# Concentration of measure for quantum states with a fixed expectation value (arXiv:1003.4982)

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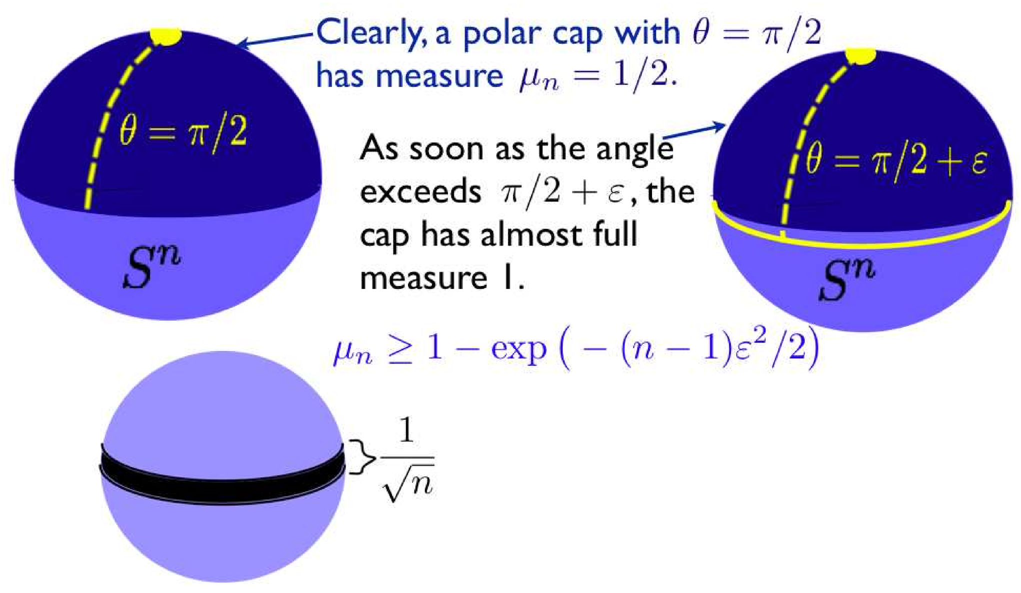
Given some observable $H$ on a finite-dimensional quantum system, we investigate the typical properties of random state vectors $|\psi\rangle$ that have a fixed expectation value $\langle\psi|H|\psi\rangle = E$ with respect to $H$. Under some conditions on the spectrum, we prove that this manifold of quantum states shows a concentration of measure phenomenon: any continuous function on this set is almost everywhere close to its mean. We also give a method to estimate the corresponding expectation values analytically, and we prove a formula for the reduced density matrix in the case that $H$ is a sum of local observables.

## Concentration of measure in quantum information theory...

- What do random quantum states look like?
- Drawing a pure state in $\mathbb{C}^d$ randomly wrt. the unitarily invariant measure corresponds to picking a point on the unit sphere $S^{2d-1}$ in $\mathbb{R}^{2d}$. In high dimensions, most of the uniform measure on the sphere is strongly concentrated around any equator.




- Consequence: Lévy's Lemma. *Let $f : S^n \to \mathbb{R}$ be a function with $\|\nabla f\| \leq \eta$ and a point $x \in S^n$ chosen uniformly at random. Then,*

$$\mathrm{Prob}\{|f(x) - \mathbb{E}f| > \varepsilon\} \leq 2\exp\left(-c(n+1)\varepsilon^2/\eta^2\right),$$

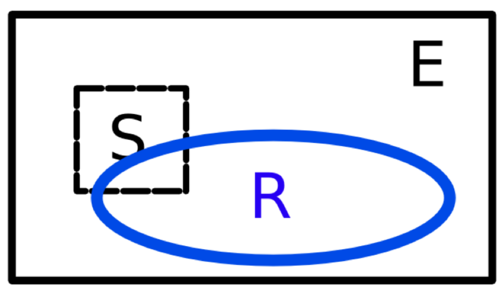
*where $c := (9\pi^3 \ln 2)^{-1}$ is a constant.*
- Sample application: Most bipartite quantum states are highly entangled [1]. *Let $|\psi\rangle$ be a random pure state on $A \otimes B$, with $d_B \geq d_A \geq 3$. Then*

$$\mathrm{Prob}\left\{S(\psi_A) < \log d_A - \alpha - \beta\right\} \leq \exp\left(-\frac{(d_A d_B - 1)c\alpha^2}{(\log d_A)^2}\right),$$

*where $\beta = \frac{d_A}{d_B \ln 2}$, and $c = (8\pi^2 \ln 2)^{-1}$.*
- Most famous application: M. Hastings' counterexample to the additivity conjecture [2].

## ... and typicality in statistical mechanics

Consider a subspace $\mathcal{H}_R \subset \mathcal{H}_S \otimes \mathcal{H}_E$. Example: S=system, E=environment, R=subspace spanned by global energy eigenstates in $[E - \Delta E, E + \Delta E]$.




Statistical mechanics recipe: equidistribution on $R$ gives "microcanonical ensemble" $\Omega_S := \mathrm{Tr}_E(\mathbf{1}_R/d_R)$. Popescu et al. [3] use measure concentration to prove the following:

- Given fixed $|\psi\rangle \in \mathcal{H}_R$, the reduced state is $\psi_S := \mathrm{Tr}_E|\psi\rangle\langle\psi|$.
- It turns out that for "almost all" $|\psi\rangle$, it holds $\psi_S \approx \Omega_S$. In more detail, if $|\psi\rangle$ is drawn randomly in $R$, then

$$\mathrm{Prob}\left\{\|\psi_S - \Omega_S\|_1 \geq \varepsilon + \frac{d_S}{\sqrt{d_R}}\right\} \leq 2\exp\left(-Cd_R\varepsilon^2\right),$$

where $C = 1/18\pi^3$, $d_R = \dim \mathcal{H}_R$, $d_S = \dim \mathcal{H}_S$.
- In this sense, most single pure quantums states locally look like the ensemble average.

Under additional assumptions on the spectrum, Goldstein et al. [4] show that $\Omega_S$ is a Gibbs state, i.e. $\Omega_S \sim e^{-\beta H}$.

However, treating all states in subspaces $R$ in equal footing is sometimes criticized as unphysical ("nature lives in a small corner of Hilbert space"). Hence it makes sense to ask for similar results for more natural subsets of states $R$:

Problem: What if the restriction $R$ is not given by a subspace, but by a *nonlinear* constraint?
As a physical example, *what if the mean energy $\langle\psi|H|\psi\rangle$ is fixed instead* – do similar results hold?

## Our result: a simple example

On the bipartite Hilbert space $A \otimes B$ with $A = \mathbb{C}^3$ and $B = \mathbb{C}^n$ and Hamiltonian $H = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} \otimes \mathbf{1}_B$, choose a state $|\psi\rangle$ in $\mathbb{C}^3 \otimes \mathbb{C}^n$ randomly under the constraint $\langle\psi|H|\psi\rangle = \frac{3}{2}$.

- With high prob., reduced state $\psi^A := \mathrm{Tr}_B|\psi\rangle\langle\psi|$ is close to

$$\psi^A \approx \rho_c := \frac{1}{12}\begin{pmatrix} 5+\sqrt{7} & 0 & 0 \\ 0 & 2(4-\sqrt{7}) & 0 \\ 0 & 0 & -1+\sqrt{7} \end{pmatrix}.$$

- More in detail, we have

$$\mathrm{Prob}\left\{\|\psi^A - \rho_c\|_2 > 3\sqrt{8}\left(t + \frac{59}{\sqrt[4]{n}}\right)\right\} \leq 369960\, n^{\frac{3}{2}} \times \times e^{-\frac{3}{64}n\left(t-\frac{1}{4n}\right)^2 + 4\sqrt{n}}.$$

- Note: reduced state is not a Gibbs state in general!

Special case of generalization of Lévy's Lemma for quadratic submanifolds. Ready to generalize to other non-linear constraints.

- Physics siginificance: Several authors [5, 6] have suggested to define a "quantum microcanonical ensemble" as in this example above: given a Hamiltonian $H$, fix the energy expectation value $E$ and consider the "mean energy ensemble"

$$M_E := \{|\psi\rangle \in \mathbb{C}^n \mid \langle\psi|H|\psi\rangle = E, \quad \||\psi\rangle\| = 1\}.$$

- In physics terms: we prove typicality for this mean energy ensemble (under some conditions on the spectrum); for some models, we show that typicality does not hold (see Ising model).

## Main result in detail

If $H$'s eigenvalues are $E_1, E_2, \ldots, E_n$, then $M_E$ is invariant wrt. energy shifts $E'_k := E_k + s$, $E' := E + s$. While the arithmetic mean $E_A := \frac{1}{n}\sum_k E_k$ is shifted as well (i.e. $E'_A = E_A + s$), the harmonic mean $E_H := \left(\frac{1}{n}\sum_k \frac{1}{E_k}\right)^{-1}$ changes in a non–linear way. Choosing $s$ appropriately, we can shift the energies such that $E' = E'_H$ if $E_{min} < E < E_A$.

**Main Theorem 1.** Suppose that $E > E_{min}$ is an arbitrary energy value such that $E$ is not too close to the "infinite temperature" energy $E_A$, i.e.

$$E \leq E_A - \frac{\pi(E_{max} - E_{min})}{\sqrt{2(n-1)}}.$$

If $f : M_E \to \mathbb{R}$ is any function with $\|\nabla f\| \leq \lambda$ and median $\bar{f}$, then the value $f(\psi)$ evaluated on a randomly chosen state $|\psi\rangle \in M_E$ satisfies

$$\mathrm{Prob}\left\{|f(\psi) - \bar{f}| > \lambda t\right\} \leq a \cdot n^{\frac{3}{2}} \cdot e^{-cn\left(t-\frac{1}{4n}\right)^2 + \varepsilon\sqrt{n}}.$$

The constants $a$, $c$ and $\varepsilon$ depend on the spectrum. They can be determined in the following way:

- Find an energy shift (which is always possible, see above) such that $E' = \left(1+\frac{1}{n}\right)\left(1+\frac{\varepsilon}{\sqrt{n}}\right)E'_H$ for some $\varepsilon > 0$ which is arbitrary, but must be large enough such that the constant $a$ (described below) is positive.
- Compute $c = \frac{3E'_{min}}{64E'}$, $E'_Q := \left(\frac{1}{n}\sum_k {E'_k}^{-2}\right)^{-\frac{1}{2}}$, and $a = 3040{E'_{max}}^2\left[E'^2\left(1 - \frac{E'^2}{\varepsilon^2 {E'_Q}^2}\right)\right]^{-1}$.

Moreover, we have a formula to estimate the value of the median $\bar{f}$ which appears above.

## No concentration in the Ising model

Let $H = \frac{1}{2}\sum_{i=1}^m (1 + Z_i)$, i.e. $m$ non-interacting $\frac{1}{2}$-spins (Ising model). Choose a state $|\psi\rangle$ randomly under $\langle\psi|H|\psi\rangle = \alpha m$ with $0 < \alpha \leq \frac{1}{2}$ fixed. Then:

The resulting mean energy ensemble does *not* concentrate exponentially in the dimension $n = 2^m$ (as in Lévy's Lemma) unless $\alpha = \frac{1}{2}$.

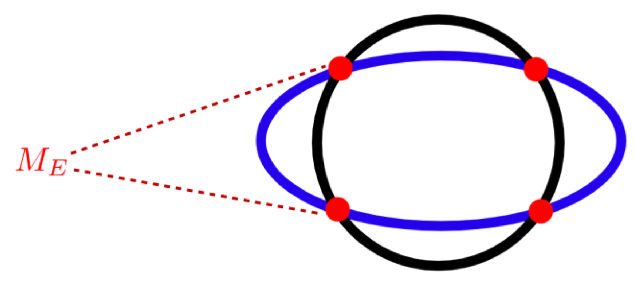
Instead, the best we can hope for is concentration of the form

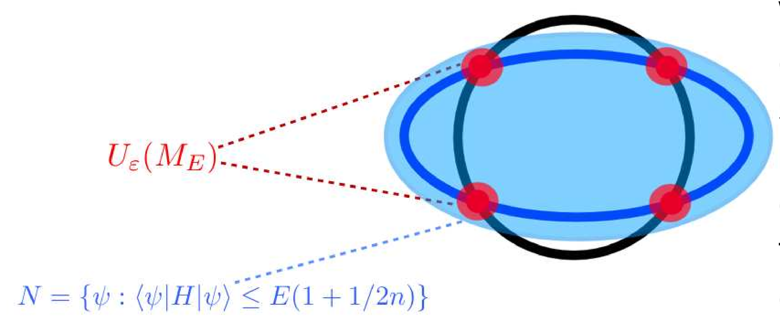
$$\mathrm{Prob}\left\{|f - \bar{f}| > t\right\} \leq b \cdot \exp\left(-\mathcal{O}(n^{H(\alpha)})\, t^2\right),$$

where $H(\alpha) = -\alpha\log_2\alpha - (1-\alpha)\log_2(1-\alpha) \leq 1$ is the binary entropy function.

## Proof idea and tools: integral geometry

Geometrically, $M_E$ is the intersection of a sphere (normalization) and an ellipsoid (energy expectation value).




We get solid objects by considering the $\varepsilon$-neighbourhood of $M_E$, and the full (enlarged) ellipsoid.

- By standard results, we have concentration of measure in the full ellipsoid $N$.
- If $U_\varepsilon(M_E)$ covers a lot of $N$, then $M_E$ "inherits" measure concentration from the surrounding ellipsoid $N$.
- This is the case if $\mathbb{E}_N\|x\|^2 \approx 1$, such that "most" points in $N$ are close to the sphere. It turns out that $E$ must be close to the harmonic energy $E_H$ such that this is true.

This proof strategy is inspired by M. Gromov [7]. To estimate the volume of $U_\varepsilon(X)$ for $X \subseteq M_E$, we use an analog of Buffon's needle experiment: the Crofton formula [8]

$$\int_{L_r} \mu_{r+q-n}(M \cap L_r)\, dL_r = \sigma\mu_q(M)$$

relates the volume of $q$-dim. submanifolds $M \subset \mathbb{R}^n$ with the average vol. of intersections with random hyperplanes $L_r$.

## References

[1] P. Hayden, D.W. Leung, and A. Winter, Comm. Math. Phys. **265**, 95 (2006).
[2] M.B. Hastings, Nature Physics **5**, 255 (2009).
[3] S. Popescu, A.J. Short, and A. Winter, Nature Physics **2**, 754 (2006).
[4] S. Goldstein, J.L. Lebowitz, R. Tumulka, N. Zanghi, Phys. Rev. Lett. **96**, 050403 (2006).
[5] D.C. Brody, D.W. Hook, and L.P. Hughston, Proc. R. Soc. A **463**, 2021 (2007).
[6] C.M. Bender, D.C. Brody, and D.W. Hook, J. Phys. A **38**, L607 (2005).
[7] M. Gromov, *Metric structures for Riemannian and Non-Riemannian spaces* (Mod. Birkhäuser Classics, 2007).
[8] L.A. Santaló, *Integral geometry and geometric probability* (Addison-Wesley, 1972).