Undecidability of Quantum Measurement Occurrence

Jens Eisert[1], Markus P. Müller[2], and Christian Gogolin[1]

[1] Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany

[2] Perimeter Institute for Theoretical Physics, 31 Caroline St N, Waterloo, ON N2L 2Y5, Canada

Abstract

A famous result by Alan Turing dating back to 1936 is that a general algorithm solving the halting problem on a Turing machine for all possible inputs and programs cannot exist – the halting problem is undecidable.
Formally, an undecidable problem is a decision problem for which one cannot construct a single algorithm that will always provide a correct answer in finite time. In [1], M. Wolf et al. have initiated a discussion whether undecidability occurs in quantum information theory, and have shown that reachability of fidelity thereholds falls into the class of undecidable problems.
In this work, we show that very natural, apparently simple problems in quantum measurement theory can be undecidable even if their classical analogues are decidable. Undecidability appears as a genuine quantum property. The problem we consider is to determine whether sequentially used identical Stern-Gerlach-type measurement devices, giving rise to a tree of possible outcomes, have outcomes that never occur. Finally, we sketch implications for measurement-based quantum computing and studies of quantum many-body models.

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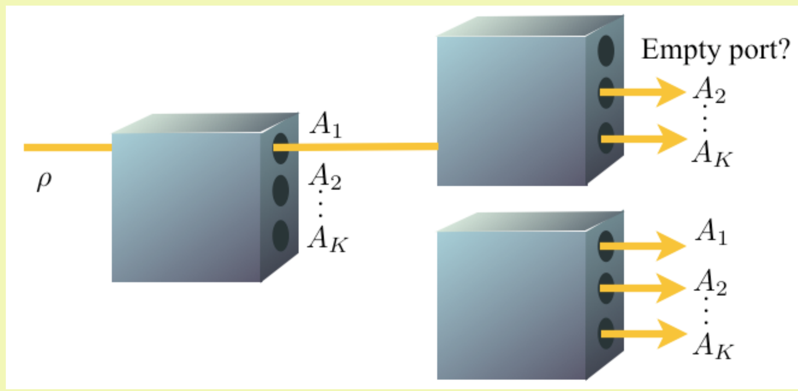
The Setup



- Consider a quantum measurement device with K outcomes, described by Kraus operators $\{A_1,\ldots,A_K\}$. In a sequence of n measurements, this device is applied iteratively: every output is fed into an identical device as input.
- Every run of the experiment gives a sequence of outcomes $(j_1,j_2,\ldots,j_n)$ with all $j_i\in\{1,\ldots,K\}$. The probability of each sequence depends on the input ρ.
- **Question:** Is there any sequence of outcomes $(j_1,j_2,\ldots,j_n)$ that has probability zero for every input? This problem turns out to be **undecidable** in the quantum case**,** but **decidable** in the classical case.

Main Theorem

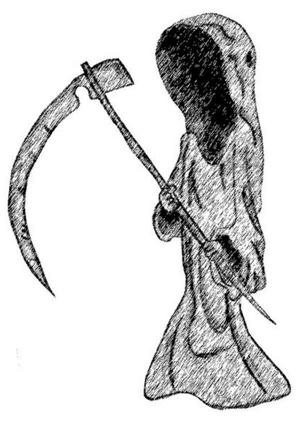
- **The QMOP is undecidable**. That is, there exists no algorithm which gives the correct answer in finite time in all instances of the problem.
- However, **the analogous classical problem is decidable**, even in its most general version (slightly more general than the quantum setup).

The main reason for the more complex quantum behaviour is **destructive interference**: classical transitition matrices have non-negative entries, which reduces the MOP to the simpler matrix mortality problem for matrices with non-negative entries ($MMP_{\geq 0}$).

Quantum MOP	Classical MOP
↕	↕
MMP	$MMP_{\geq 0}$
↕	↕
Destructive interference - undecidable	Only constructive interference - decidable

A taste of the proof: Matrix Mortality

$$A=\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n}\\ a_{2,1} & a_{2,2} & \cdots & a_{2,n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m,1} & a_{m,2} & \cdots & a_{m,n}\end{pmatrix}=(a_{ij})$$



The proof uses the classical result that the so-called *Matrix Mortality Problem* [2,3] is undecidable. In more detail:

- **Matrix Mortality Problem (MMP)**. Given a finite set of matrices $\{M_1,\ldots,M_k\}$, decide whether any finite product will give the zero matrix -- that is, whether there is any finite sequence $(j_1,\ldots,j_n)$ such that the matrix product $M_{j_n}\ldots M_{j_2}M_{j_1}=0$.

Depending on what kind of matrices are allowed, the problem turns out to be either decidable or undecidable:

- The MMP is undecidable for integer matrices.

In other words, *there is no single algorithm which, given a finite list of integer matrices of arbitrary size, decides whether they generate the zero matrix* -- at least no algorithm which supplies the correct answer in *all* instances of the problem.

It turns out that the problem is already undecidable for fixed matrix size und number:

- The MMP is undecidable for 8 integer 3x3-matrices.

This fact is the main ingredient in the proof that the QMOP (quantum measurement occurrence problem) is undecidable: the probability of obtaining outcome sequence $(j_1,\ldots,j_n)$ on input state ρ is

$$\text{Prob}(j_1,\ldots,j_n)=\text{tr}\left(A_{j_n}\ldots A_{j_1}\rho A_{j_1}^\dagger\ldots A_{j_n}^\dagger\right).$$

This is zero if and only if the matrix product $A_{j_n}\ldots A_{j_1}=0$. The main technical difficulty is to encode a given instance of the MMP into valid Kraus operators (describing the QMOP) which are *normalized*.

Surprisingly, it turns out that

- The MMP is *decidable* for non-negative matrices.

This observation proves that the *classical* version of the measurement problem is decidable.

Formal decision problem:

Definition [Quantum Measurement Occurrence Problem (QMOP)]:

Given a description of a measurement device in terms of K Kraus operators $\{A_1,\ldots,A_K\}\subset\mathbb{Q}^{d\times d}$, decide whether, in the setting described above, there exists any finite sequence $\{j_1,\ldots,j_n\}$ which can never be observed (i.e. has probability zero), even if the input state has full rank.

The Classical Version:

- The (non-selective) action of a classical channel on probability vectors is given by a $d\times d$ stochastic matrix Q :

$$(p_1,\ldots,p_d)^T\mapsto Q(p_1,\ldots,p_d)^T$$

- If the measurement is selective with K outcomes, we have a decomposition of Q into „substochastic" matrices $Q_1,\ldots,Q_k$ with non-negative entries:

$$Q=\sum_{i=1}^K Q_i.$$

- Measuring repeatedly, the probability of obtaining outcomes $(j_1,\ldots,j_n)$ on input distribution p is

$$\text{Prob}(j_1,\ldots,j_n)=\sum_{i=1}^d\left(Q_{j_n}\ldots Q_{j_1}p\right)_i.$$

Outlook: Undecidability, a frequent phenomenon?

- As shown in [1], it is undecidable whether a small set of noisy gates allows to create a state which overcomes some pre-given fidelity threshold with respect to a given target state. Several other candidates for undecidable questions in QIT are suggested.
- Resources for quantum computation [4,5] - or ground states in many-body-sytems - are often described by matrix product states (MPS),

$$|\psi_n\rangle=\sum_{x_1,\ldots,x_n}\langle x_n|A[x_{n-1}]\ldots A[x_1]|0\rangle|x_1,\ldots,x_n\rangle$$

  These involve products of matrices, similar as in the setting above. Thus, some natural properties of measurements on those states are undecidable as well.
- Due to superactivation, computation of many quantum channel capacities involves infinite-dimensional optimization problems. This suggests that some channel capacities might be noncomputable.

References

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