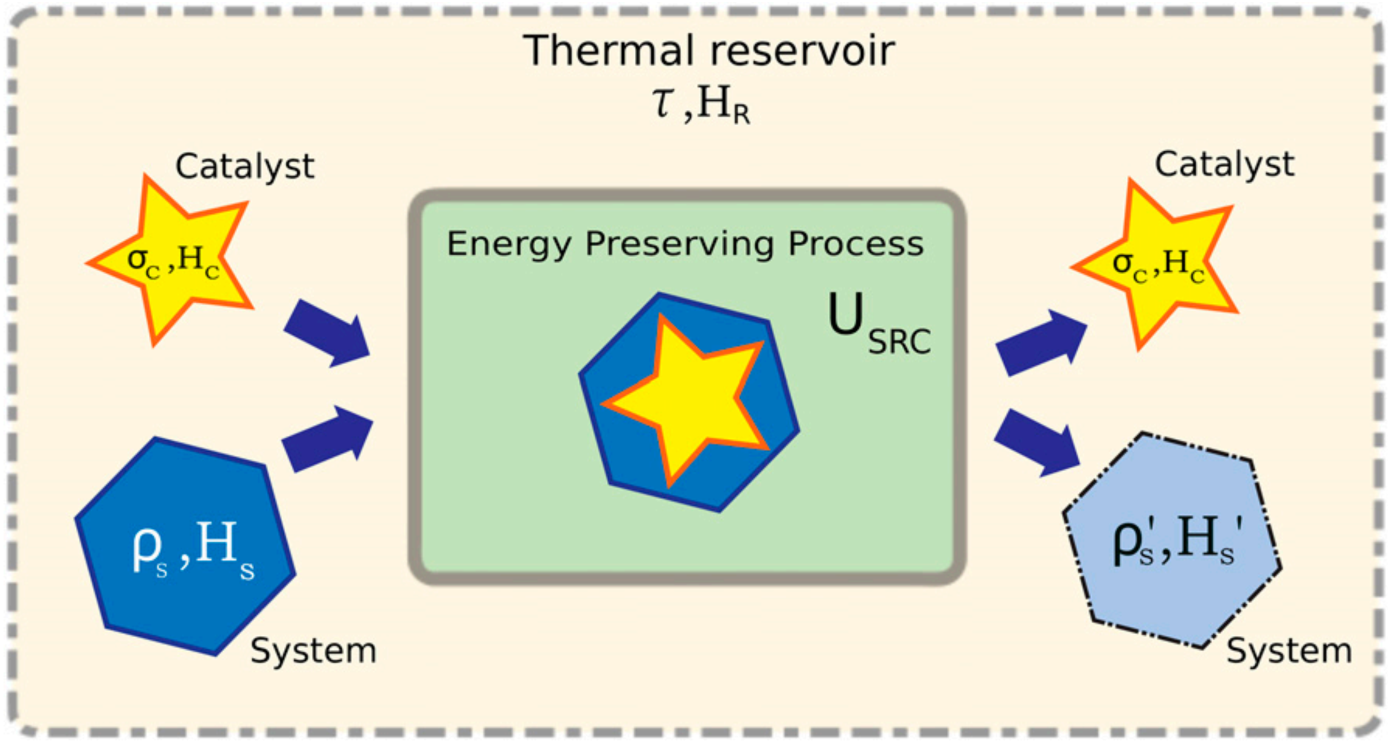
Theorem: Possible if and only if $F_\alpha(\rho_S) \geq F_\alpha(\rho'_S)$ for all $\alpha \geq 0$.

"**Second laws**" of thermodynamics. Note: $F_{\alpha=1} = F$.

One-shot interpretation of non-equilibrium **F**



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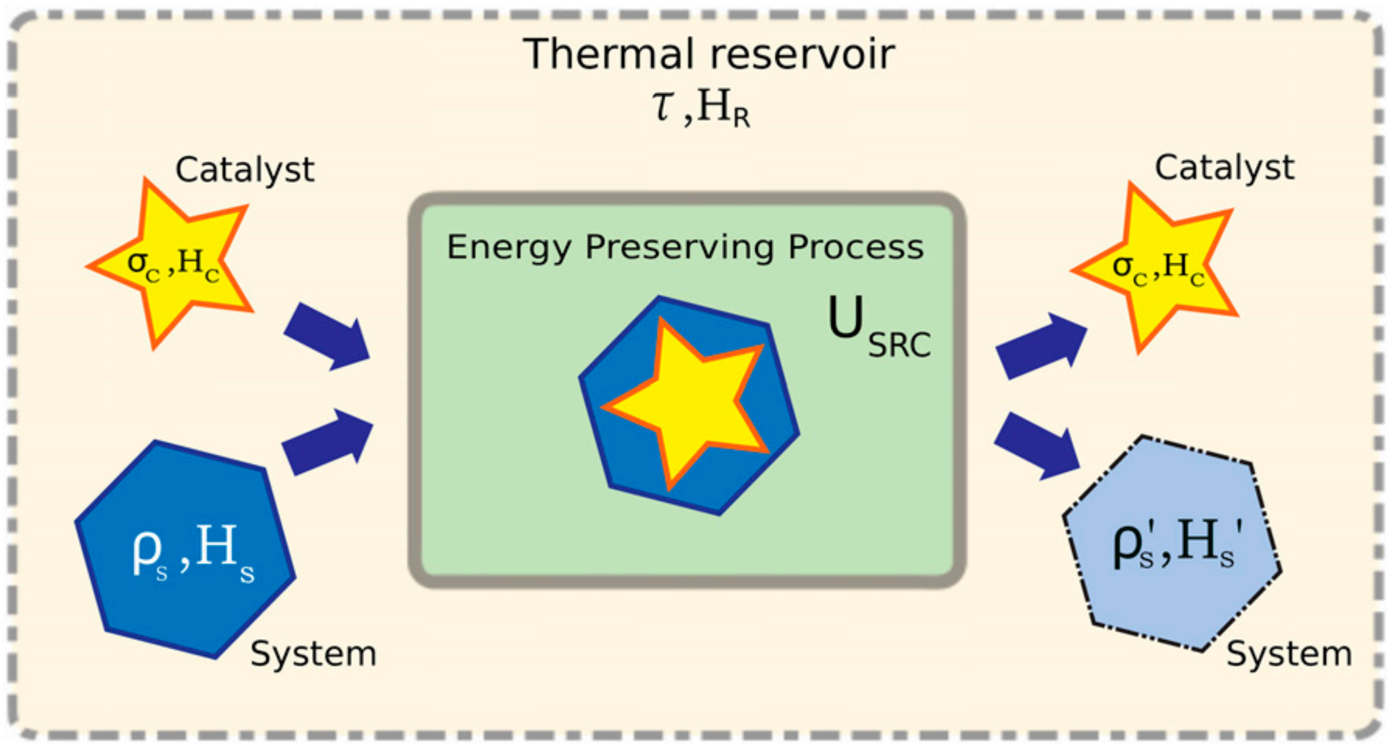
Own work: **allow correlations** between catalyst and system.

[1] MM, Phys. Rev. X **8**, 041051 (2018).



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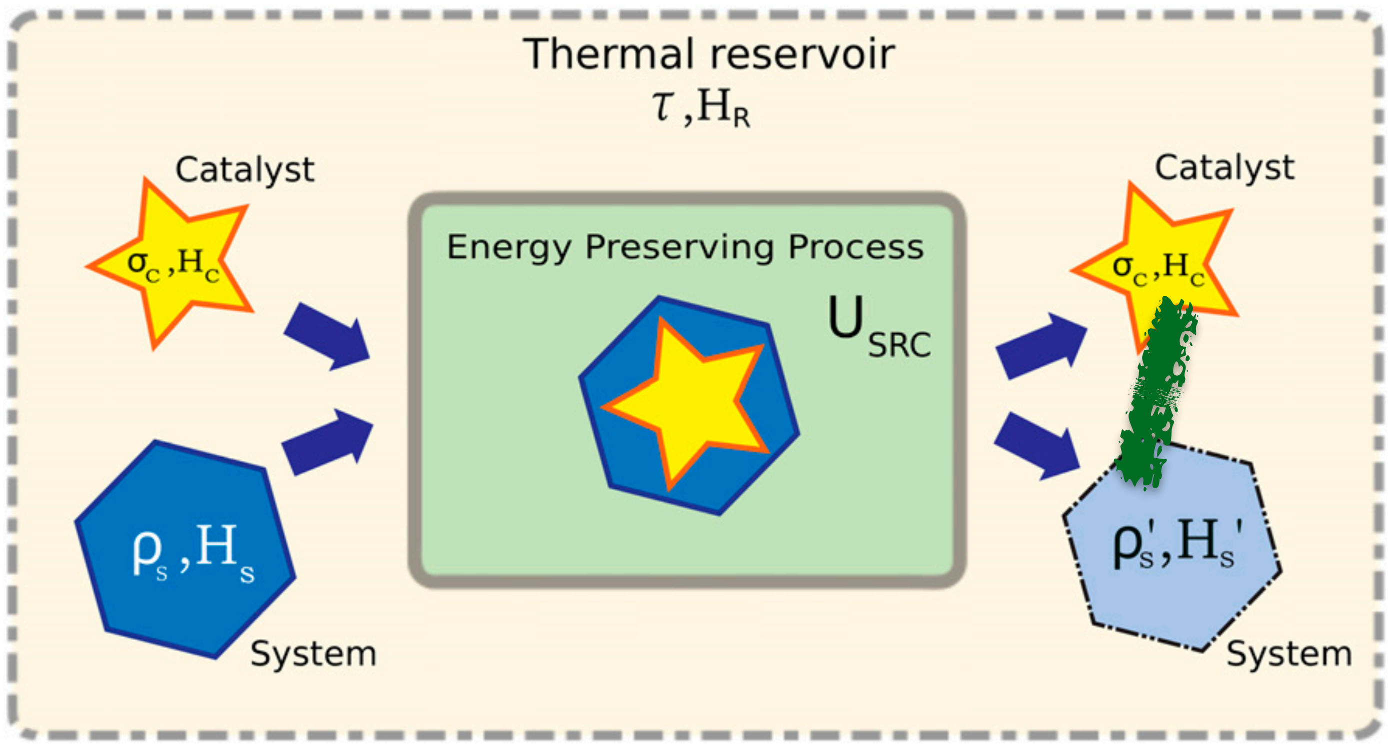

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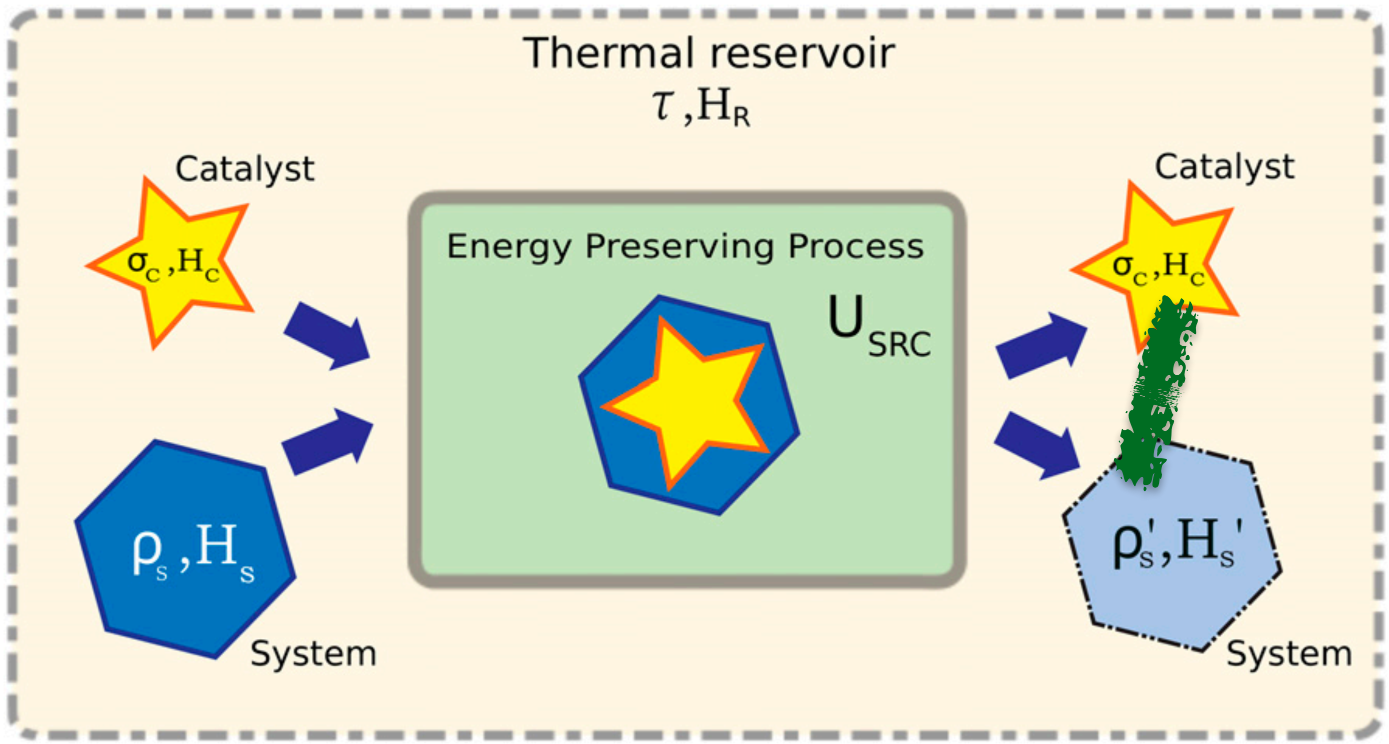

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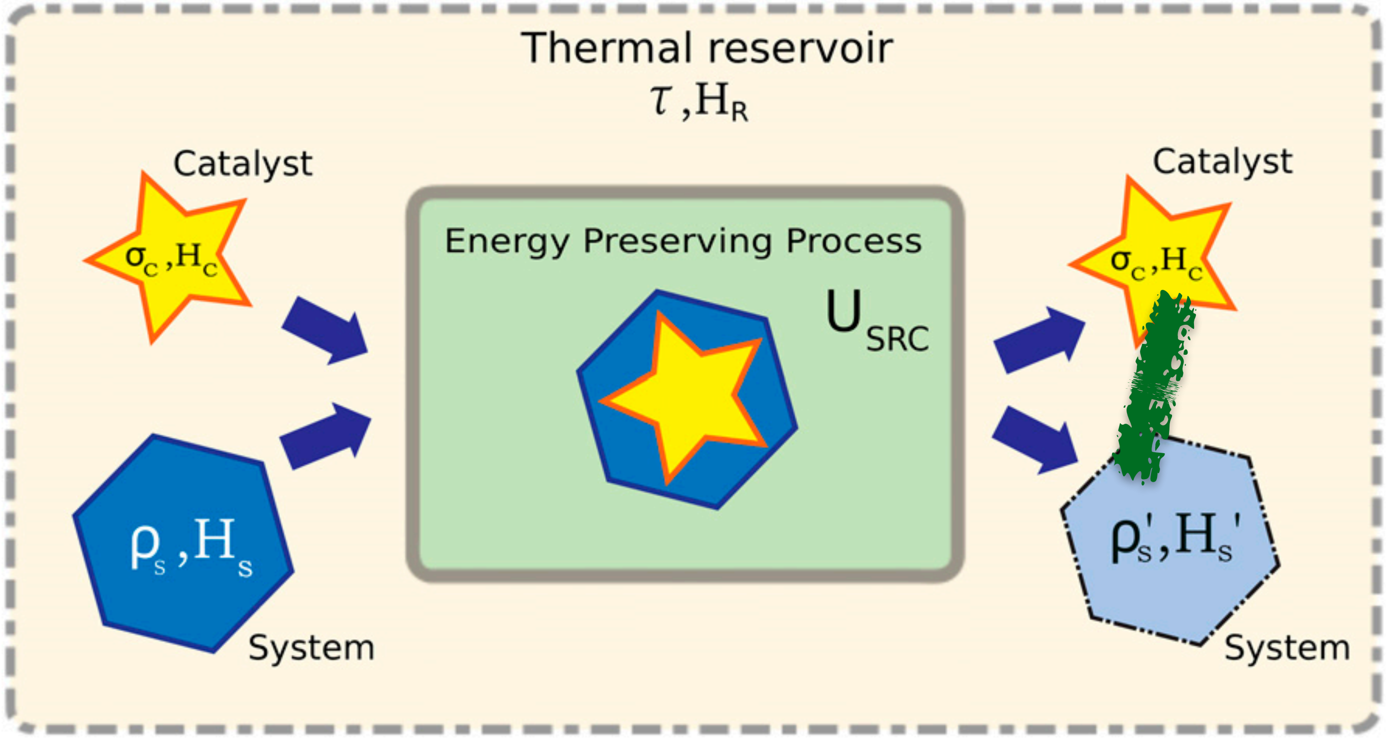

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Theorem [1]: Possible if and only if $F(\rho_S) \geq F(\rho'_S)$.

One-shot interpretation of the free energy *F*.

One-shot interpretation of non-equilibrium **F**

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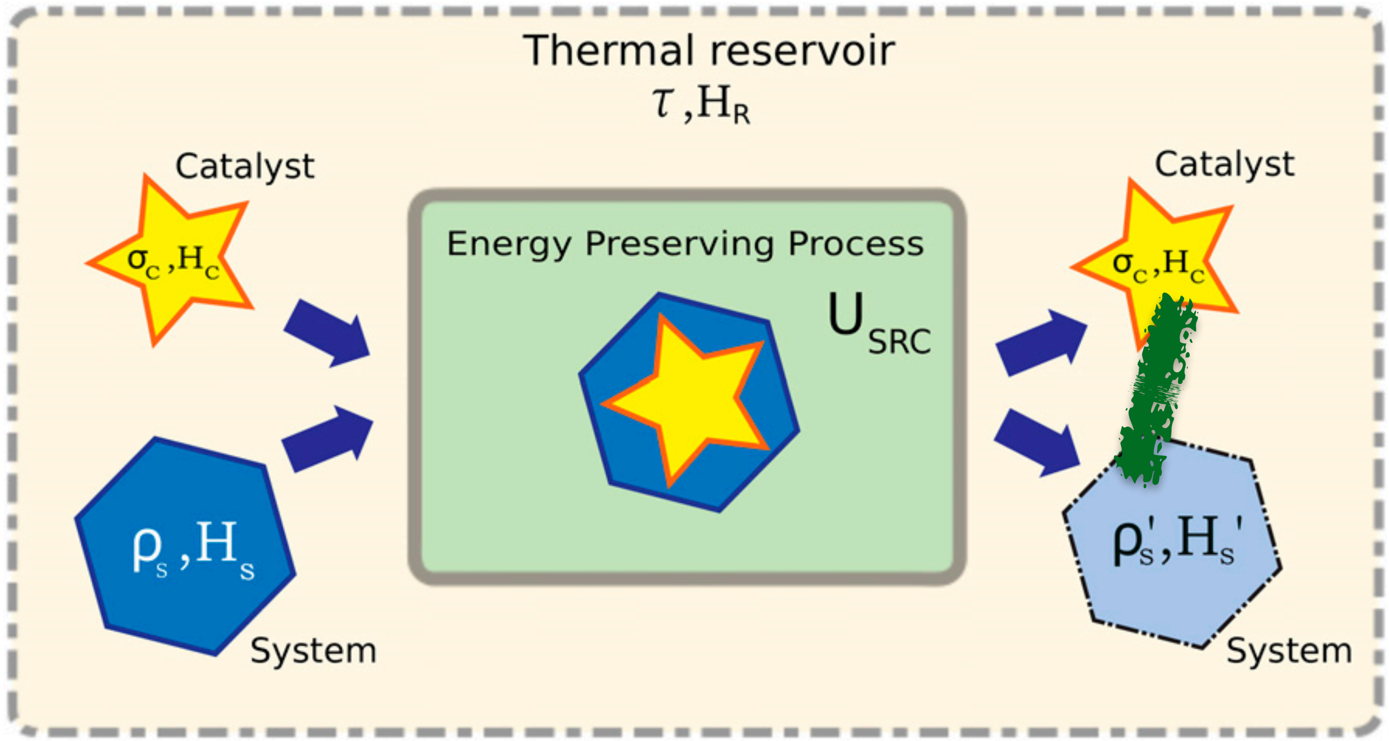

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When is a transition $\rho_S \to \rho'_S$ possible?

- Fluctuation-free **work cost:** If
  then transition possible while work bit $|e\rangle_W \mapsto |g\rangle_W$.
- *Almost* fluct.-free **work extraction:** If $F(\rho_A) - F(\rho'_A) > \Delta > 0$,
  then transition possible while work bit (for arbitrary $\delta > 0$)
  $$|g\rangle\langle g|_W \mapsto (1-\delta)|e\rangle\langle e|_W + \delta\,\mathbf{1}/d.$$

Questions

- Can the resource-theoretic approach be applied to "one-shot regime" of classical reversible computation?

  E.g. if irreversibility cannot be completely avoided, can it at least be implemented in the "least costly" way?

- Can it be suitably modified to incorporate realistic constraints arising in classical reversible computation?

  C. Perry, P. Ćwikliński, J. Anders, M. Horodecki, and J. Oppenheim, *A Sufficient Set of Experimentally Implementable Thermal Operations for Small Systems*, Phys. Rev. X **8**, 041049 (2018).

Thank you!