Thermalization and canonical typicality
in translation-invariant quantum lattice systems

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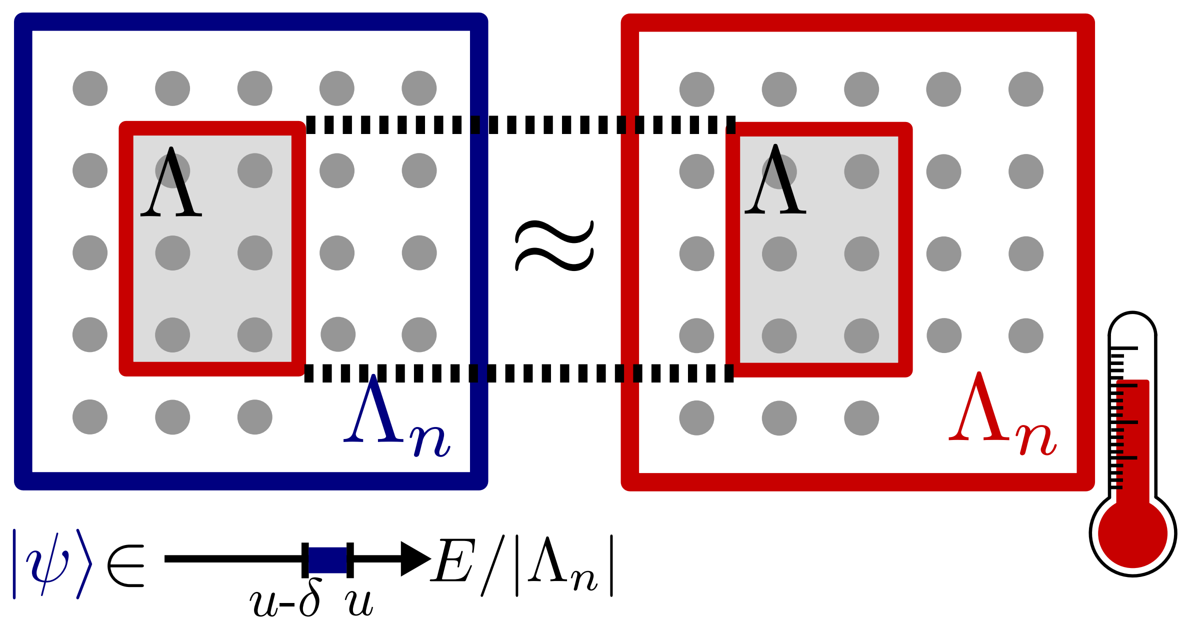
$$\mathrm{Tr}_{\Lambda_n \setminus \Lambda} |\psi\rangle\langle\psi| \approx \mathrm{Tr}_{\Lambda_n \setminus \Lambda} \frac{\exp(-\beta H^p_{\Lambda_n})}{Z}$$

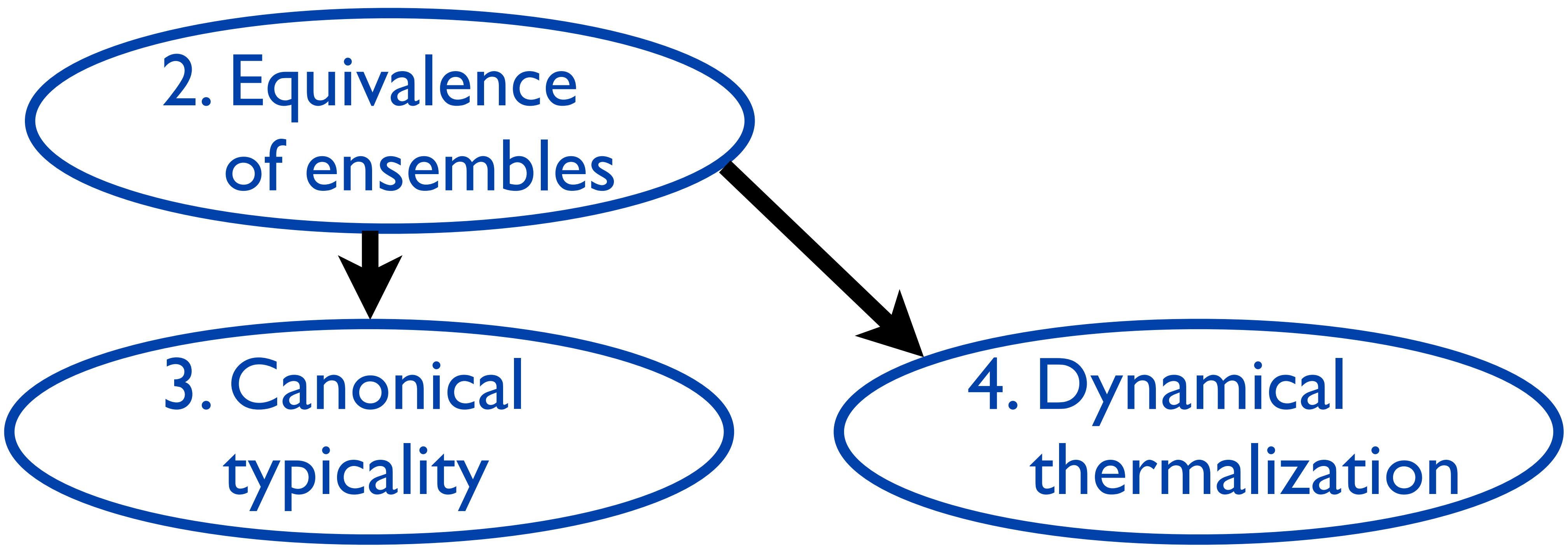


joint work with Emily Adlam, Lluís Masanes, and Nathan Wiebe.

Outline

1. The setup

Quantum lattice systems with interaction.



5. Eigenstate thermalization hypothesis

A weak version that holds for all models.

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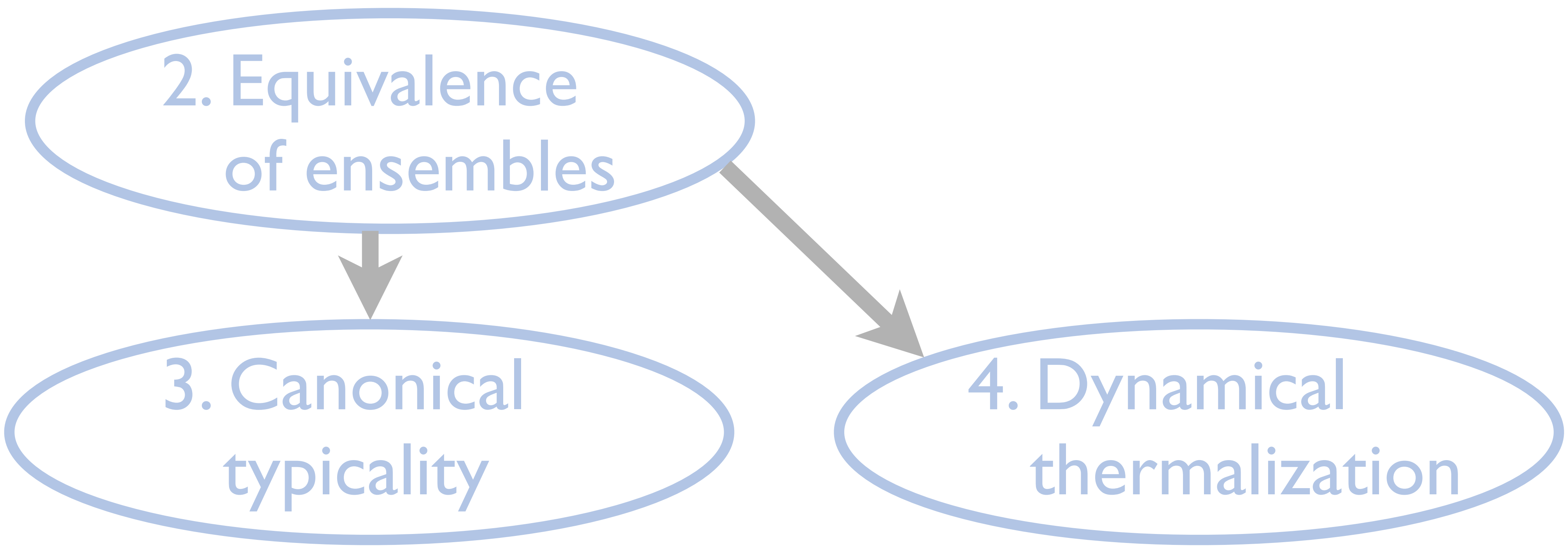

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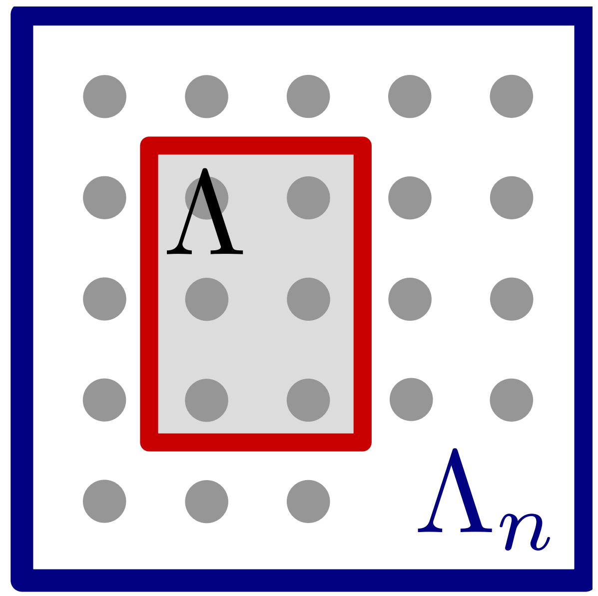

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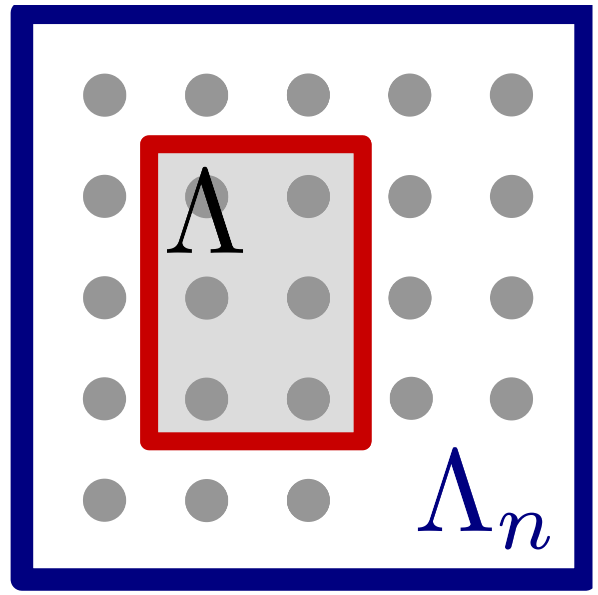

For example $\Lambda_n = [-n, n]^\nu$.

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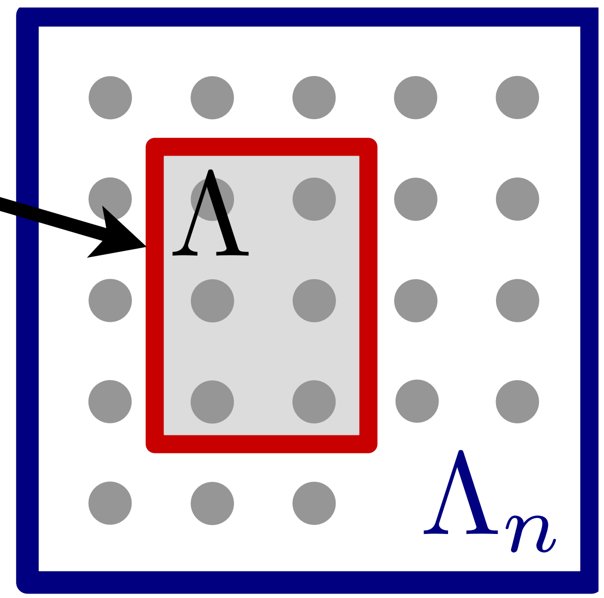
Small sub-system Λ



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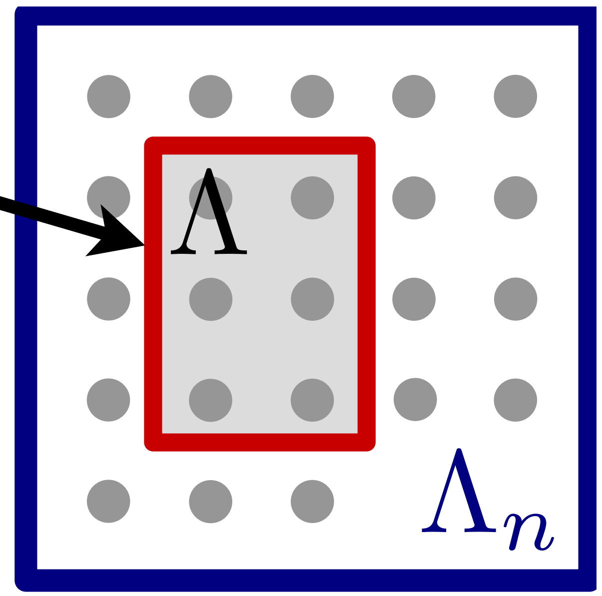

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Some Hamiltonian H_{Λ_n} on Λ_n, for example

$$H_{\Lambda_n} = -J \sum_{i=1}^{n} \left(X_i X_{i+1} + Y_i Y_{i+1} \right) - h \sum_{i=1}^{n} Z_i .$$

Finite-range, translation-invariant; otherwise arbitrary.

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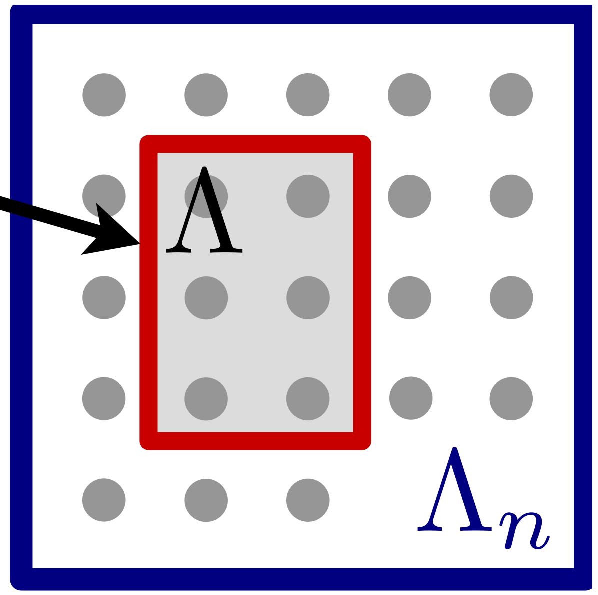

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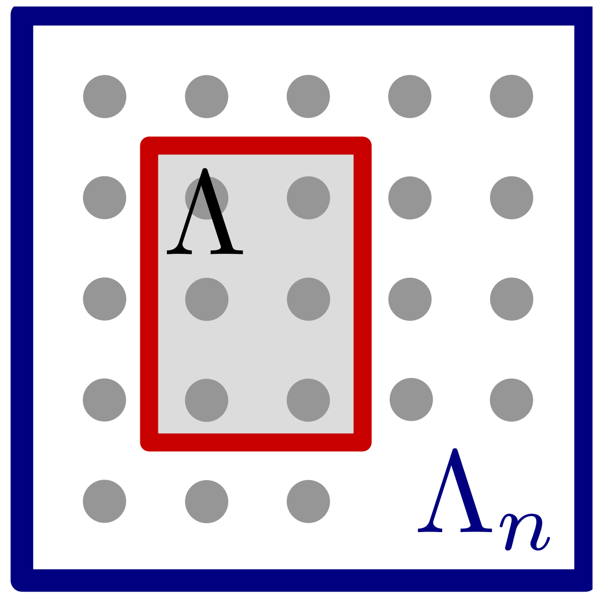
In this talk: periodic boundary conditions.

Why translation-invariant lattice systems?



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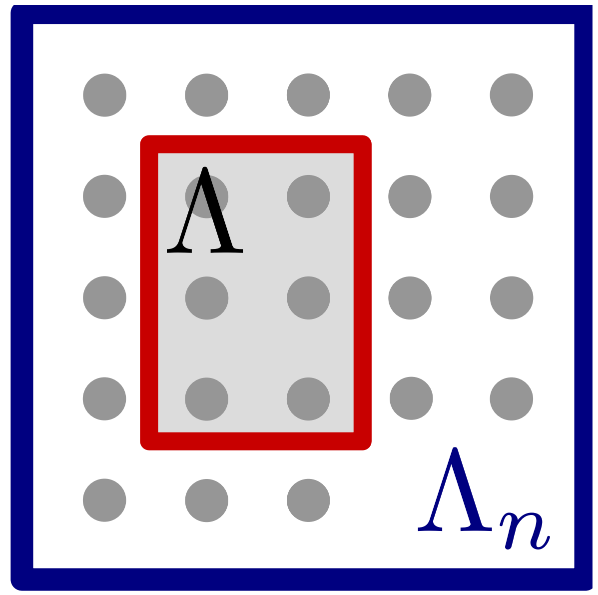

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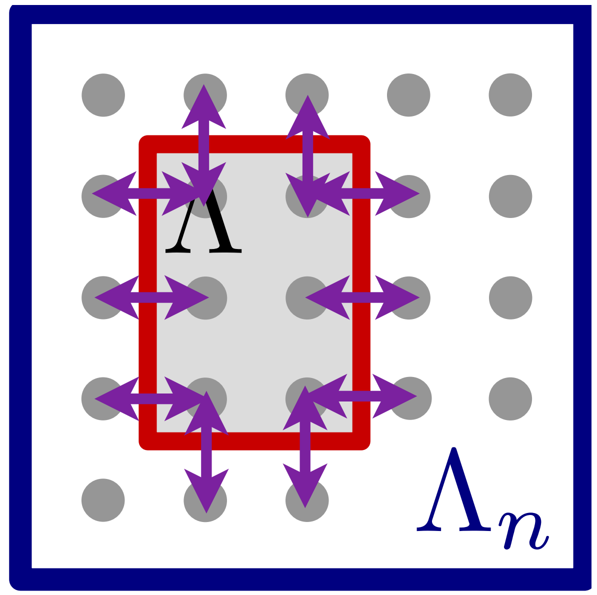

$$\|H_{\text{int}}\| \ll k_B T$$

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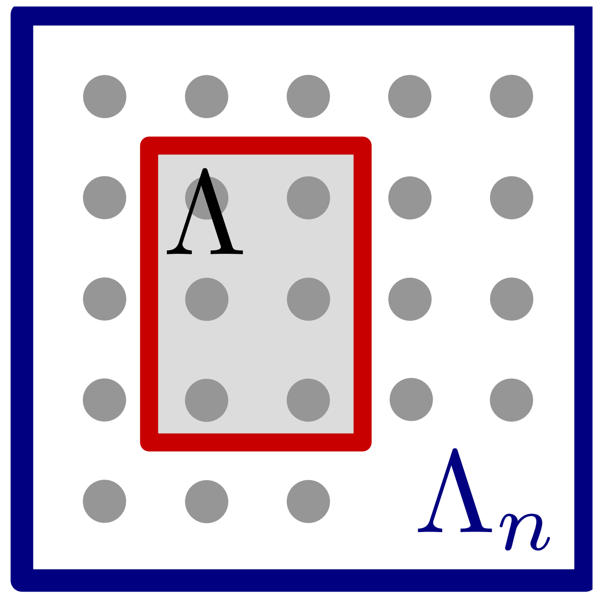

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- We show: T.I. + F.R. ⇒ thermality, also for strong interaction.

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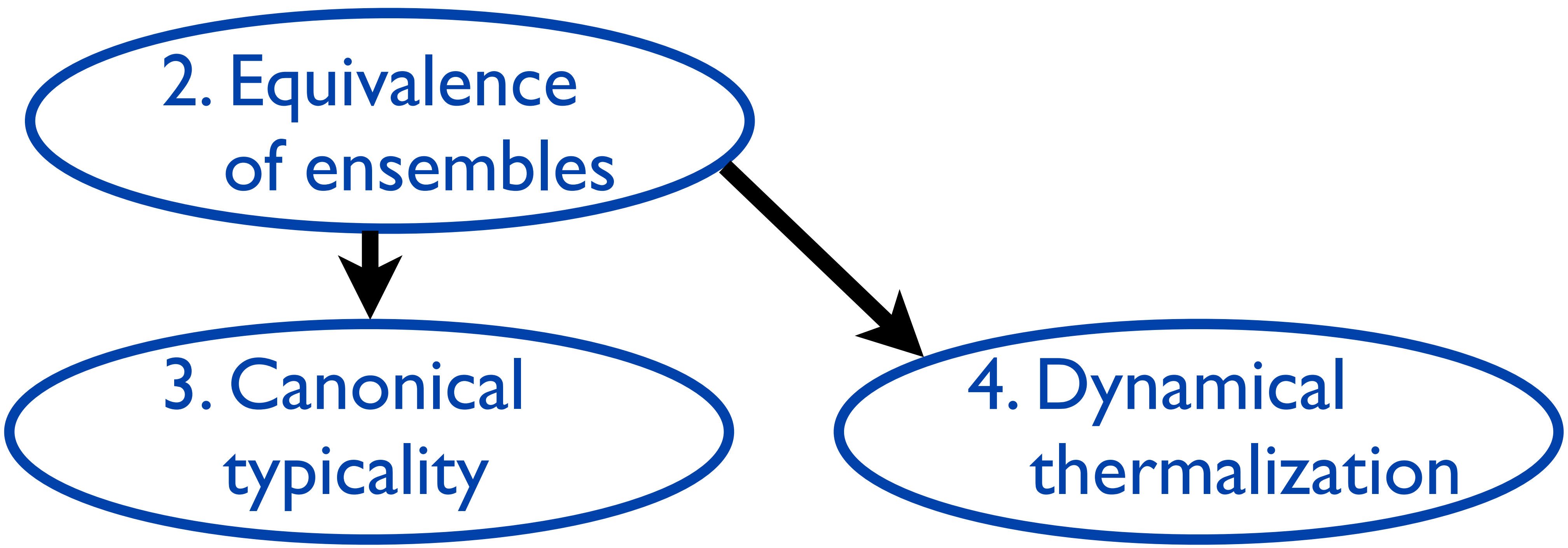

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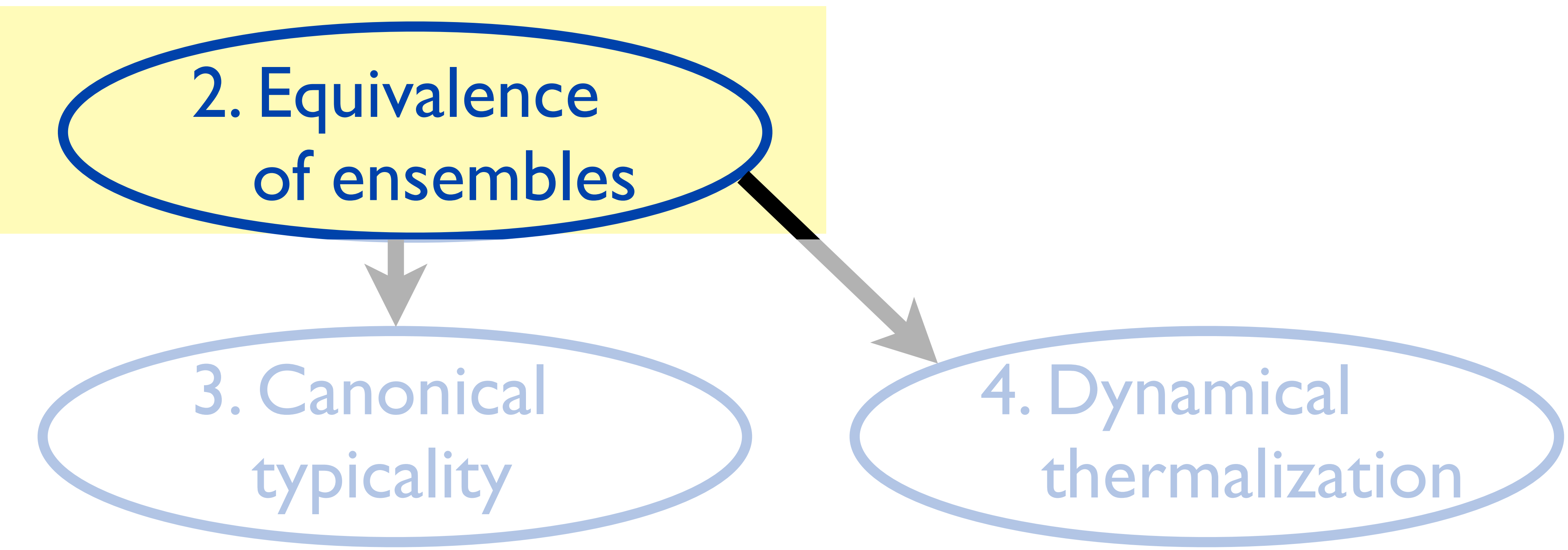

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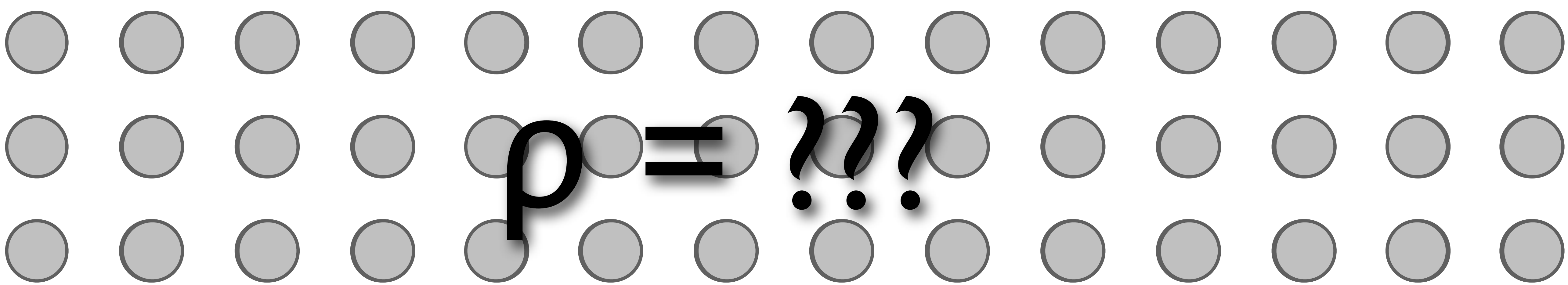
Proof goes via infinite-lattice Gibbs states.



Definition: (cf. Barry Simon, Stat. Mech. of Lattice Gases)

A state ω on the infinite lattice is a family of density matrices

$$(\omega_\Lambda)_{\Lambda\subset\mathbb{Z}^\nu\,\mathrm{finite}}$$

satisfying $\Lambda'\subset\Lambda \;\Rightarrow\; \omega_{\Lambda'} = \mathrm{Tr}_{\Lambda\setminus\Lambda'}\omega_\Lambda.$

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Local Gibbs states $\rho_{\Lambda_n}(\beta) := \exp(-\beta H^p_{\Lambda_n})/Z$
minimize the free energy functional

$$F(\rho) := \mathrm{tr}(\rho H^p_{\Lambda_n}) - S(\rho)/\beta \;\; (= U - TS).$$

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Use analogous definition on infinite lattice via densities

$$u(\omega) \quad := \quad \lim_{n\to\infty}\frac{1}{|\Lambda_n|}\mathrm{tr}(\omega_{\Lambda_n}H^p_{\Lambda_n}),$$

$$s(\omega) \quad := \quad \lim_{n\to\infty}\frac{1}{|\Lambda_n|}S(\omega_{\Lambda_n}).$$

Definition: A translation-invariant state ω on the infinite lattice is a Gibbs state at inv.temp. β if it minimizes the F.E. density, i.e.

$$u(\omega) - s(\omega)/\beta \leq f_{\rm th}(\beta),$$

where $f_{\rm th}(\beta)$ is the limiting F.E. density of local Gibbs states.

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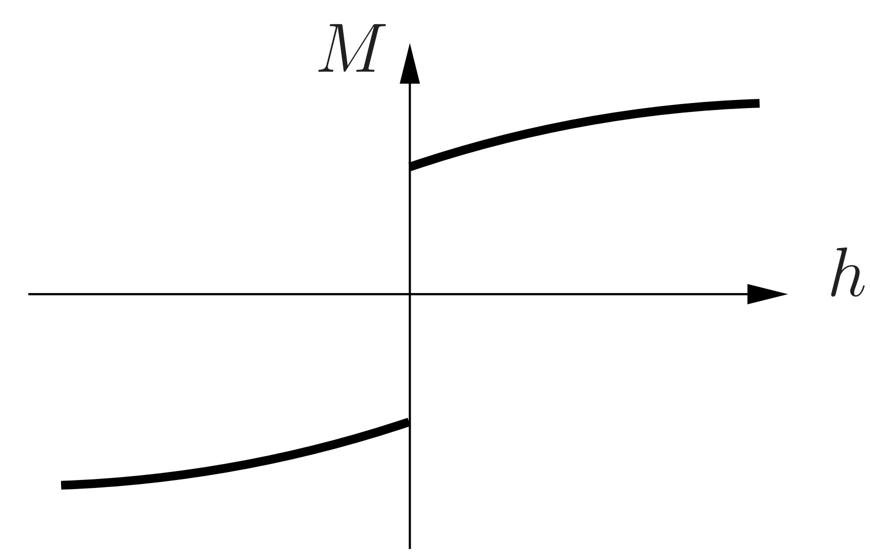
! If $\nu \geq 2$ then there can be more than one Gibbs state
⇒ several phases / phase transition!



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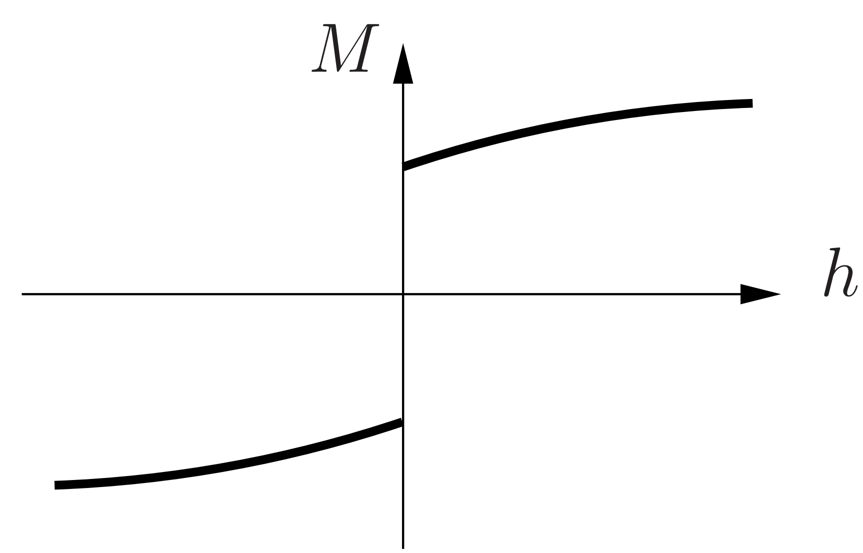
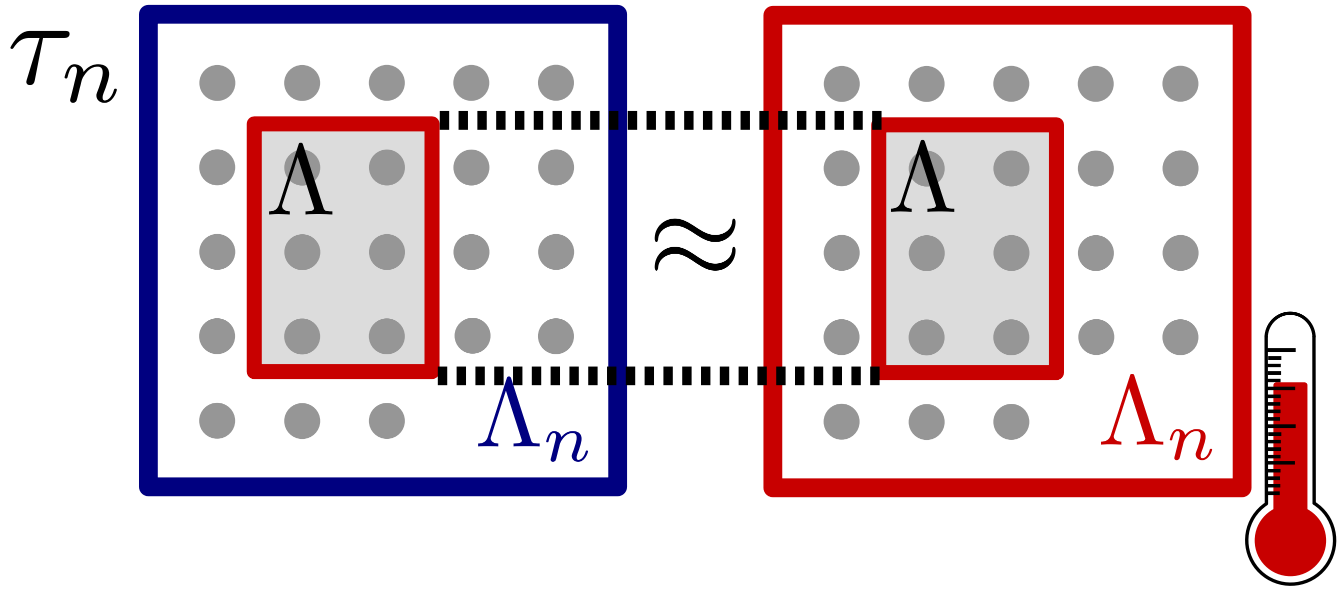


Now show:

Every sequence of states with asymptotically minimal free energy density is equivalent to the canonical ensemble.

$$\mathrm{Tr}_{\Lambda_n\setminus\Lambda}\tau_n \approx \mathrm{Tr}_{\Lambda_n\setminus\Lambda}\frac{\exp(-\beta H^p_{\Lambda_n})}{Z}$$



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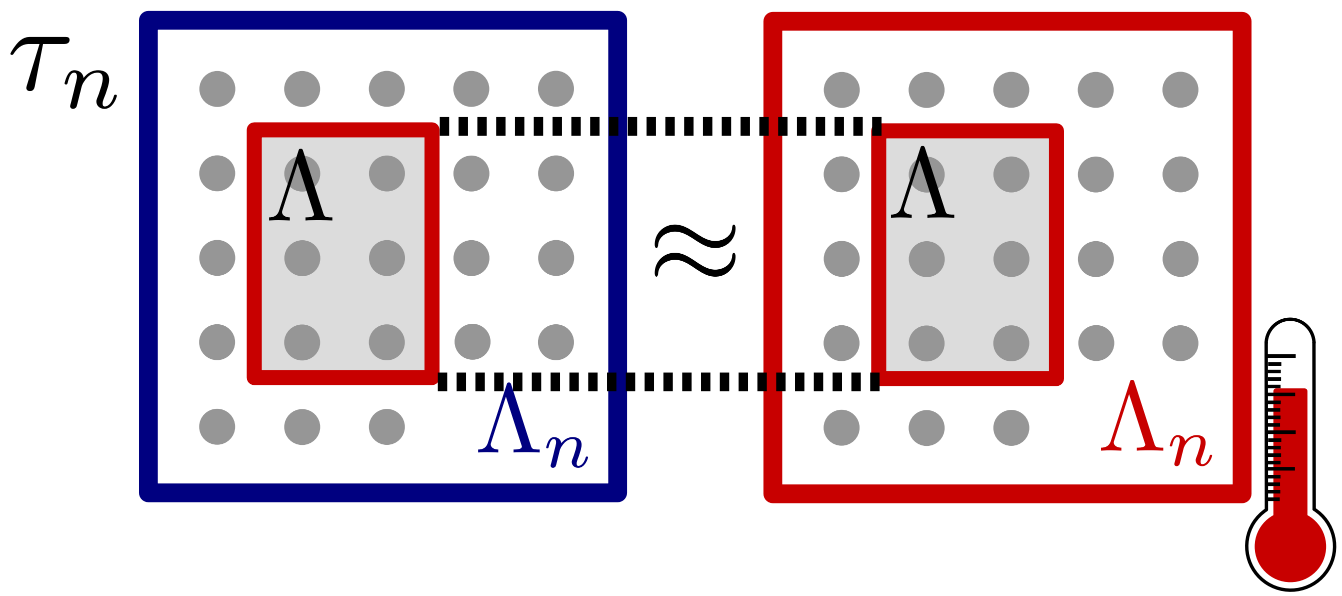

Theorem 2. *Suppose that $(\tau_n)_{n\in\mathbb{N}}$ is any sequence of Λ_n-translation-invariant states on Λ_n, and $\beta > 0$ such that there is a unique phase around inverse temperature β. If*

$$\limsup_{n\to\infty}\frac{1}{|\Lambda_n|}\left(\mathrm{tr}(\tau_n H^{BC}_{\Lambda_n}) - S(\tau_n)/\beta\right) \leq f_{\mathrm{th}}(\beta) \qquad (2)$$

for some choice of boundary conditions BC, then

$$\lim_{n\to\infty}\left\|\mathrm{Tr}_{\Lambda_n\setminus\Lambda}\tau_n - \mathrm{Tr}_{\Lambda_n\setminus\Lambda}\frac{\exp(-\beta_n H^p_{\Lambda_n})}{Z_n}\right\|_1 = 0, \qquad (3)$$

where we may set β_n either equal to the fixed value β, or equal to the solution of $\mathrm{tr}(H^p_{\Lambda_n}\rho^p_{\Lambda_n}(\beta_n))/|\Lambda_n| = u(\beta)$.

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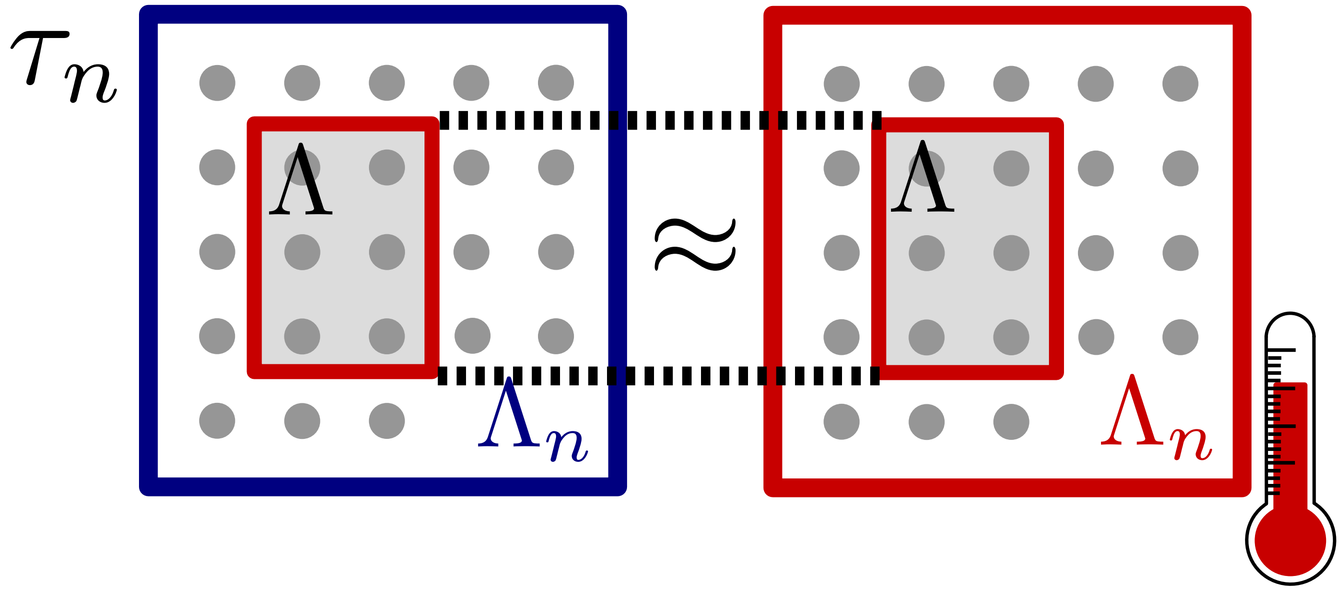

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Example: $\tau_n =$ mixture on

$\mathrm{span}\,\{|E\rangle \mid u-\delta \leq E/|\Lambda_n| \leq u\}$

Microcanonical ensemble!

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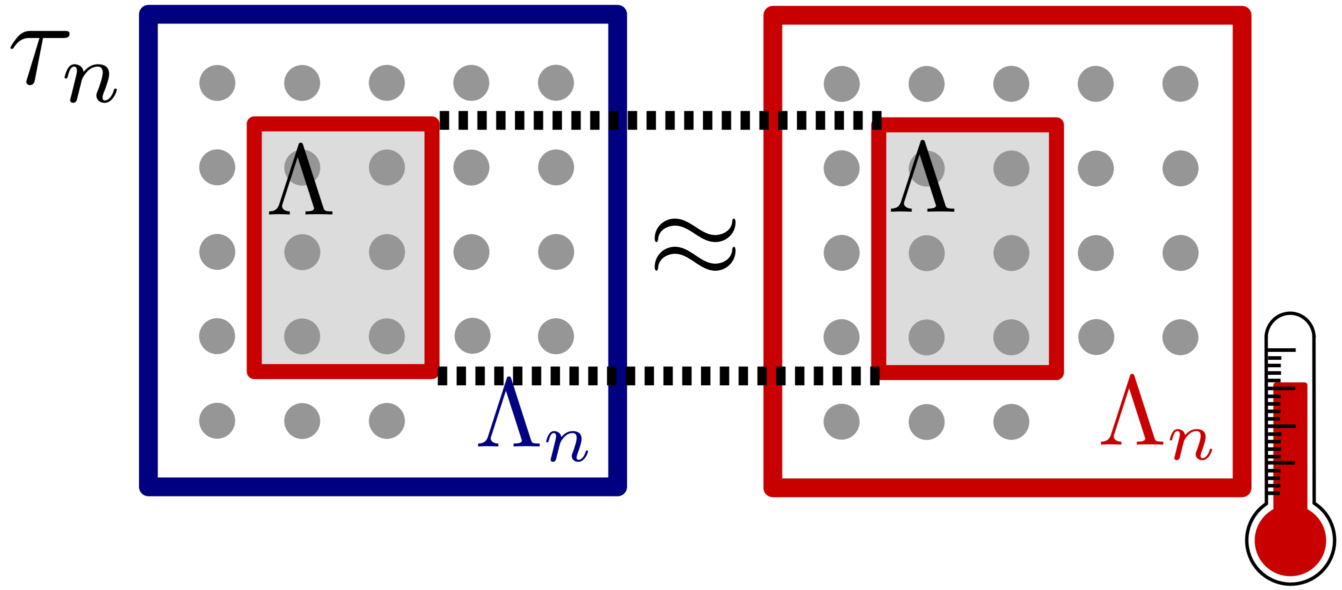

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Proof: they both converge locally to the unique infinite-volume Gibbs state.

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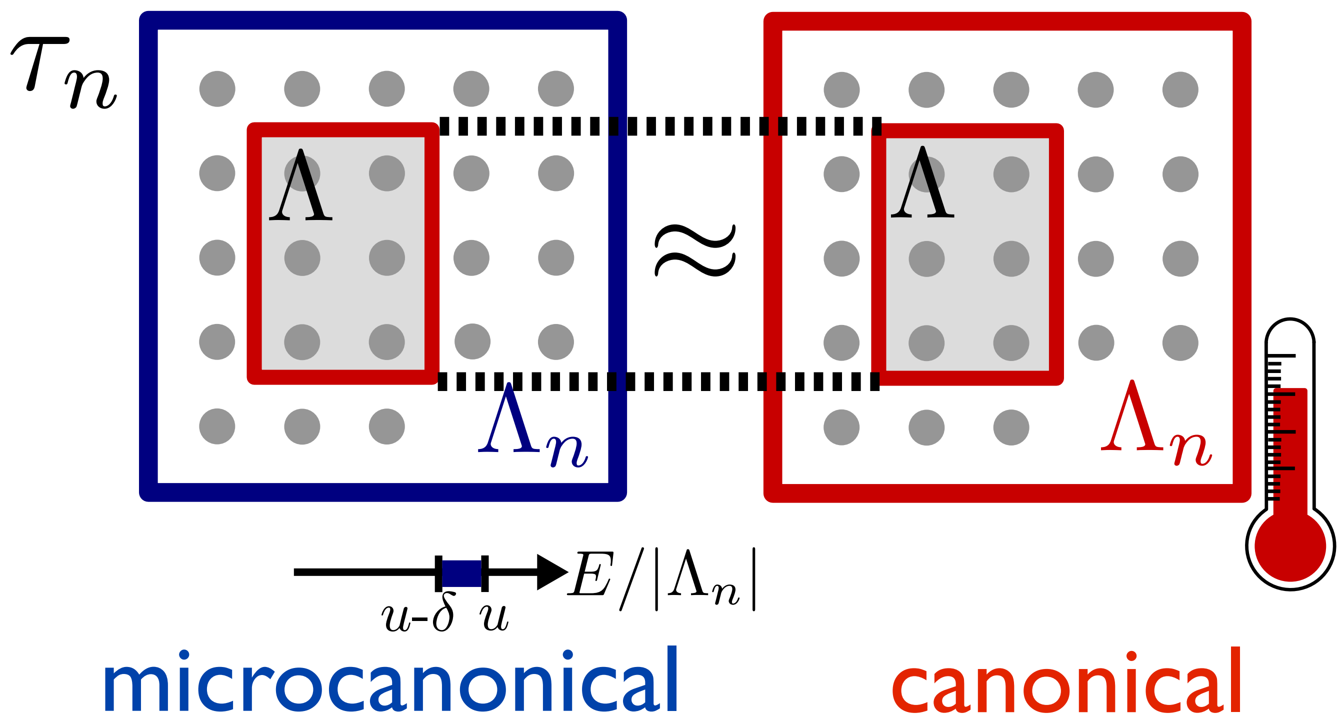

For local qubits, without interaction, δ=0 (de Finetti Theorem):

$$\left\| \mathrm{Tr}_{\Lambda_n \setminus \Lambda} \tau_n - \mathrm{Tr}_{\Lambda_n \setminus \Lambda} \frac{\exp(-\beta H^p_{\Lambda_n})}{Z} \right\|_1 \leq \frac{4|\Lambda|}{|\Lambda_n|}$$

Similar analytic bound for δ>0, and numerically with interaction.

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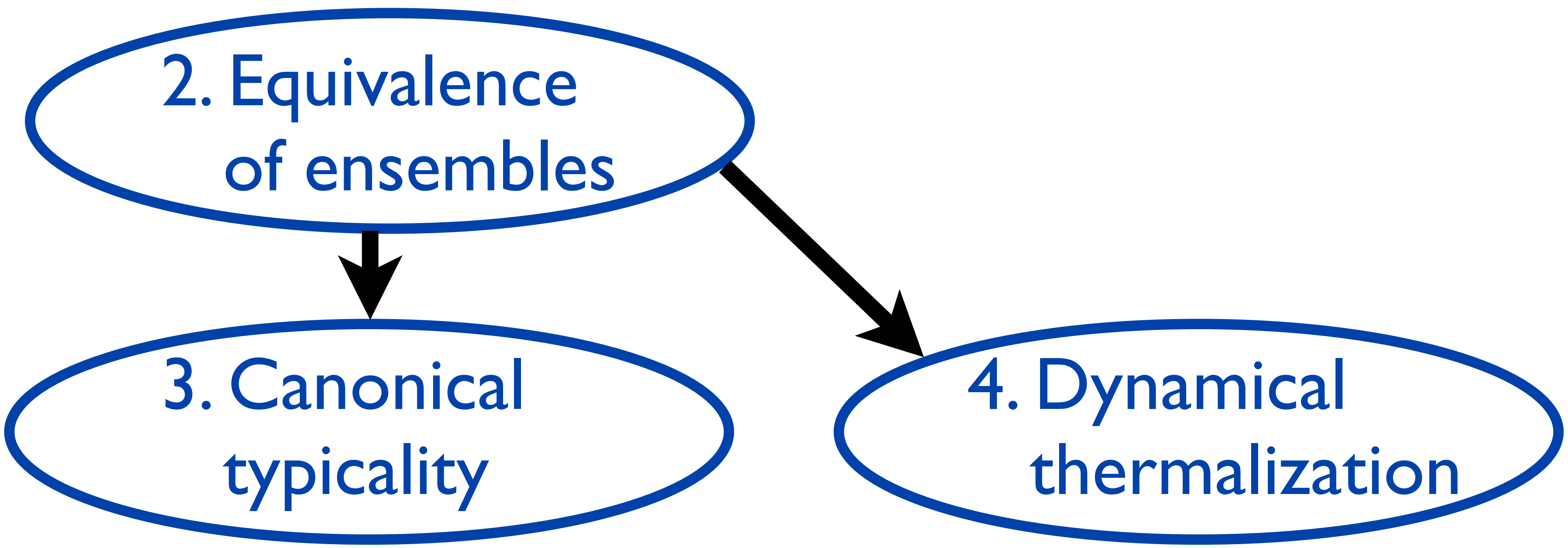

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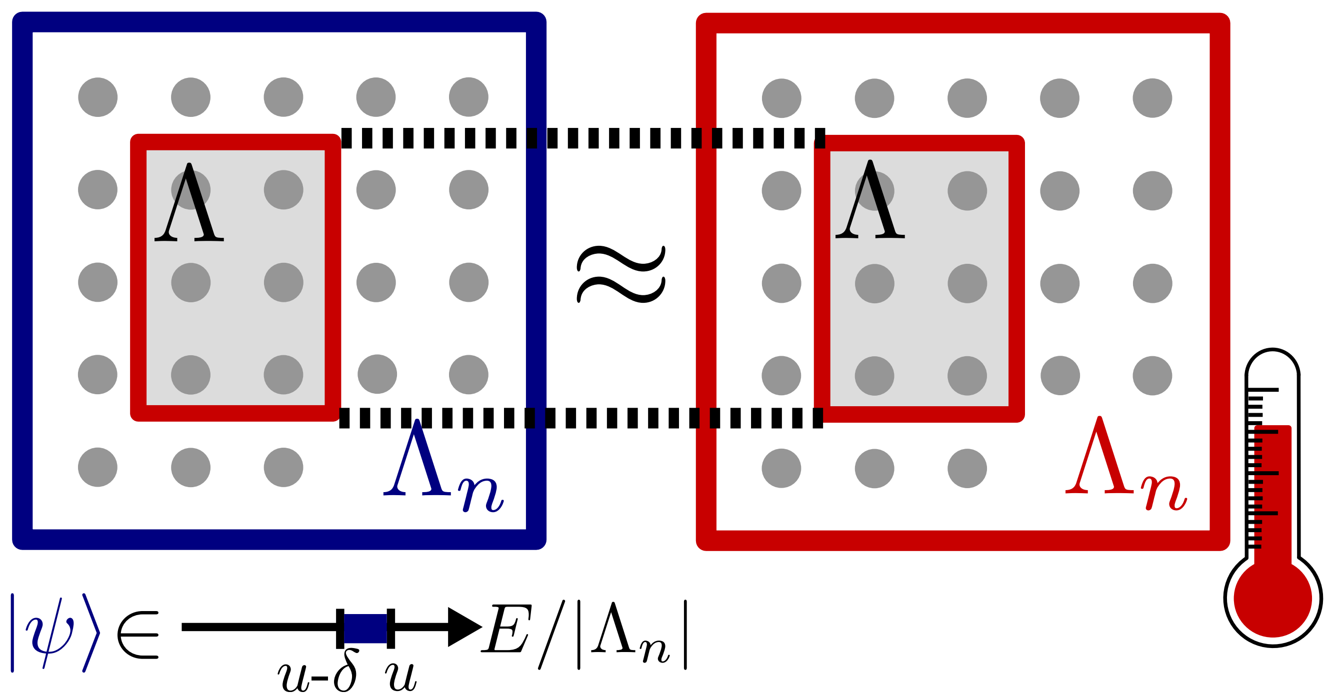

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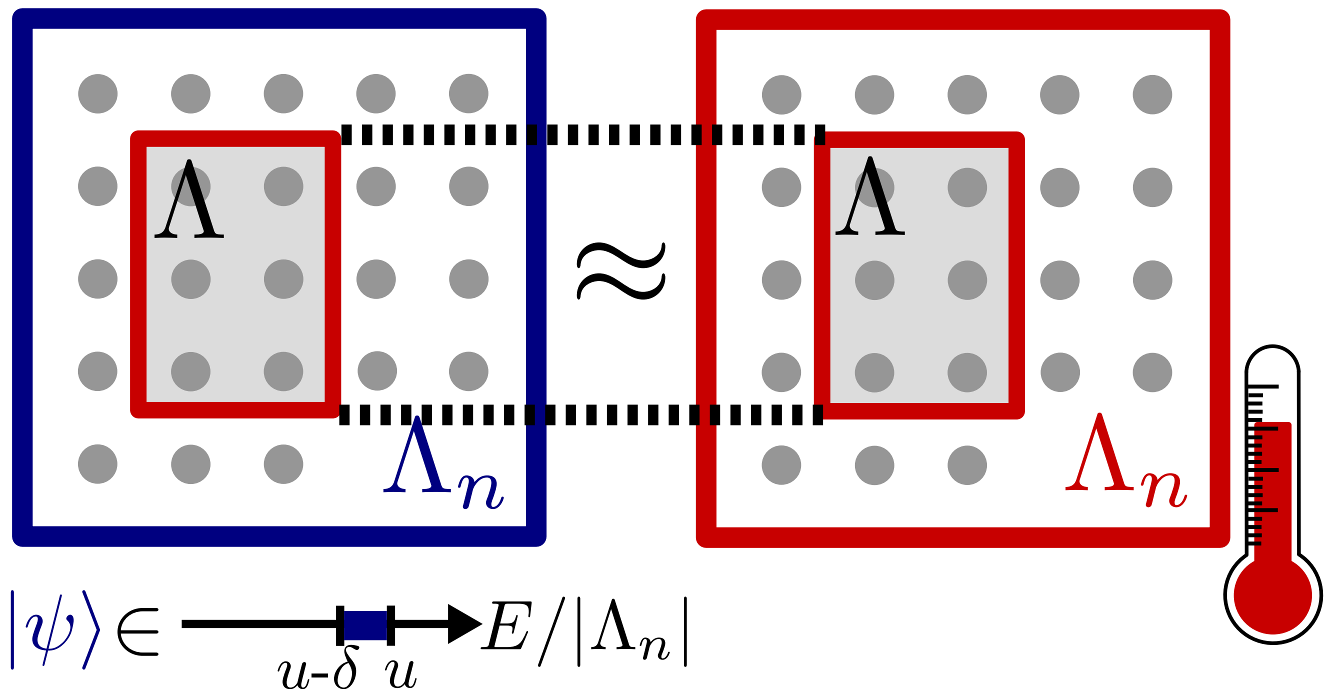

Draw a pure state $|\psi\rangle$ at random from the microcanonical subspace

$$\mathrm{span}\,\{|E\rangle\ |\ u-\delta\leq E/|\Lambda_n|\leq u\}$$

If there is a unique phase around $\beta(u)$ then:

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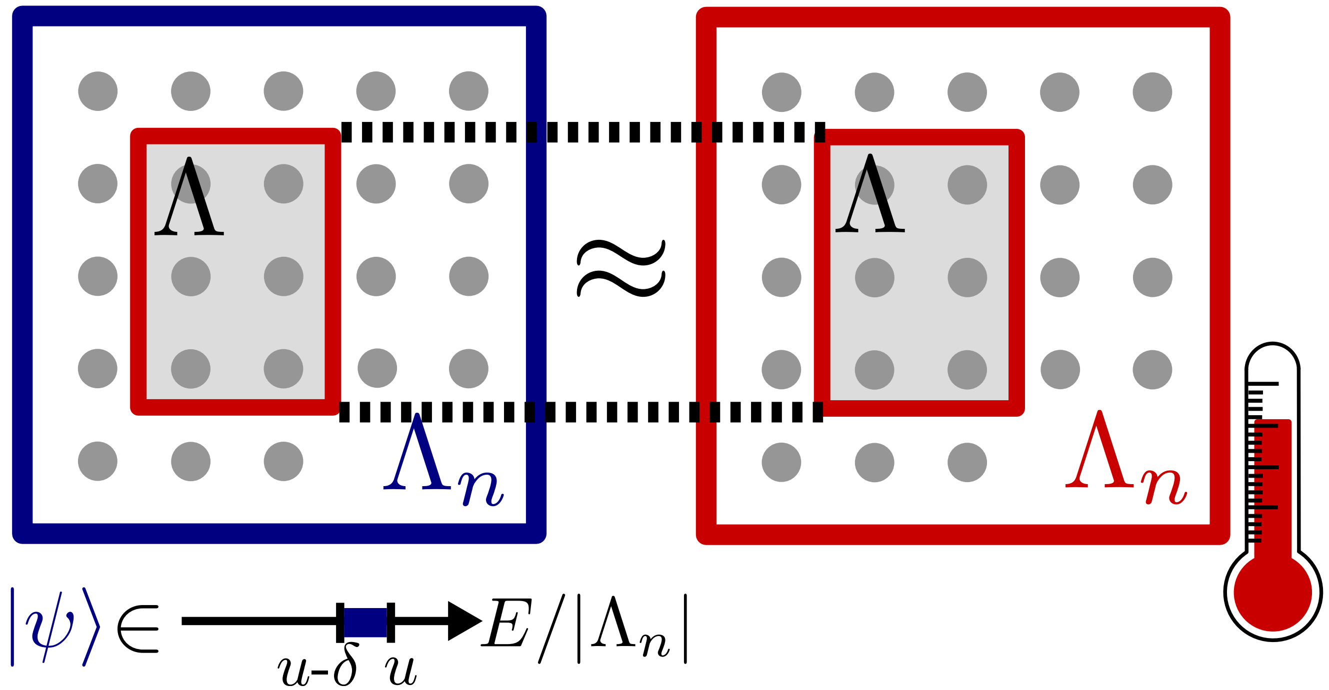

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If there is a unique phase around $\beta(u)$ then:

Theorem 1. *For any $\varepsilon \geq 0$, the probability p that a state $|\psi\rangle \in T^p_n$ sampled according to the unitarily invariant measure satisfies*

$$\left\|\mathrm{Tr}_{\Lambda_n\setminus\Lambda}|\psi\rangle\langle\psi| - \mathrm{Tr}_{\Lambda_n\setminus\Lambda}\frac{\exp(-\beta H^p_{\Lambda_n})}{Z}\right\|_1 \geq \varepsilon + \Delta_{n,\Lambda}$$

is doubly-exponentially small in the lattice size $|\Lambda_n|$; that is, $p \leq \exp\left(-\varepsilon^2 \exp(|\Lambda_n| s + o(|\Lambda_n|))\right)$, where $s = s(\omega_\beta)$ is the entropy density of the corresponding Gibbs state, and $\Delta_{n,\Lambda}$ is a sequence of positive real numbers with $\lim_{n\to\infty}\Delta_{n,\Lambda} = 0$ for every fixed Λ. Here, β can either be set equal to $\beta(u)$ as defined above, or equal to the solution of $\mathrm{tr}(H^p_{\Lambda_n}\rho^p_{\Lambda_n}(\beta))/|\Lambda_n| = u$ (which depends on n).

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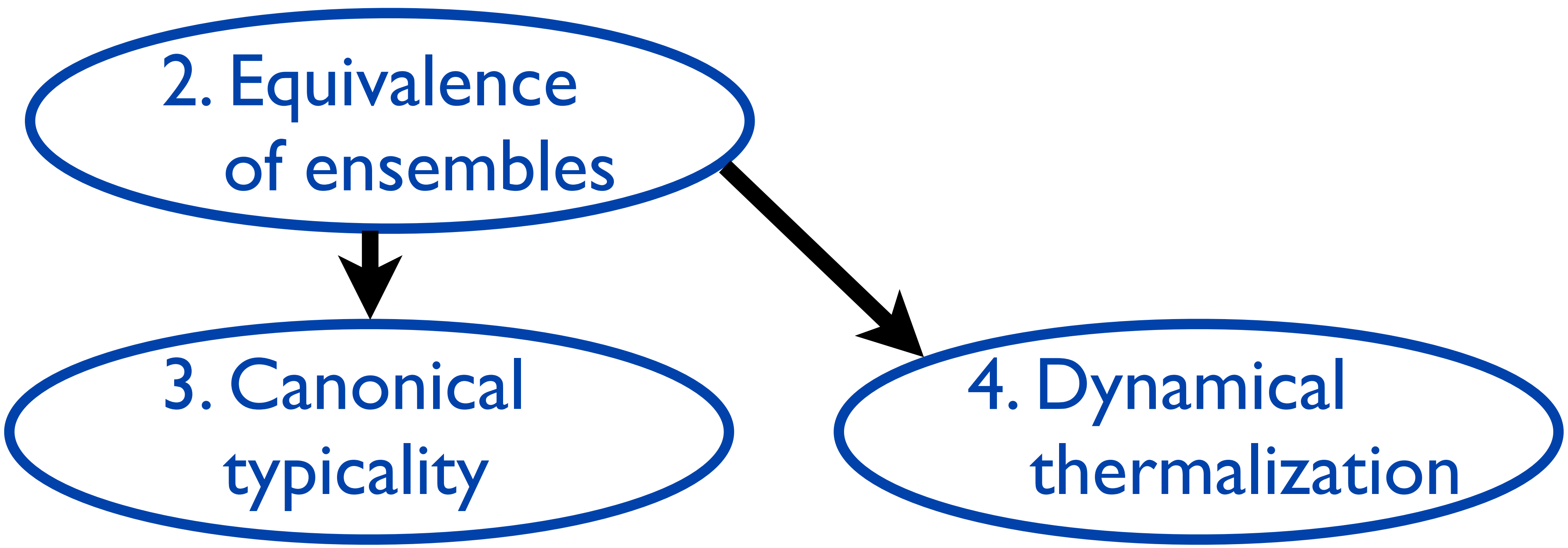

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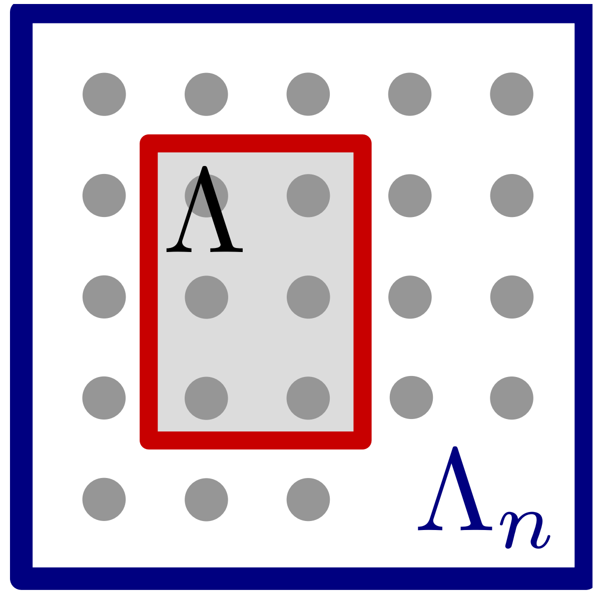
Full lattice Λ_n evolves unitarily: $|\psi(t)\rangle = \exp(-itH^p_{\Lambda_n})|\psi(0)\rangle$



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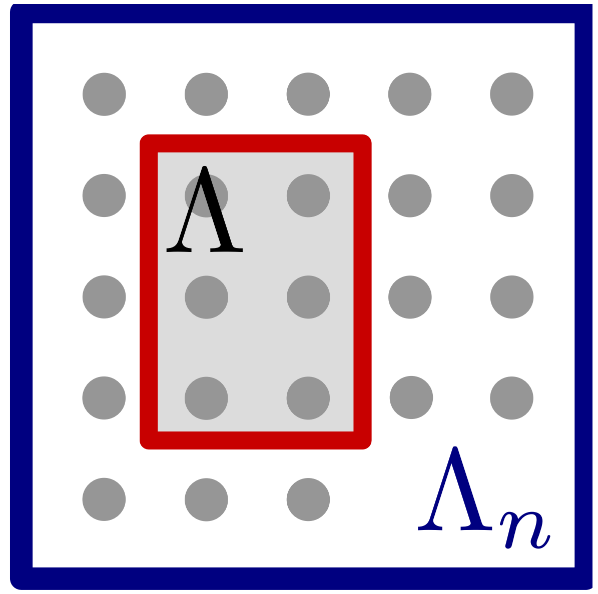

We show: if Shannon entropy of initial occupation numbers

$$p_n = |\langle E_n|\psi(0)\rangle|^2$$

is close to maximal, in first order in $|\Lambda_n|$,
then

$$\mathrm{Tr}_{\Lambda_n\setminus\Lambda}|\psi(t)\rangle\langle\psi(t)| \approx \mathrm{Tr}_{\Lambda_n\setminus\Lambda}\frac{\exp(-\beta H^p_{\Lambda_n})}{Z}$$

for most times *t*.

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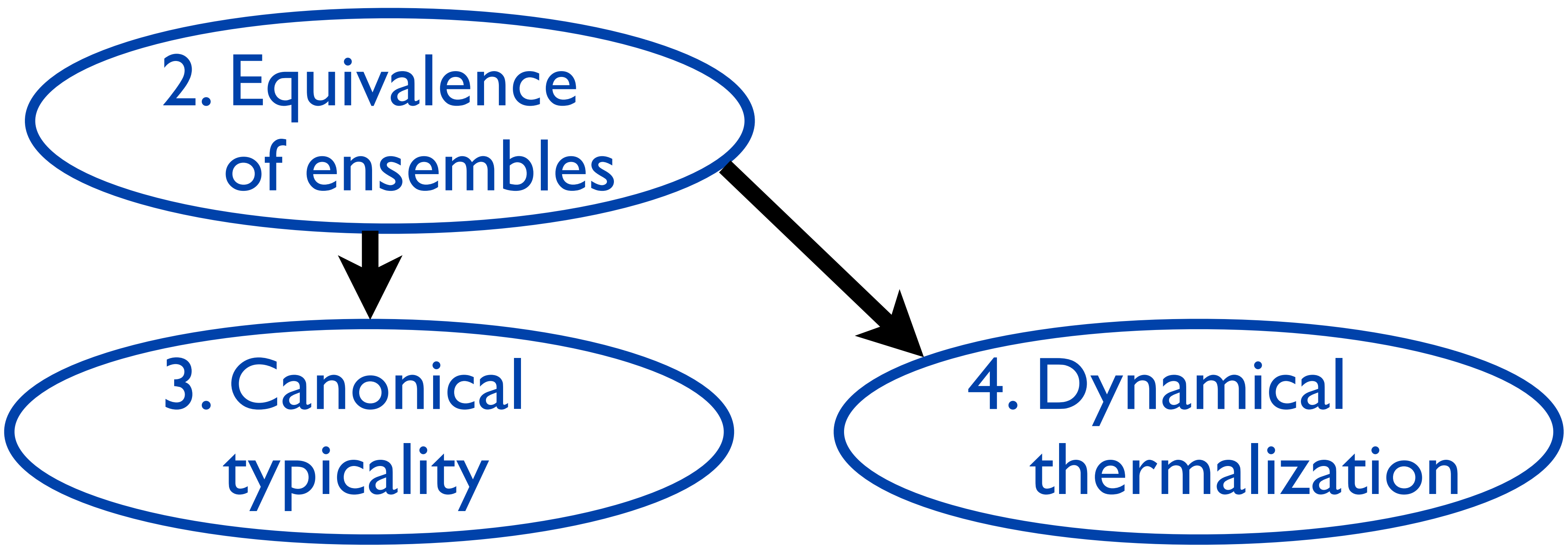

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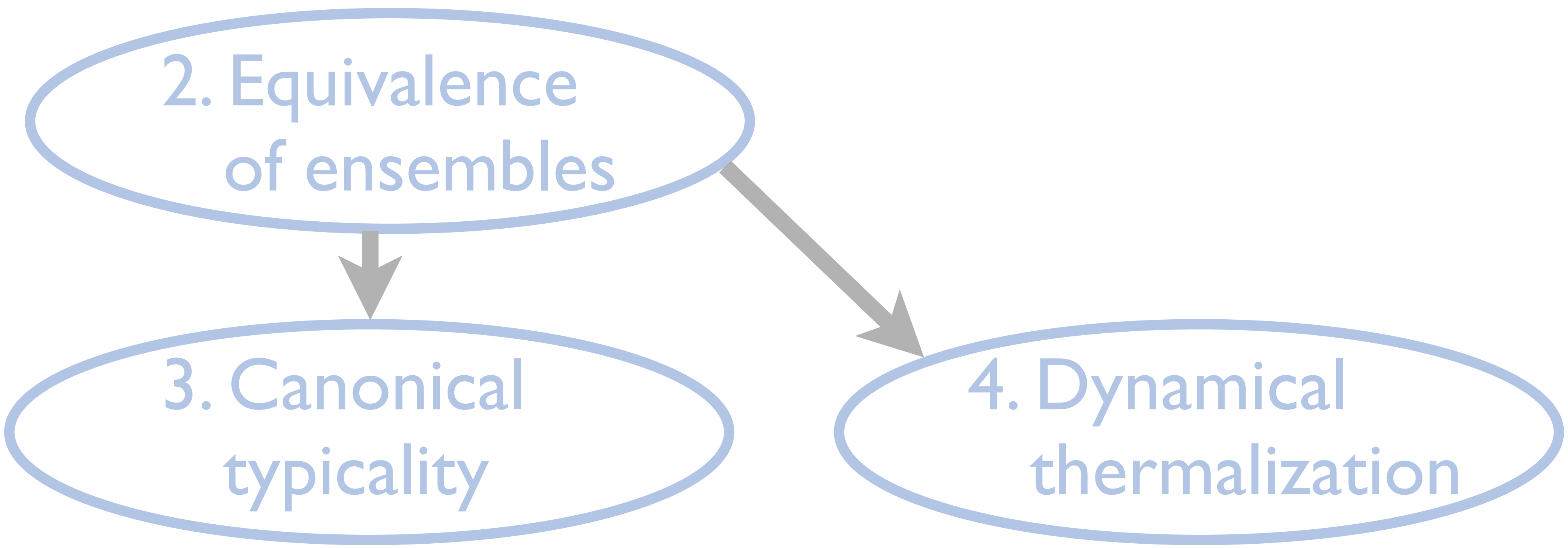

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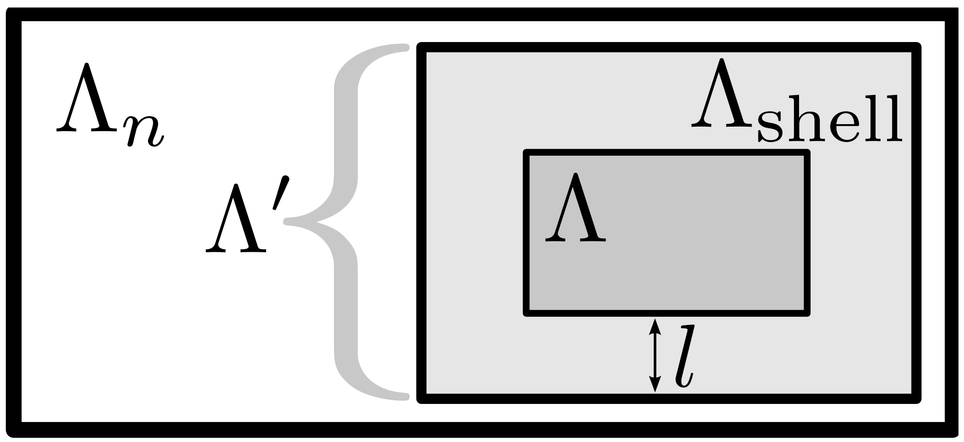

Natural conjecture: If $|E\rangle$ is an energy eigenstate on Λ_n , then

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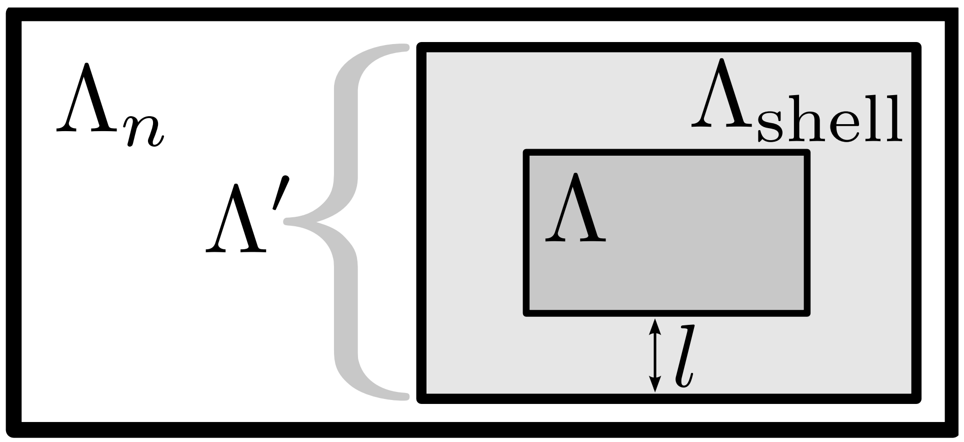

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Theorem 4. *There is a state ω_E on Λ' such that*

$$\left\|\mathrm{Tr}_{\Lambda_{\text{shell}}}(\omega_E) - \mathrm{Tr}_{\Lambda_n\setminus\Lambda}|E\rangle\langle E|\right\|_1 \leq \kappa\cdot e^{-c(l-r)/2}$$

which is weakly diagonal in the eigenbasis $\{|e\rangle\}$ of $H_{\Lambda'}$, i.e.

$$|\langle e_1|\omega_E|e_2\rangle| \leq e^{-(l-r)(e_1-e_2)^2/(8cv^2)},$$

where $\kappa, c, v > 0$ are constants, and r the range of interaction.

Conclusions

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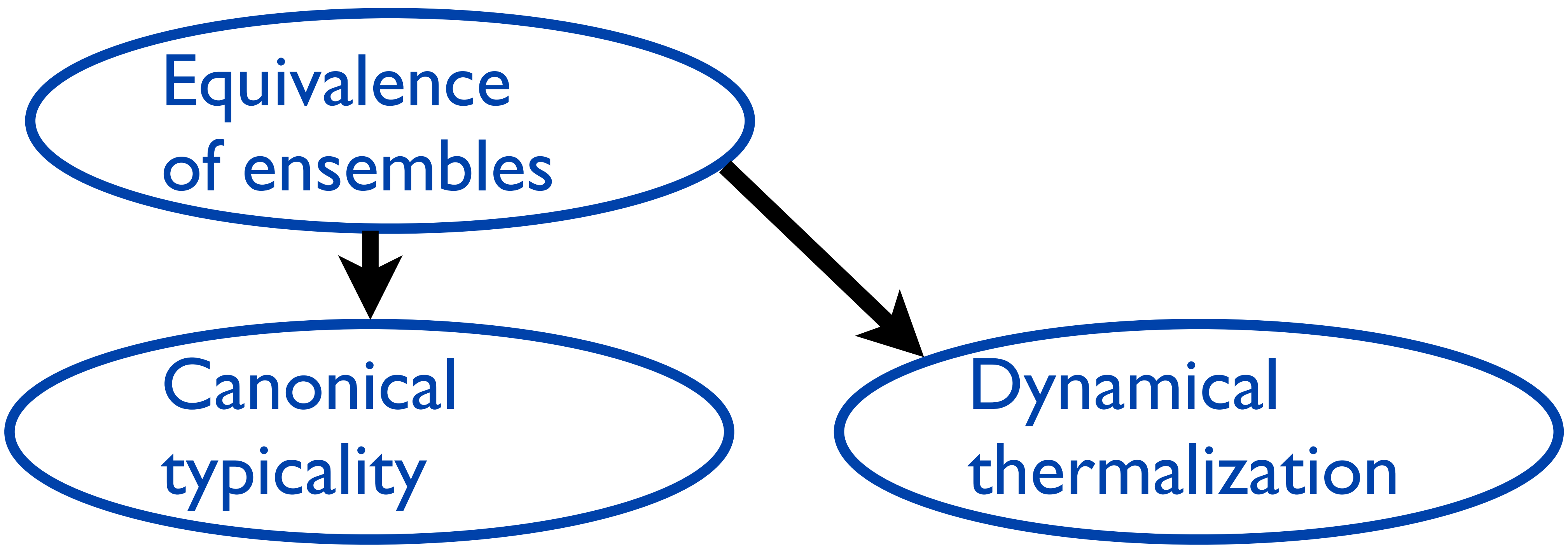

Open problem: Can we prove a strong version of the ETH? Integrability?

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