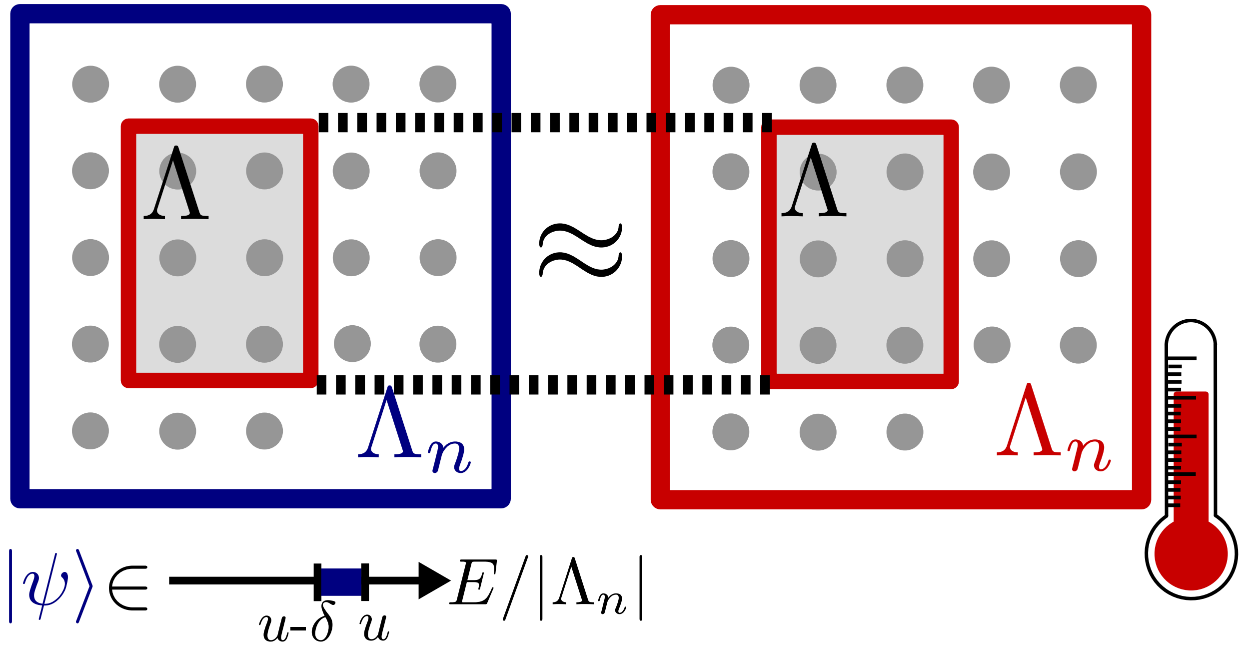
Thermalization and canonical typicality in translation-invariant quantum lattice systems

Markus Müller*
Institute for Theoretical Physics, Heidelberg University (Germany)

joint work with Emily Adlam, Lluís Masanes, Nathan Wiebe

arXiv:1312.7420



Outline

1. How do quantum systems thermalize?

New approaches to old questions
Canonical typicality
Dynamical thermalization

2. Weak eigenstate thermalization

Lieb-Robinson bounds
Weak ETH: physical interpretation
Weak ETH: proof sketch

3. Some math. details on part 1

Detailed theorems and proof sketches
Finite-size bounds for non-interacting systems

Outline

1. How do quantum systems thermalize?

New approaches to old questions
Canonical typicality
Dynamical thermalization

very sketchy overview

2. Weak eigenstate thermalization

Lieb-Robinson bounds
Weak ETH: physical interpretation
Weak ETH: proof sketch

3. Some math. details on part 1

Detailed theorems and proof sketches
Finite-size bounds for non-interacting systems

Outline

1. How do quantum systems thermalize?

New approaches to old questions
Canonical typicality
Dynamical thermalization

very sketchy overview

2. Weak eigenstate thermalization

Lieb-Robinson bounds
Weak ETH: physical interpretation
Weak ETH: proof sketch

3. Some math. details on part 1

Detailed theorems and proof sketches
Finite-size bounds for non-interacting systems

on blackboard

Outline

We prove our results by combining

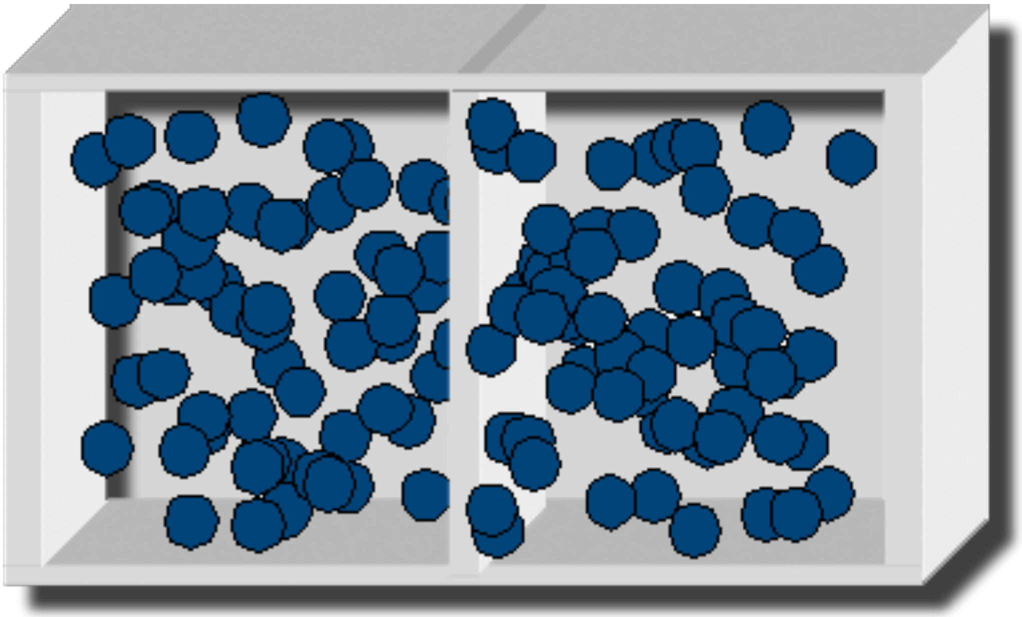
Traditional mathematical physics techniques

- Quasilocal algebra
- KMS-Gibbs states
- Equivalence of ensembles
- Thermodynamic limit

More recent quantum information techniques

- Random pure quantum states
- Concentration of measure
- Quantum pseudorandomness
- Lieb-Robinson bounds

1. How do quantum systems thermalize?

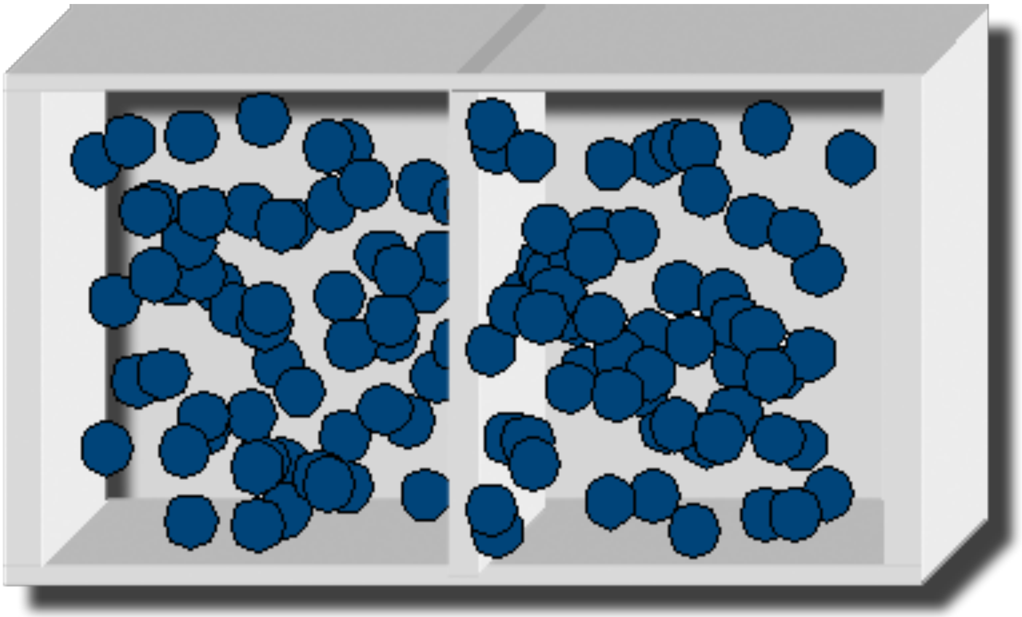


$$|\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle$$

E. Schrödinger

J. von Neumann

1. How do quantum systems thermalize?



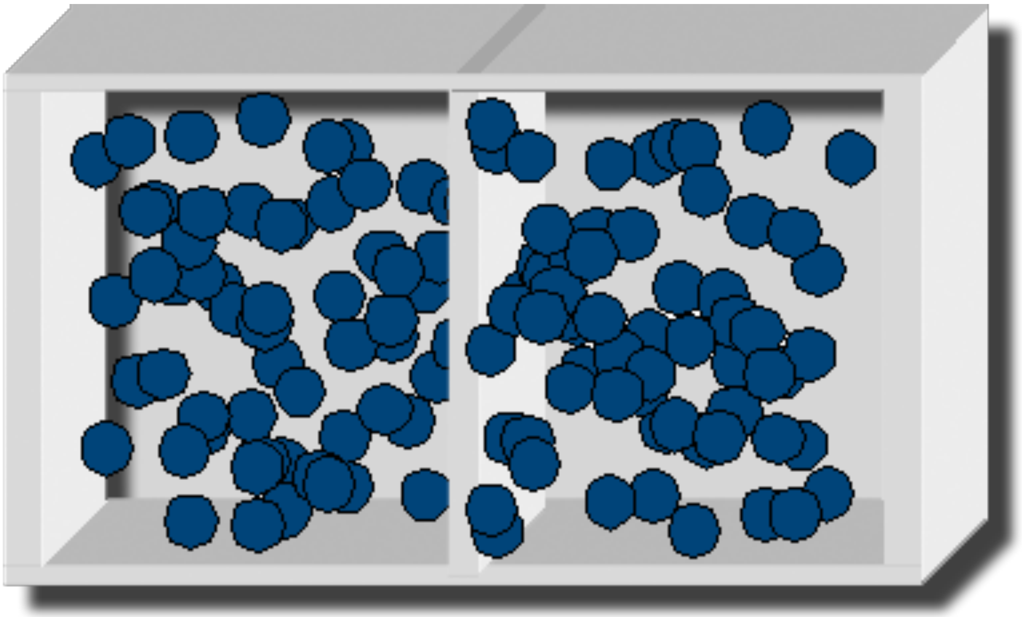
$$|\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle$$

E. Schrödinger

J. von Neumann

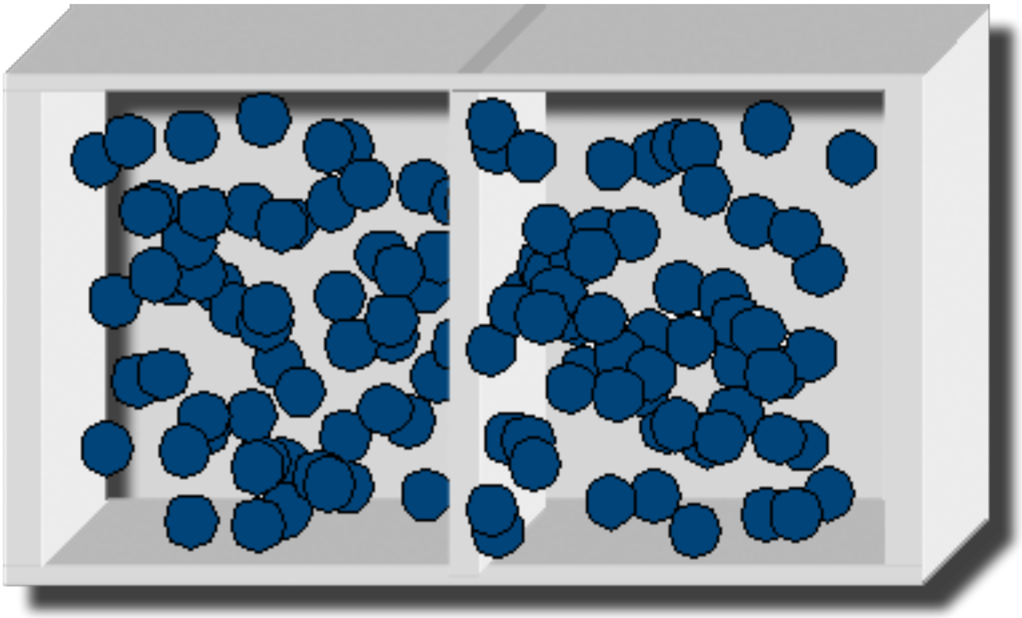
- New experimental methods (cold atoms in optical lattices),
- novel numerical techniques,
- new mathematical insights from quantum information theory.

1. How do quantum systems thermalize?



$$|\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle$$

Canonical typicality



$$|\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle$$

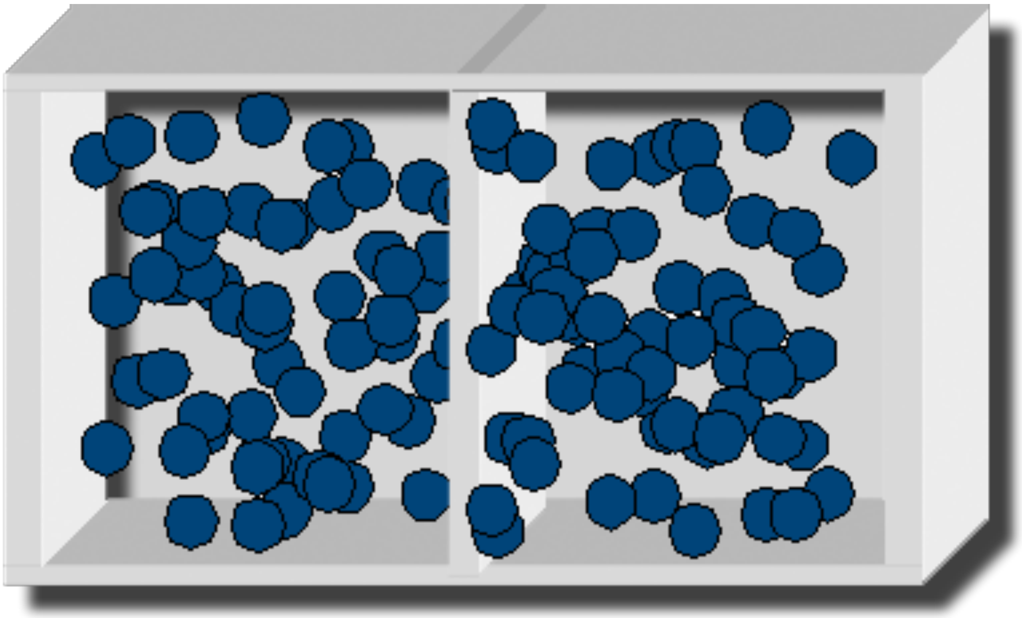
$$|\psi(t)\rangle \in R,$$

for example

$$R = \mathrm{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}.$$

S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett. **96**, 050403 (2006).

Canonical typicality



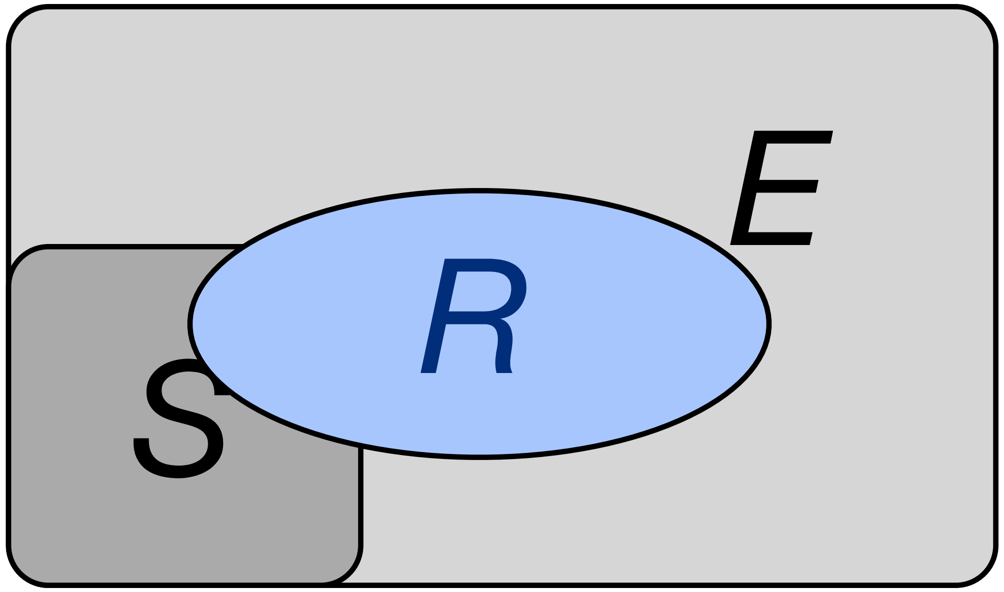
$$|\psi\rangle \in R,$$

for example

$$R = \mathrm{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}$$

S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett. **96**, 050403 (2006).

Canonical typicality



$$|\psi\rangle \in R \subset S \otimes E,$$

for example

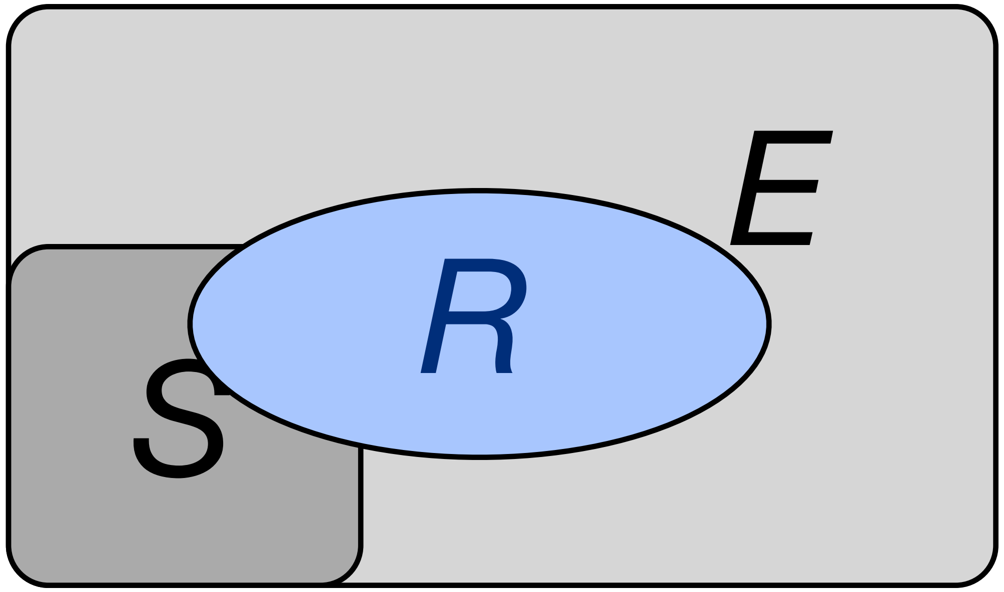
$$R = \mathrm{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}$$

$$H = H_S + H_E + H_{\mathrm{int}}$$

Consider a "typical" / random state $|\psi\rangle \in R.$

S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett. **96**, 050403 (2006).

Canonical typicality



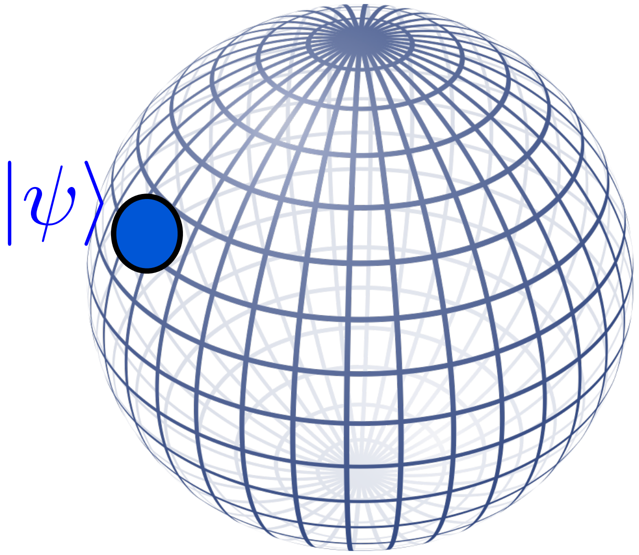
$$|\psi\rangle \in R \subset S \otimes E,$$

for example

$$R = \mathrm{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}$$

$$H = H_S + H_E + H_{\mathrm{int}}$$

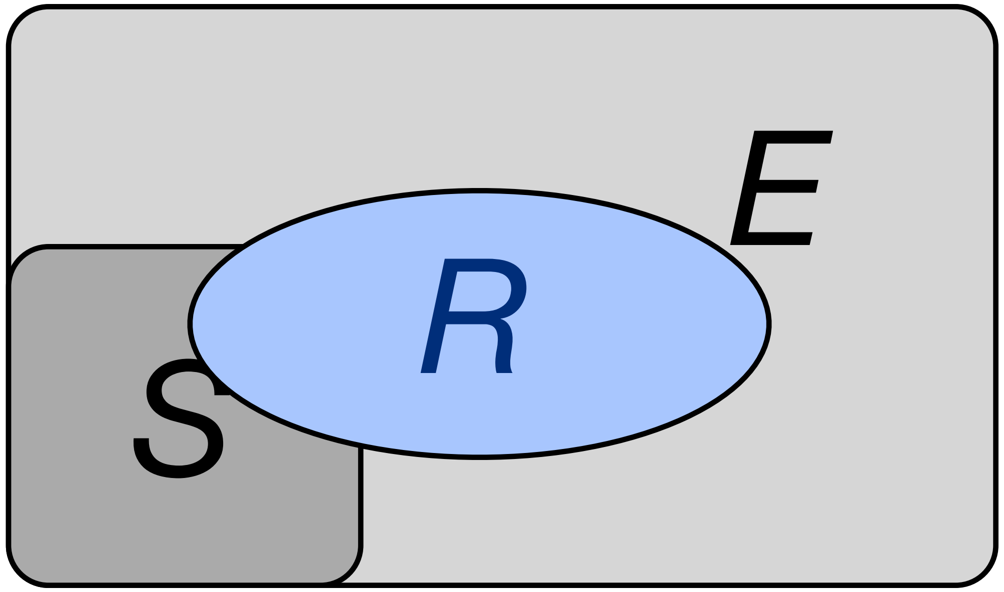
Consider a "typical" / random state $|\psi\rangle \in R.$



All pure states in *R*:
complex sphere.
Can draw a random state
by picking a random point.

S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett. **96**, 050403 (2006).

Canonical typicality



$$|\psi\rangle \in {\color{blue}R} \subset S \otimes E,$$

for example

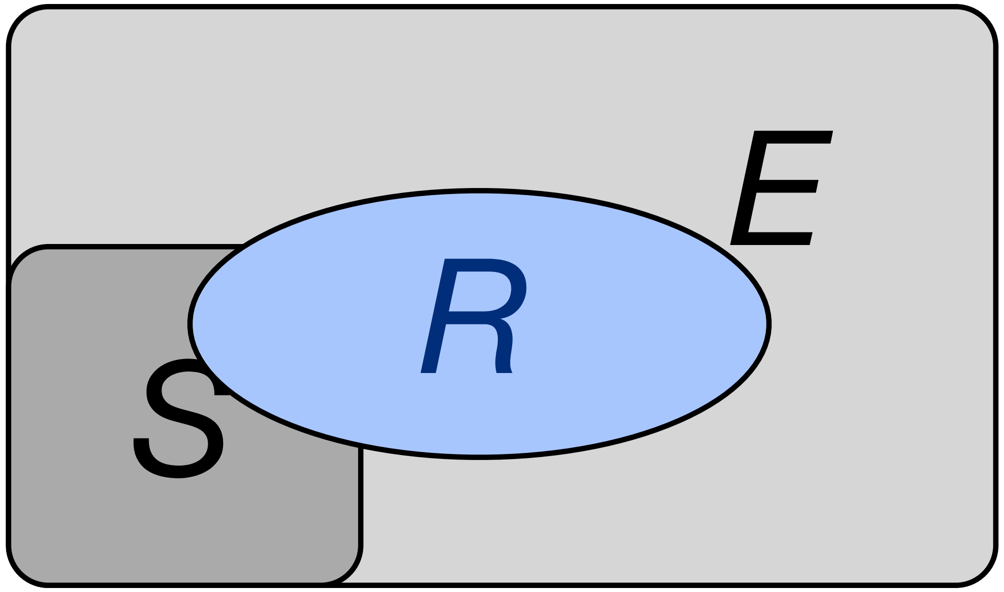
$$R = \text{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}.$$

$$H = H_S + H_E + H_{\text{int}}$$

Consider a "typical" / random state $|\psi\rangle \in {\color{blue}R}.$

S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett. **96**, 050403 (2006).

Canonical typicality



$|\psi\rangle \in R \subset S \otimes E,$

for example

$$R = \mathrm{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}$$

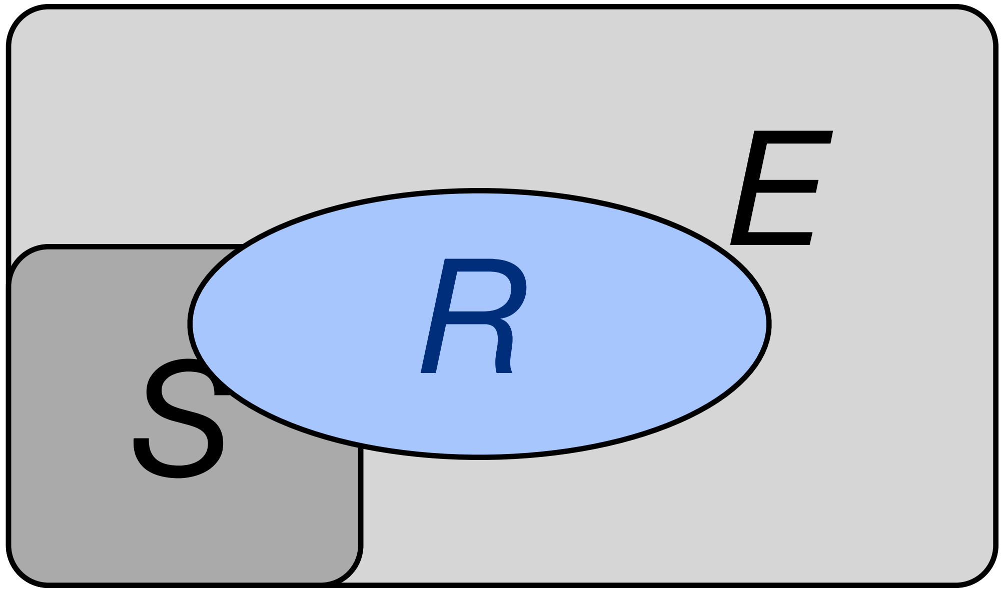
$$H = H_S + H_E + H_{\mathrm{int}}$$

Consider a "typical" / random state $|\psi\rangle \in R.$

Since it is entangled, $\rho_S := \mathrm{Tr}_E |\psi\rangle\langle\psi|$ is typically mixed.

S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett. **96**, 050403 (2006).

Canonical typicality



$|\psi\rangle \in {\color{blue}R} \subset S \otimes E,$

for example

$$R = \mathrm{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}$$

$$H = H_S + H_E + H_{\mathrm{int}}$$

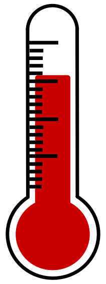
Thermalization from entanglement

Consider a "typical" / random state $|\psi\rangle \in {\color{blue}R}.$

Since it is entangled, $\rho_S := \mathrm{Tr}_E |\psi\rangle\langle\psi|$ is typically mixed.

Goldstein et al.: This state is actually thermal:

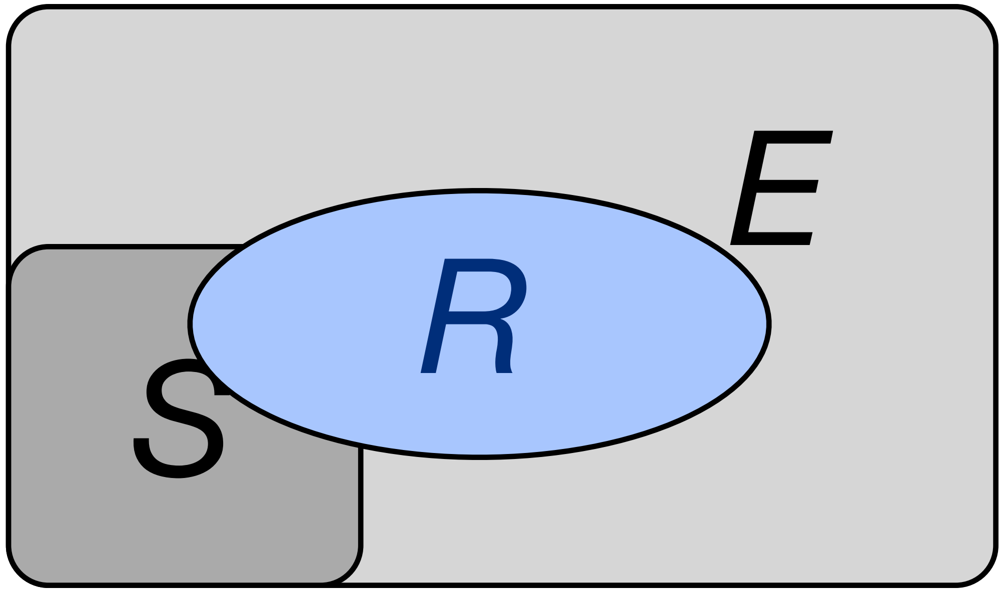
$$\rho_S \approx \exp(-\beta H_S)/Z.$$



$$\beta = \frac{1}{k_B T}$$

S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett. **96**, 050403 (2006).

Canonical typicality



$$|\psi\rangle \in R \subset S \otimes E,$$

for example

$$R = \text{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}$$

$$H = H_S + H_E + H_{\text{int}}$$

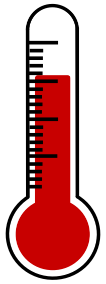
Thermalization from entanglement

Consider a "typical" / random state $|\psi\rangle \in R.$

Since it is entangled, $\rho_S := \text{Tr}_E|\psi\rangle\langle\psi|$ is typically mixed.

Goldstein et al.: This state is actually thermal:

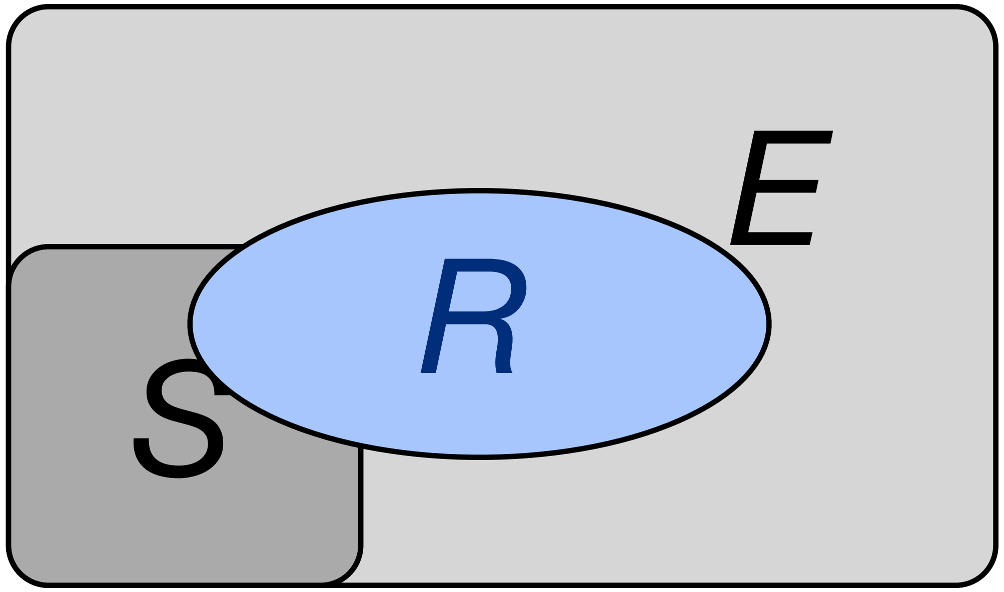
$$\rho_S \approx \exp(-\beta H_S)/Z.$$



not true in general!

S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett. **96**, 050403 (2006).

Canonical typicality: what is known



$$|\psi\rangle \in R \subset S \otimes E,$$

for example

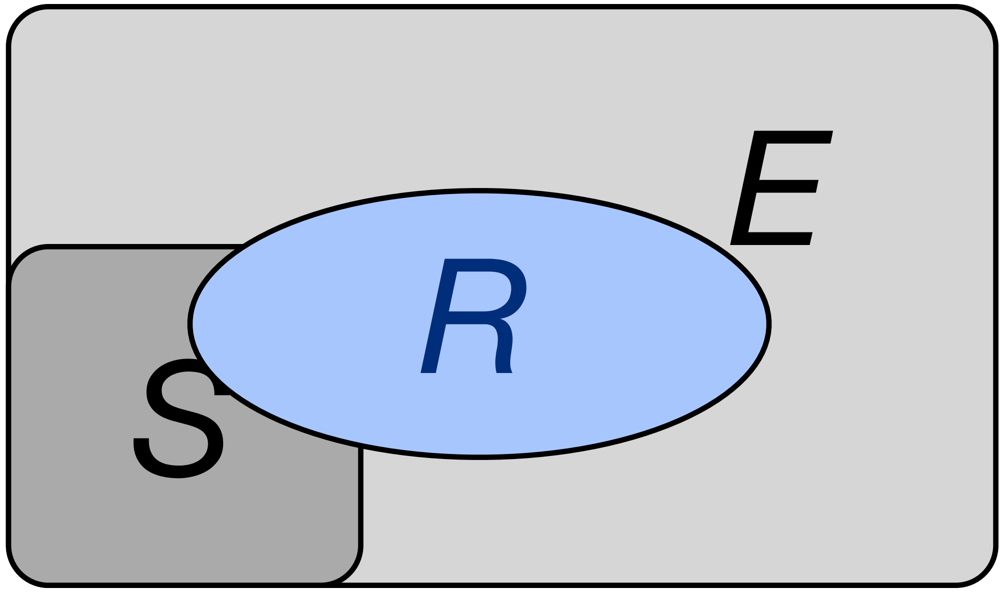
$$R = \mathrm{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}$$

$$H = H_S + H_E + H_{\mathrm{int}}$$

Consider a random state $|\psi\rangle \in R.$ Let $\rho_S := \mathrm{Tr}_E |\psi\rangle\langle\psi|$.

S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett. **96**, 050403 (2006).

Canonical typicality: what is known



$|\psi\rangle \in R \subset S \otimes E,$

for example

$$R = \mathrm{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}$$

$$H = H_S + H_E + H_{\mathrm{int}}$$

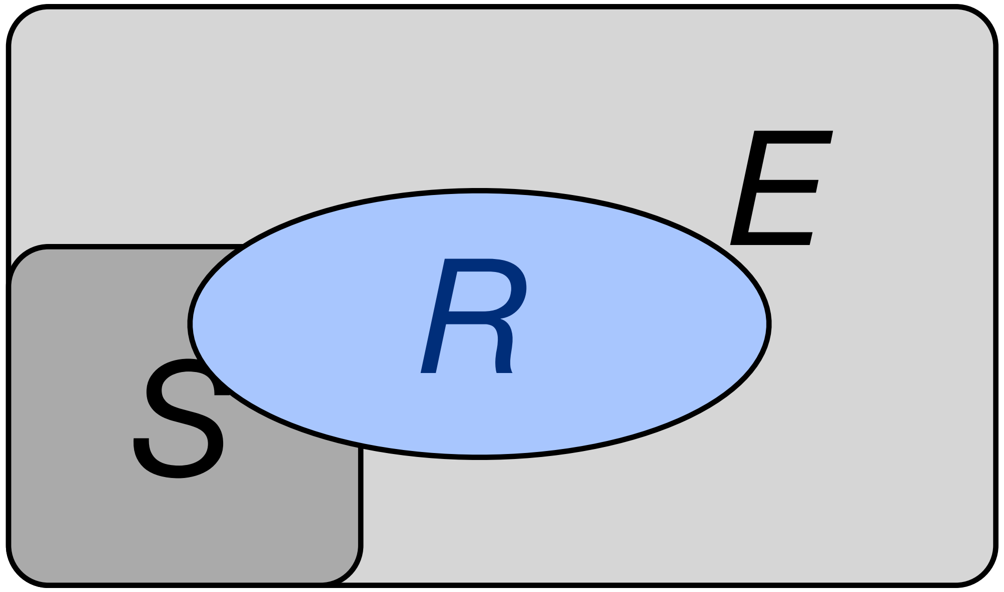
Consider a random state $|\psi\rangle \in R$. Let $\rho_S := \mathrm{Tr}_E|\psi\rangle\langle\psi|$.

Theorem (Popescu et al.): There is a state Ω_S such that

$$\mathrm{Prob}\left[\|\rho_S - \Omega_S\|_1 \geq \varepsilon + \frac{d_S}{\sqrt{d_R}}\right] \leq 2\, \exp(-d_R\varepsilon^2/559).$$

S. Popescu, A. J. Short, and A. Winter, Nature Physics **2**, 754 (2006).

Canonical typicality: what is known



$$|\psi\rangle \in R \subset S \otimes E,$$

for example

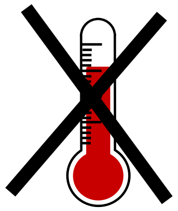
$$R = \text{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}.$$

$$H = H_S + H_E + H_{\text{int}}$$

Consider a random state $|\psi\rangle \in R$. Let $\rho_S := \text{Tr}_E |\psi\rangle\langle\psi|$.

Theorem (Popescu et al.): There is a state Ω_S such that

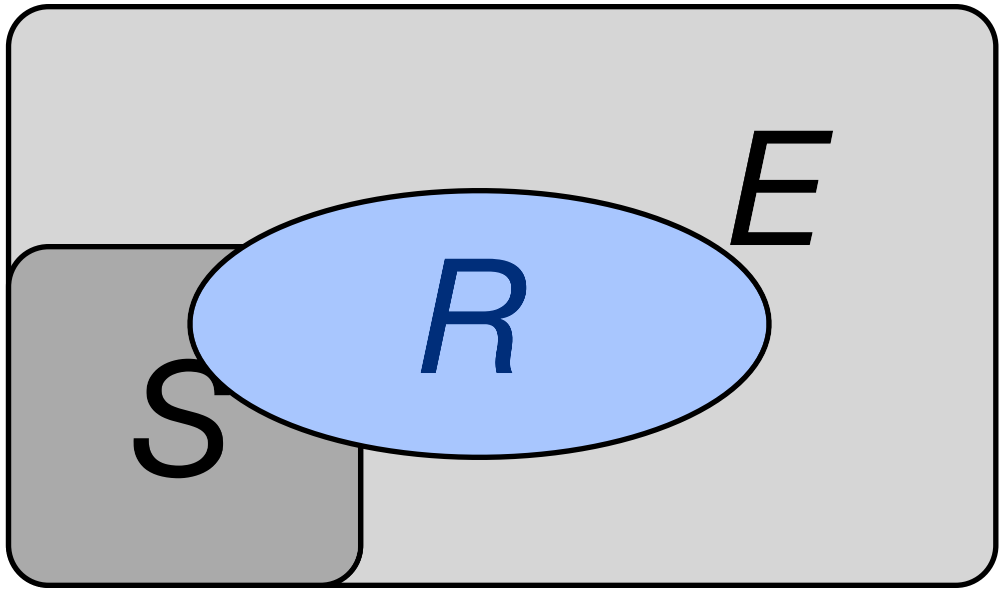
$$\text{Prob}\left[\|\rho_S - \Omega_S\|_1 \geq \varepsilon + \frac{d_S}{\sqrt{d_R}}\right] \leq 2\, \exp(-d_R \varepsilon^2/559).$$



This state Ω_S is not thermal in general.

S. Popescu, A. J. Short, and A. Winter, Nature Physics **2**, 754 (2006).

Canonical typicality: what is known



$$|\psi\rangle \in {\color{blue}R} \subset S \otimes E,$$

for example

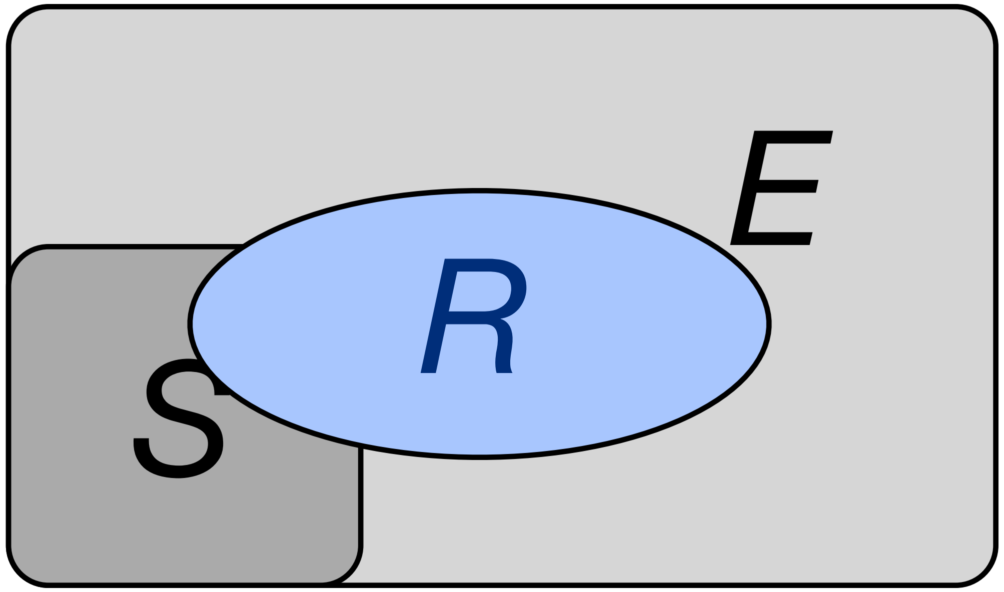
$$R = \mathrm{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}$$

$$H = H_S + H_E + H_{\mathrm{int}}$$

Consider a random state $|\psi\rangle \in {\color{blue}R}$. Let $\rho_S := \mathrm{Tr}_E |\psi\rangle\langle\psi|$.

A. Riera, C. Gogolin, and J. Eisert, Phys. Rev. Lett. **108**, 080402 (2012)

Canonical typicality: what is known



$|\psi\rangle \in R \subset S \otimes E,$

for example

$$R = \mathrm{span}\{|E\rangle \mid E_0 \leq E \leq E_0 + \Delta\}$$

$$H = H_S + H_E + H_{\mathrm{int}}$$

Consider a random state $|\psi\rangle \in R.$ Let $\rho_S := \mathrm{Tr}_E |\psi\rangle\langle\psi|.$

Theorem (Riera et al.): W/ high probability, ρ_S is close to thermal if

- the spectrum of H_E satisfies some complicated conditions, and
- the interaction strength $\|H_{\mathrm{int}}\|$ is tiny.

ϟ Conditions not satisfied in most interesting models.

A. Riera, C. Gogolin, and J. Eisert, Phys. Rev. Lett. **108**, 080402 (2012)

Canonical typicality: our result

Specialize to translation-invariant models, finite-range interaction.

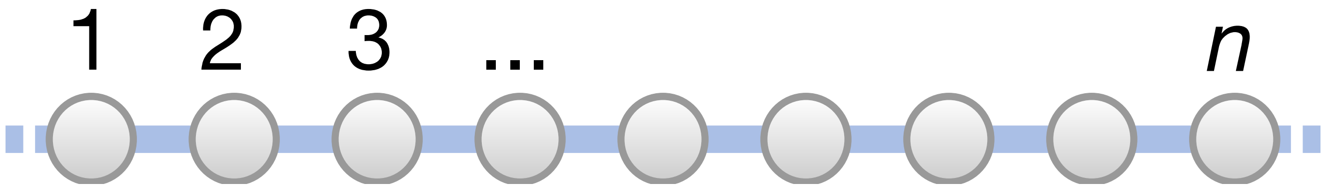
MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Canonical typicality: our result

Specialize to translation-invariant models, finite-range interaction.

For example Heisenberg model:

$$H = -J \sum_{i=1}^{n-1} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} - h \sum_{i=1}^{n} \sigma_i^Z .$$



MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

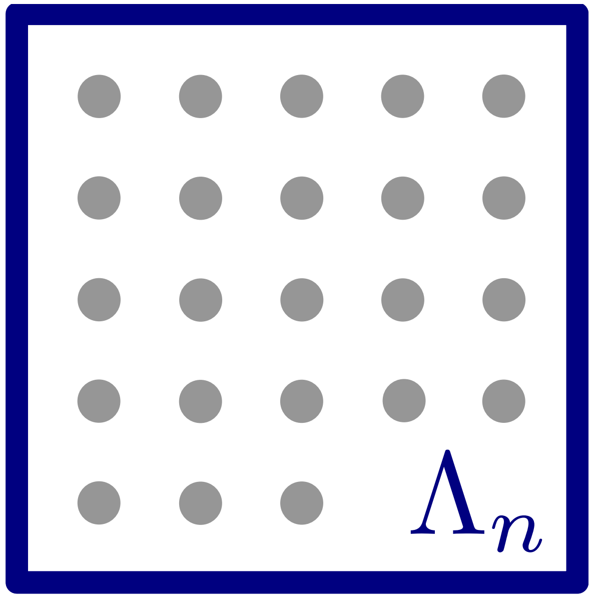
Canonical typicality: our result

Specialize to translation-invariant models, finite-range interaction.

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Canonical typicality: our result

Specialize to translation-invariant models, finite-range interaction.



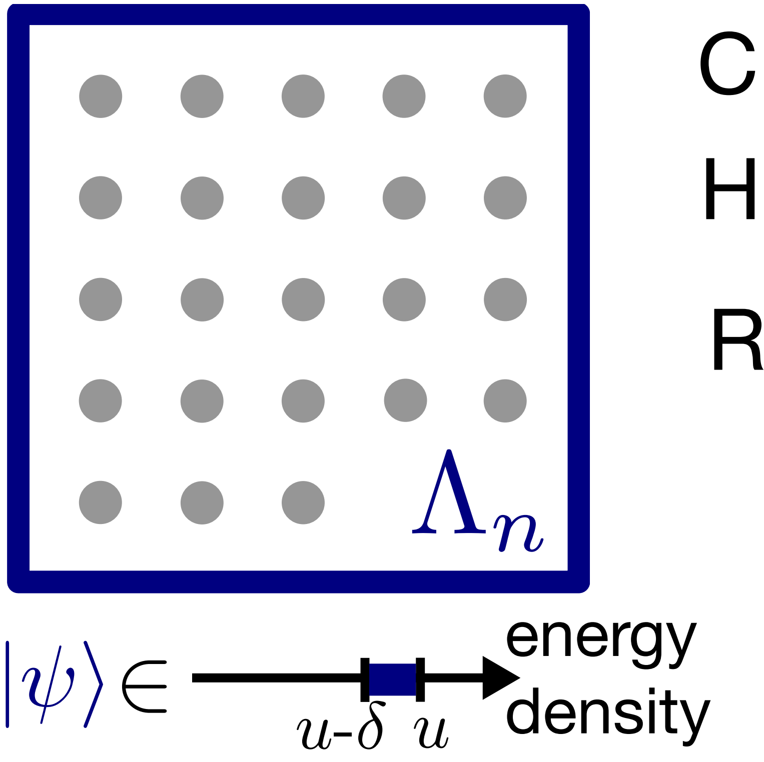
Cubic lattice; e.g. $\Lambda_n = [1,n] \times [1,n] \subset \mathbb{Z}^2$.

Hamiltonian H_{Λ_n}.

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Canonical typicality: our result

Specialize to translation-invariant models, finite-range interaction.



Cubic lattice; e.g. $\Lambda_n = [1,n] \times [1,n] \subset \mathbb{Z}^2$.

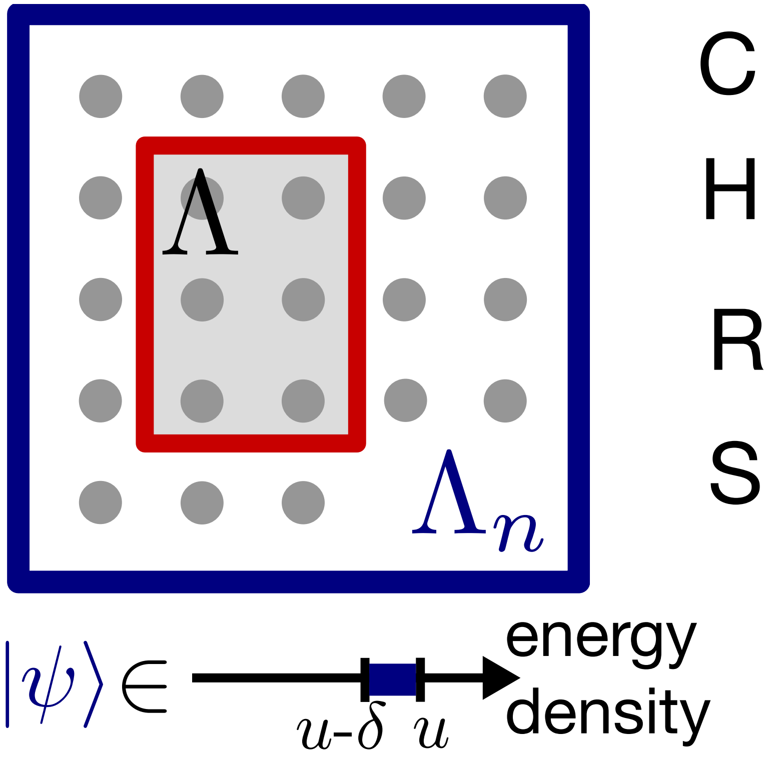
Hamiltonian H_{Λ_n}.

Random state w/ energy density close to *u*.

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Canonical typicality: our result

Specialize to translation-invariant models, finite-range interaction.



Cubic lattice; e.g. $\Lambda_n = [1,n] \times [1,n] \subset \mathbb{Z}^2$.

Hamiltonian H_{Λ_n}.

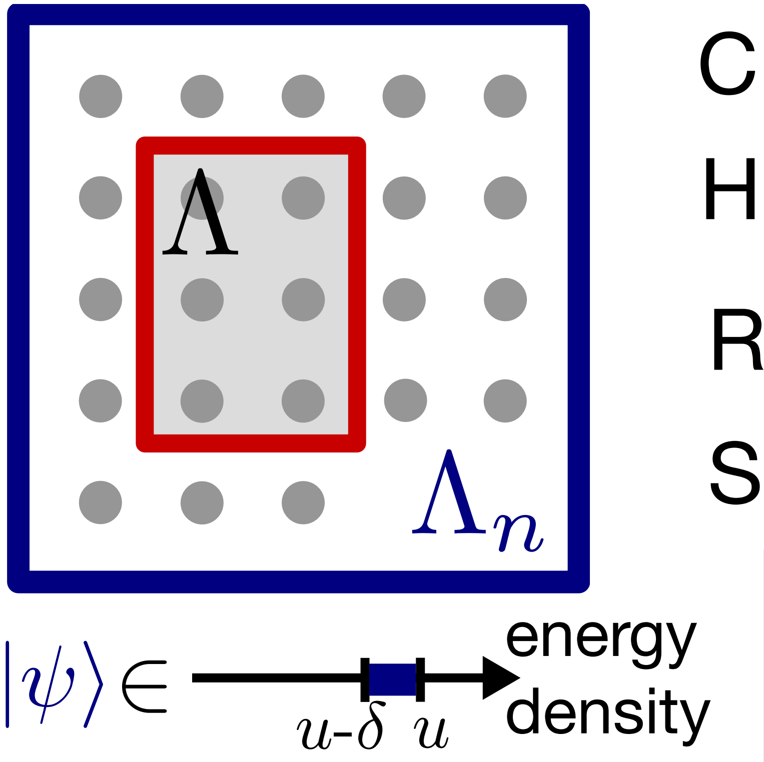
Random state w/ energy density close to *u*.

Small subsystem Λ.

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Canonical typicality: our result

Specialize to translation-invariant models, finite-range interaction.



Cubic lattice; e.g. $\Lambda_n = [1,n] \times [1,n] \subset \mathbb{Z}^2$.

Hamiltonian H_{Λ_n}.

Random state w/ energy density close to *u*.

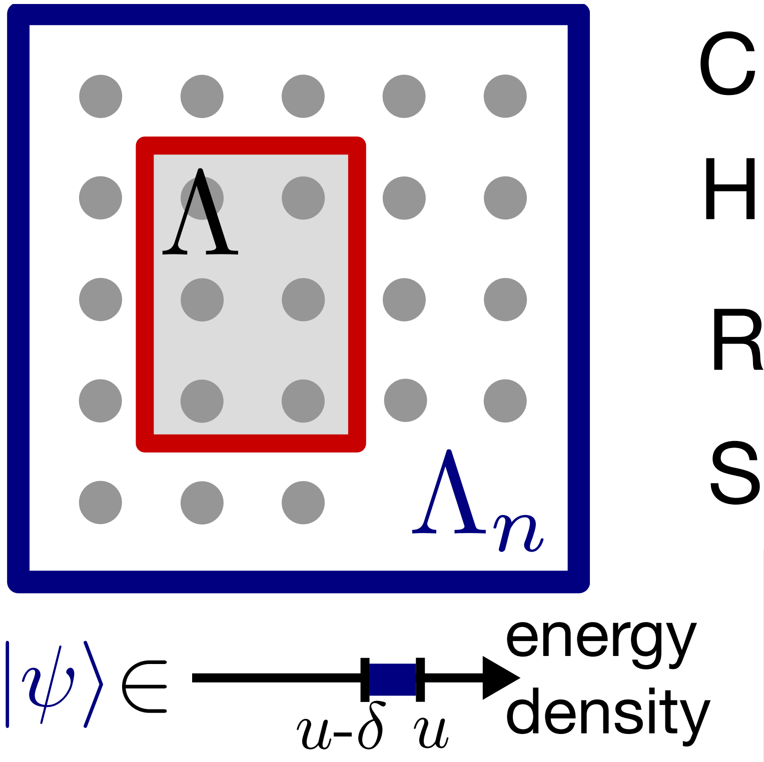
Small subsystem Λ.

Assumption: at inverse temperature *β* corresponding to *u*, there is a unique phase.

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Canonical typicality: our result

Specialize to translation-invariant models, finite-range interaction.



Cubic lattice; e.g. $\Lambda_n = [1,n] \times [1,n] \subset \mathbb{Z}^2$.

Hamiltonian H_{Λ_n}.

Random state w/ energy density close to *u*.

Small subsystem Λ.

Assumption: at inverse temperature *β* corresponding to *u*, there is a unique phase.

Theorem: Then, with high probability,

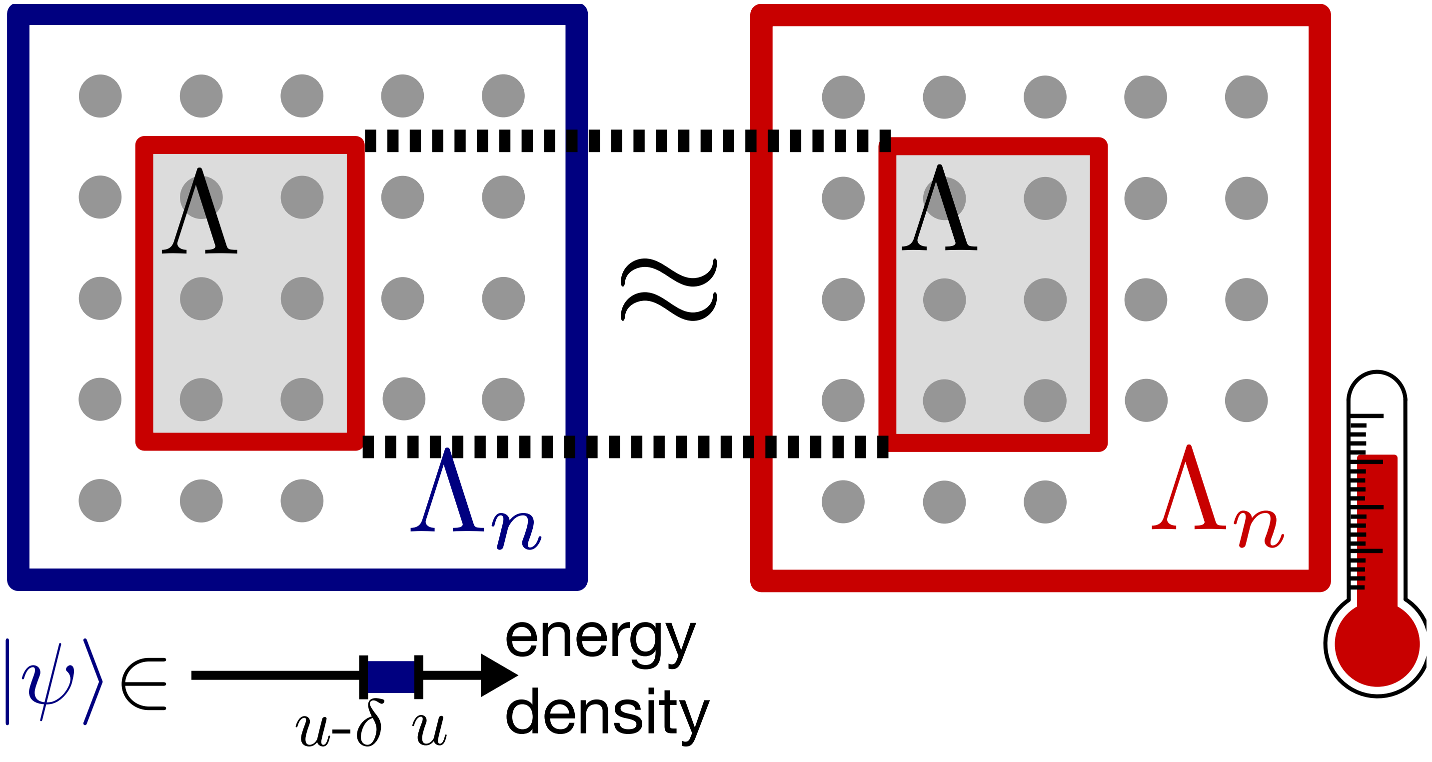
$$\mathrm{Tr}_{\Lambda_n \setminus \Lambda} |\psi\rangle\langle\psi| \approx \mathrm{Tr}_{\Lambda_n \setminus \Lambda} \frac{\exp(-\beta H_{\Lambda_n})}{Z},$$

and the distance goes to zero as $n \to \infty$.

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Canonical typicality: our result

Specialize to translation-invariant models, finite-range interaction.



Theorem: Then, with high probability,

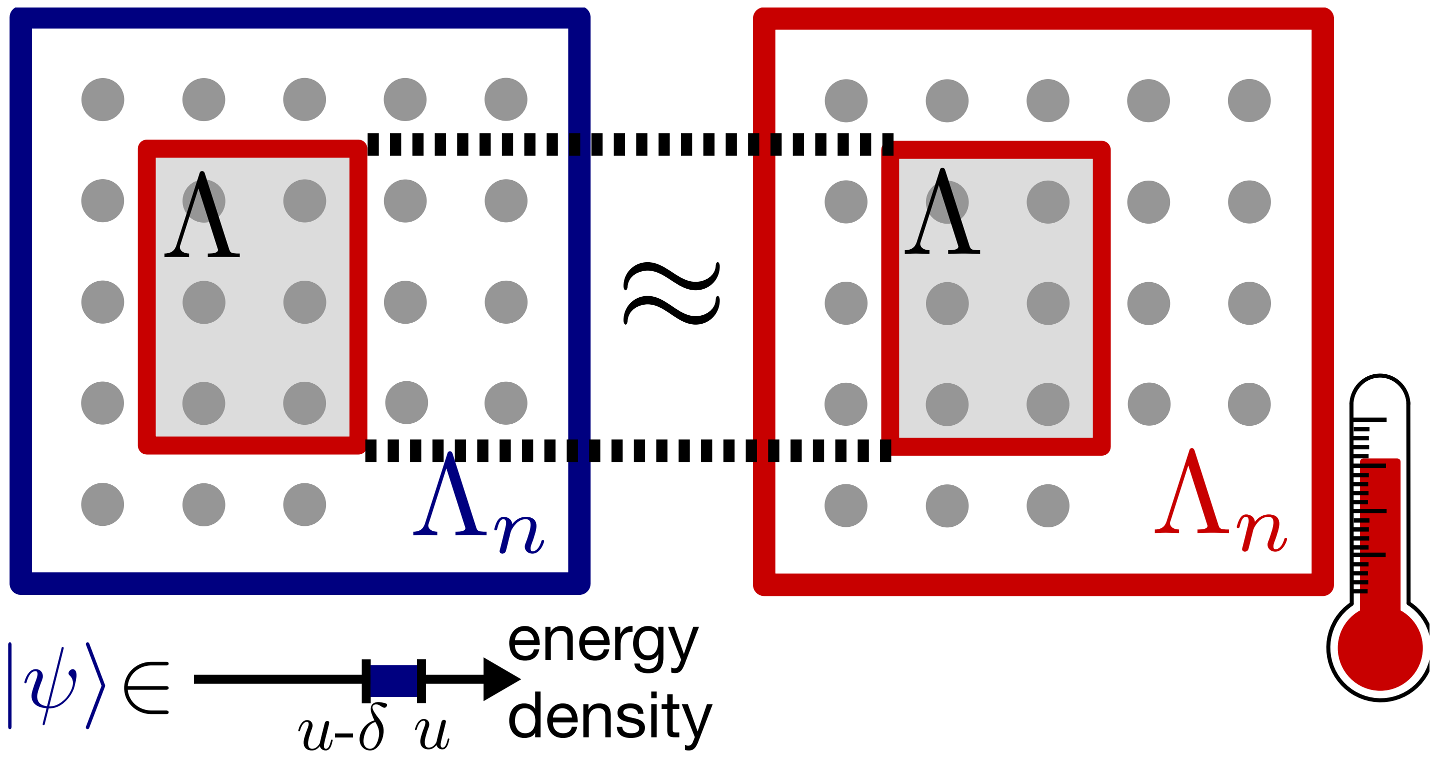
$$\mathrm{Tr}_{\Lambda_n \backslash \Lambda} |\psi\rangle\langle\psi| \approx \mathrm{Tr}_{\Lambda_n \backslash \Lambda} \frac{\exp(-\beta H_{\Lambda_n})}{Z},$$

and the distance goes to zero as $n \to \infty$.

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Canonical typicality: our result

Specialize to translation-invariant models, finite-range interaction.



Thermalization
from
entanglement

Theorem: Then, with high probability,

$$\mathrm{Tr}_{\Lambda_n \setminus \Lambda} |\psi\rangle\langle\psi| \approx \mathrm{Tr}_{\Lambda_n \setminus \Lambda} \frac{\exp(-\beta H_{\Lambda_n})}{Z},$$

and the distance goes to zero as $n \to \infty$.

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Canonical typicality: our result

Similar results can be shown for dynamical thermalization.

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Canonical typicality: our result

Similar results can be shown for dynamical thermalization.

Using A. J. Short and T. C. Farrelly, New J. Phys. **14**, 013063 (2012) we show:

Theorem: If the initial state $|\psi(0)\rangle$ occupies a large number of energy levels, and some other technical conditions are met, then

$$\mathrm{Tr}_{\Lambda_n \setminus \Lambda} |\psi(t)\rangle\langle\psi(t)|$$

is close to thermal for most times *t*.

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Canonical typicality: our result

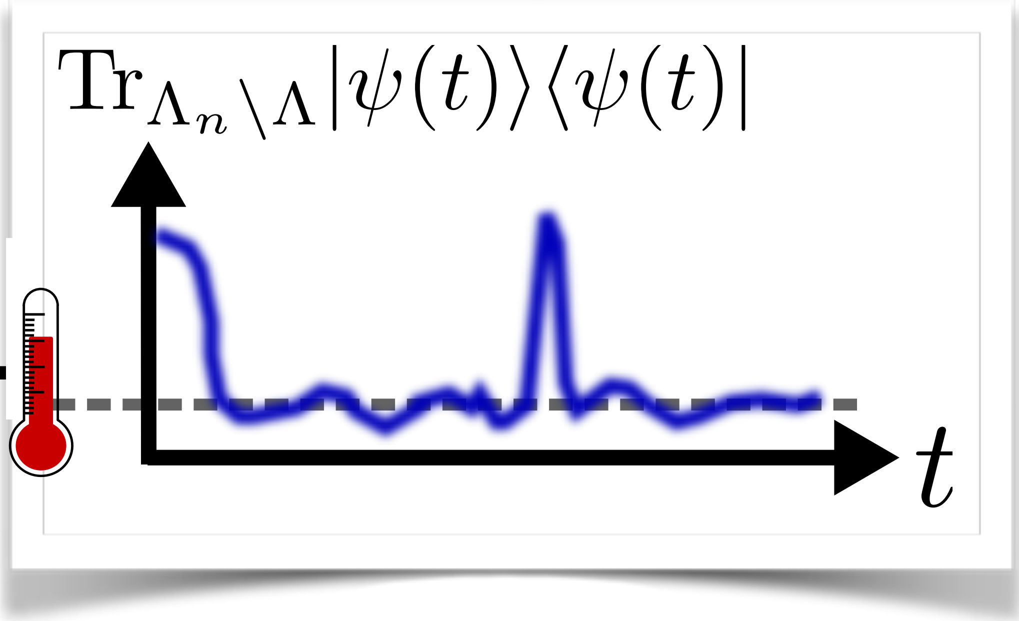
Similar results can be shown for dynamical thermalization.

Using A. J. Short and T. C. Farrelly, New J. Phys. **14**, 013063 (2012) we show:

Theorem: If the initial state $|\psi(0)\rangle$ occupies a large number of energy levels, and some other technical conditions are met, then

$$\mathrm{Tr}_{\Lambda_n\setminus\Lambda}|\psi(t)\rangle\langle\psi(t)|$$

is close to thermal for most times *t*.



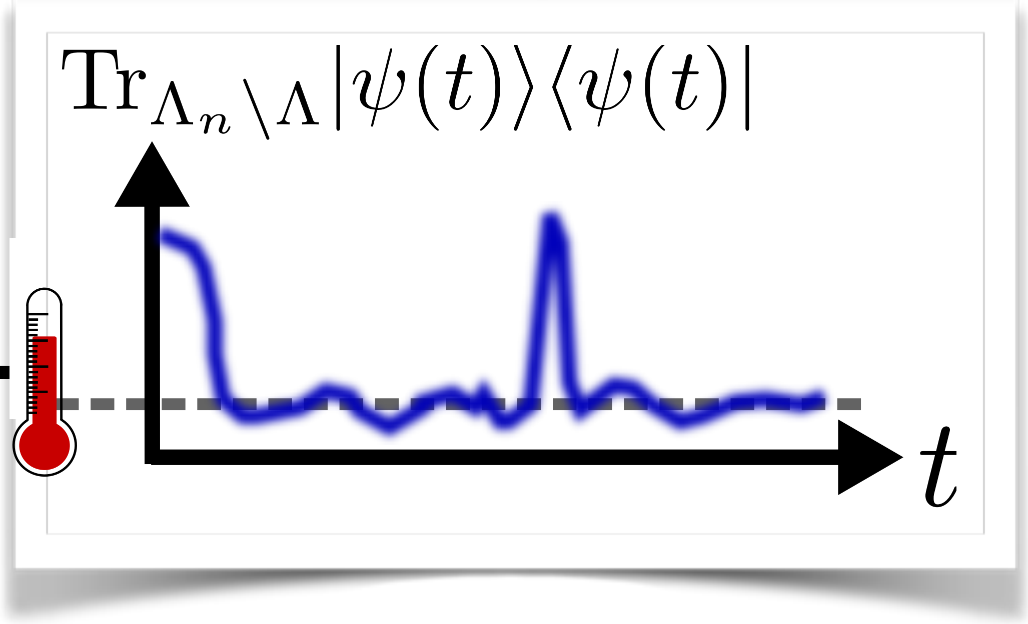
MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Natural improvement: eigenstate thermalization

Theorem: If the initial state $|\psi(0)\rangle$ occupies a large number of energy levels, and some other technical conditions are met, then

$$\mathrm{Tr}_{\Lambda_n\setminus\Lambda}|\psi(t)\rangle\langle\psi(t)|$$

is close to thermal for most times *t*.



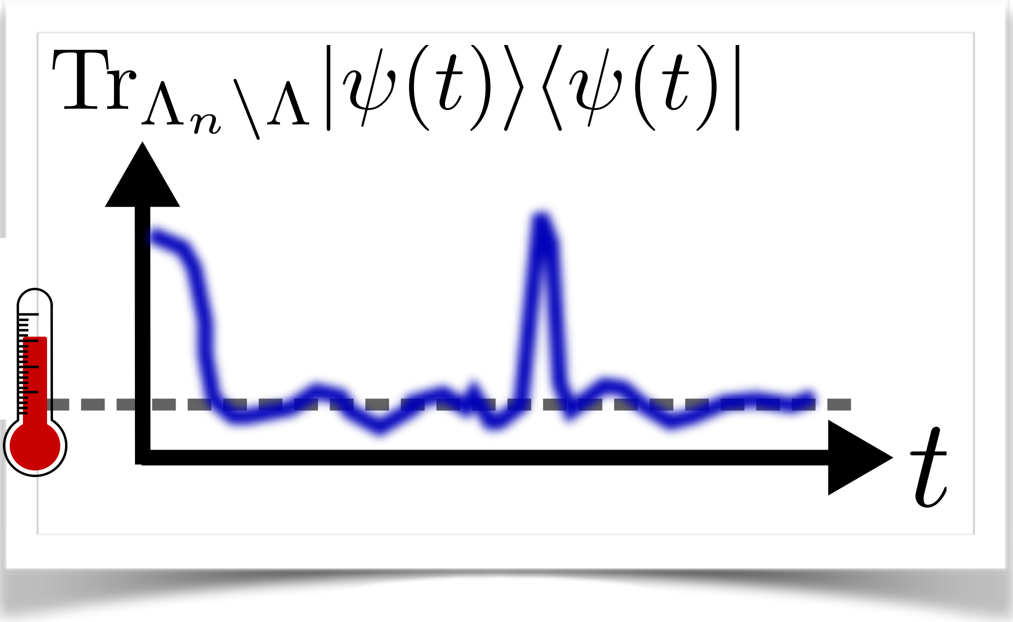
MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Natural improvement: eigenstate thermalization

Theorem: If the initial state $|\psi(0)\rangle$ occupies a large number of energy levels, and some other technical conditions are met, then

$$\mathrm{Tr}_{\Lambda_n\setminus\Lambda}|\psi(t)\rangle\langle\psi(t)|$$

is close to thermal for most times *t*.



MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

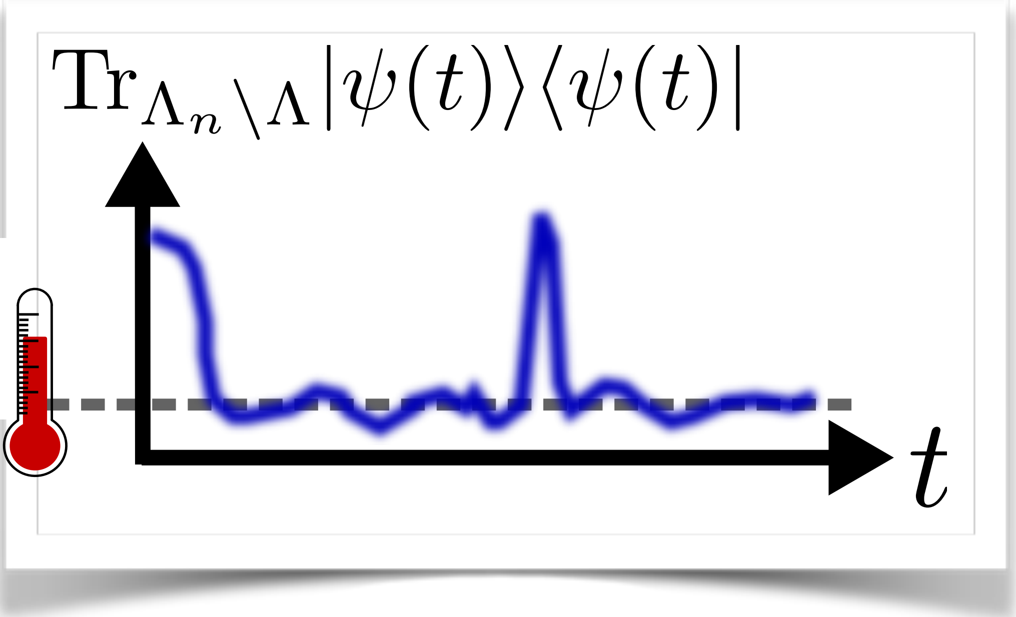
Natural improvement: eigenstate thermalization

What if only a few?
Or even a single
eigenstate?

Theorem: If the initial state $|\psi(0)\rangle$ occupies a large number of energy levels, and some other technical conditions are met, then

$$\mathrm{Tr}_{\Lambda_n\setminus\Lambda}|\psi(t)\rangle\langle\psi(t)|$$

is close to thermal for most times *t*.



MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Outline

→ 1. How do quantum systems thermalize?

New approaches to old questions
Canonical typicality
Dynamical thermalization

2. Weak eigenstate thermalization

Lieb-Robinson bounds
Weak ETH: physical interpretation
Weak ETH: proof sketch

3. Some math. details on part 1

Detailed theorems and proof sketches
Finite-size bounds for non-interacting systems

Outline

1. How do quantum systems thermalize?

New approaches to old questions
Canonical typicality
Dynamical thermalization

→ 2. Weak eigenstate thermalization

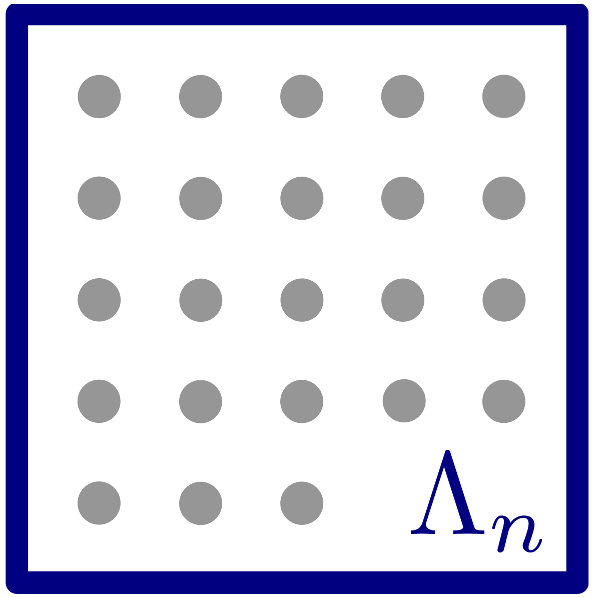
Lieb-Robinson bounds
Weak ETH: physical interpretation
Weak ETH: proof sketch

3. Some math. details on part 1

Detailed theorems and proof sketches
Finite-size bounds for non-interacting systems

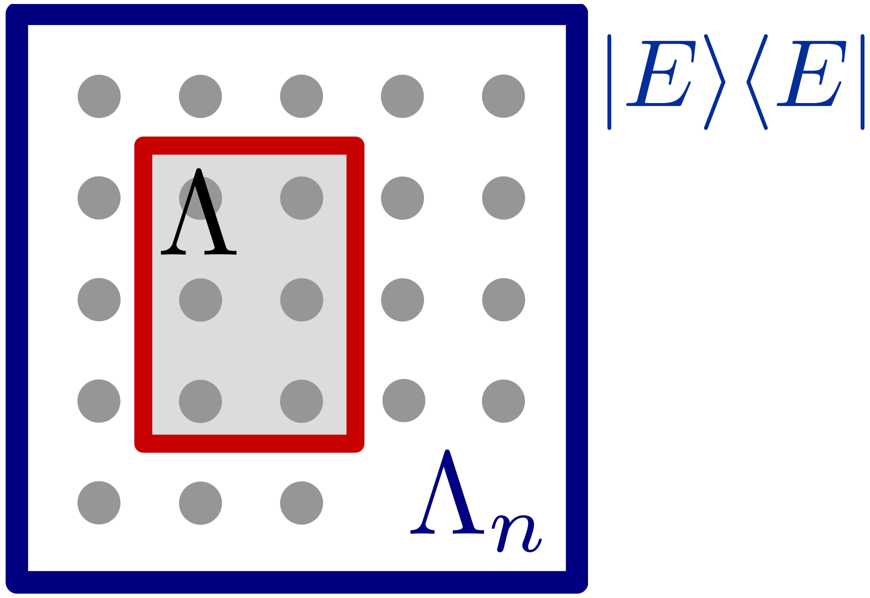
Weak eigenstate thermalization

As before: finite-range interaction on a cubic lattice;
but not necessarily translation-invariant.



Weak eigenstate thermalization

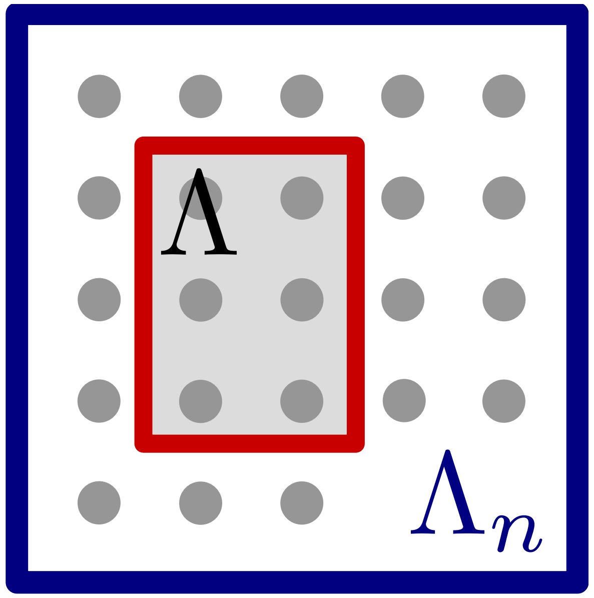
As before: finite-range interaction on a cubic lattice;
but not necessarily translation-invariant.



Do global energy eigenstates locally look thermal?

Weak eigenstate thermalization

As before: finite-range interaction on a cubic lattice;
but not necessarily translation-invariant.



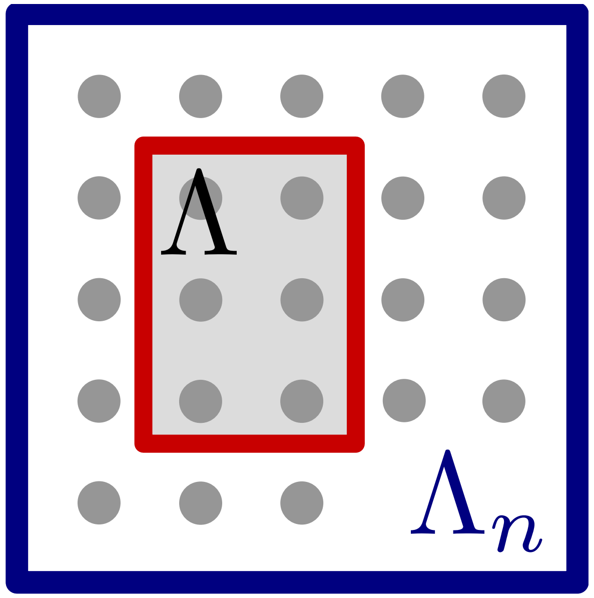
$|E\rangle\langle E|$

Before, we had random states $|\psi\rangle\langle\psi|$.
Now concrete eigenstates!

Do global energy eigenstates locally look thermal?

Weak eigenstate thermalization

As before: finite-range interaction on a cubic lattice;
but not necessarily translation-invariant.



$|E\rangle\langle E|$

Before, we had random states $|\psi\rangle\langle\psi|$.
Now concrete eigenstates!

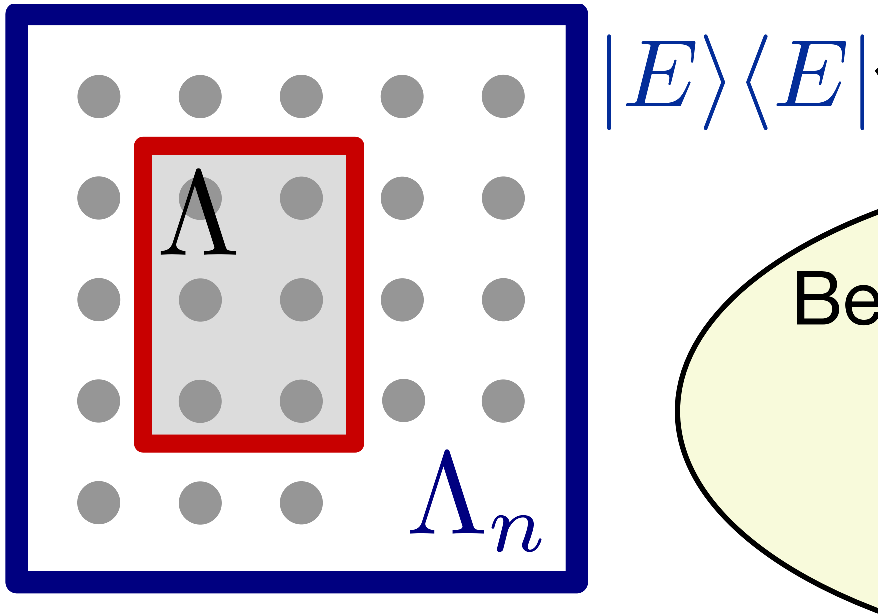
Do global energy eigenstates locally look thermal?

Eigenstate thermalization hypothesis

J. M. Deutsch, *Quantum statistical mechanics in a closed system*, Phys. Rev. A **43**, 2046 (1991).
M. Srednicki, *Chaos and quantum thermalization*, Phys. Rev. E **50**, 888 (1994).

Weak eigenstate thermalization

As before: finite-range interaction on a cubic lattice;
but not necessarily translation-invariant.

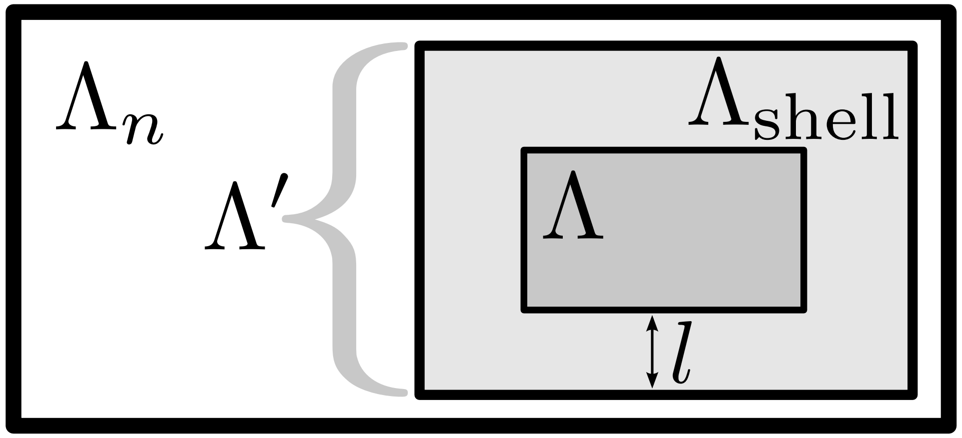


Before, we had random states $|\psi\rangle\langle\psi|$. Now concrete eigenstates!

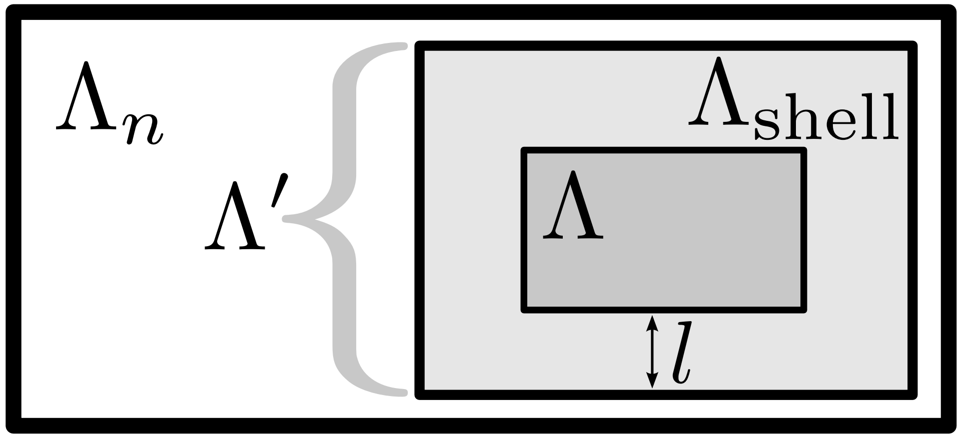
Do global energy eigenstates locally look thermal?

As before, we should *not* expect that $\mathrm{Tr}_{\Lambda_n\setminus\Lambda}|E\rangle\langle E| \approx \gamma_\Lambda$
where $\gamma_\Lambda = \exp(-\beta H_\Lambda)/Z.$

Weak eigenstate thermalization



Weak eigenstate thermalization



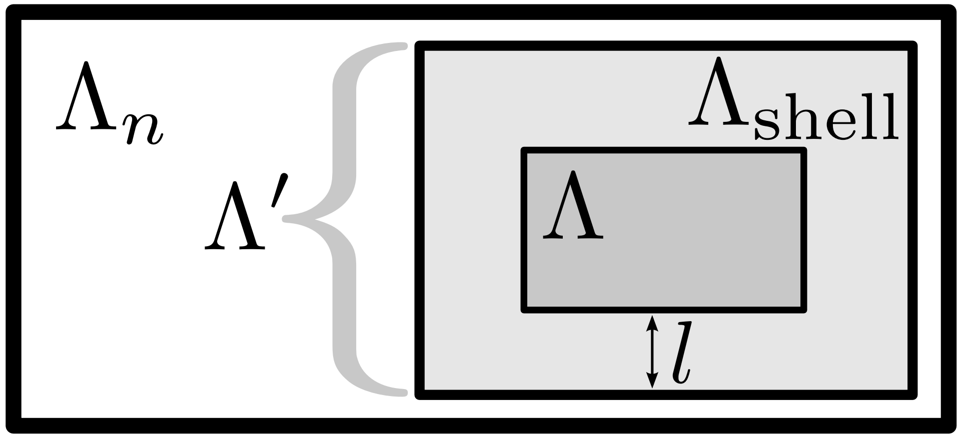
Conjecture: Under some additional assumptions on *H*,

$$\mathrm{Tr}_{\Lambda_n \setminus \Lambda} |E\rangle\langle E| \approx \mathrm{Tr}_{\Lambda_{\text{shell}}} \frac{\exp(-\beta H_{\Lambda'})}{Z},$$

and the distance goes to zero as *n*→∞ and (more slowly) *l*→∞.

We cannot prove this. But:

Weak eigenstate thermalization



Theorem 4. *There is a state ω_E on Λ' such that*

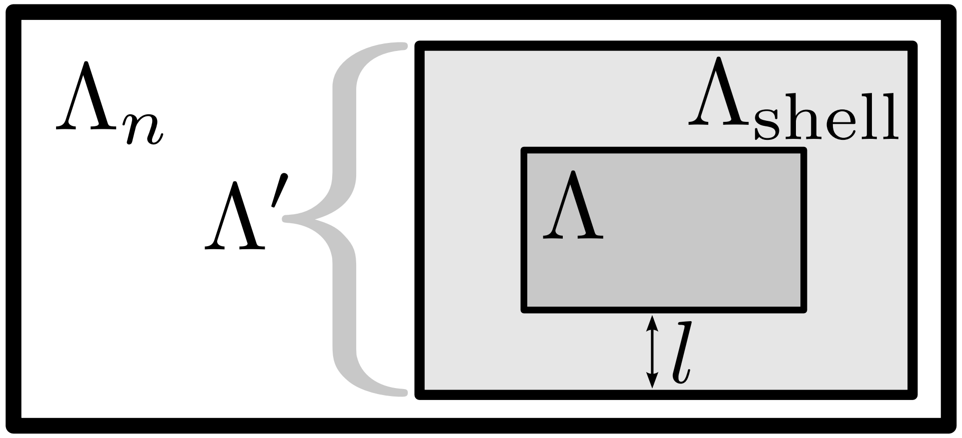
$$\left\| \mathrm{Tr}_{\Lambda_{\text{shell}}}(\omega_E) - \mathrm{Tr}_{\Lambda_n\setminus\Lambda} |E\rangle\langle E| \right\|_1 \leq \kappa \cdot e^{-c(l-r)/2},$$

where $\kappa = 2AJ(CA+2)\sqrt{\frac{l-r}{8cv^2}}$ *and* $J = \max_X \|h_X\|$, *which is weakly diagonal in the eigenbasis* $\{|e\rangle\}$ *of* $H_{\Lambda'}$, *i.e.*

$$|\langle e_1|\omega_E|e_2\rangle| \leq e^{-(l-r)(e_1-e_2)^2/(8cv^2)}.$$

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Weak eigenstate thermalization



shell width

Theorem 4. *There is a state ω_E on Λ' such that*

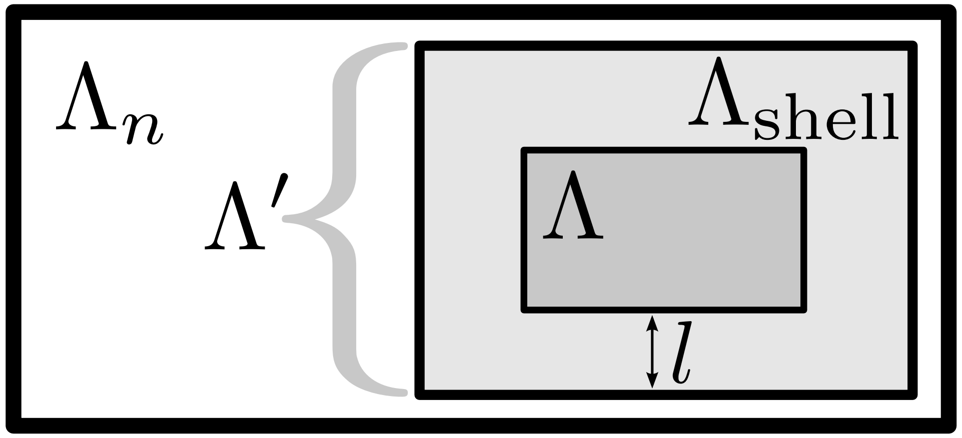
$$\left\|\mathrm{Tr}_{\Lambda_{\rm shell}}(\omega_E) - \mathrm{Tr}_{\Lambda_n\setminus\Lambda}|E\rangle\langle E|\right\|_1 \leq \kappa \cdot e^{-c(l-r)/2},$$

where $\kappa = 2AJ(CA+2)\sqrt{\frac{l-r}{8cv^2}}$ *and* $J = \max_X \|h_X\|$, *which is weakly diagonal in the eigenbasis* $\{|e\rangle\}$ *of* $H_{\Lambda'}$, *i.e.*

$$|\langle e_1|\omega_E|e_2\rangle| \leq e^{-(l-r)(e_1-e_2)^2/(8cv^2)}.$$

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Weak eigenstate thermalization



interaction range

Theorem 4. *There is a state ω_E on Λ' such that*

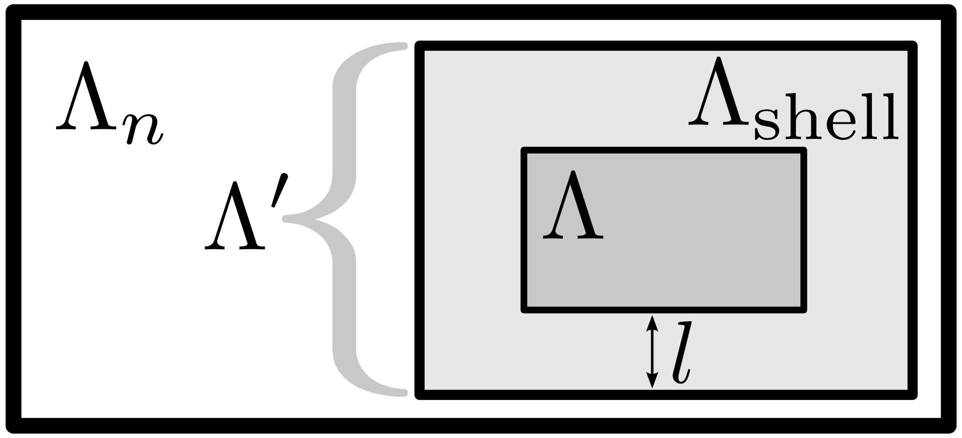
$$\left\|\mathrm{Tr}_{\Lambda_{\text{shell}}}(\omega_E) - \mathrm{Tr}_{\Lambda_n\setminus\Lambda}|E\rangle\langle E|\right\|_1 \leq \kappa\cdot e^{-c(l-r)/2},$$

where $\kappa = 2AJ(CA+2)\sqrt{\frac{l-r}{8cv^2}}$ *and* $J = \max_X \|h_X\|$, *which is weakly diagonal in the eigenbasis* $\{|e\rangle\}$ *of* $H_{\Lambda'}$, *i.e.*

$$|\langle e_1|\omega_E|e_2\rangle| \leq e^{-(l-r)(e_1-e_2)^2/(8cv^2)}.$$

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Weak eigenstate thermalization



local terms in *H*

Theorem 4. *There is a state ω_E on Λ' such that*

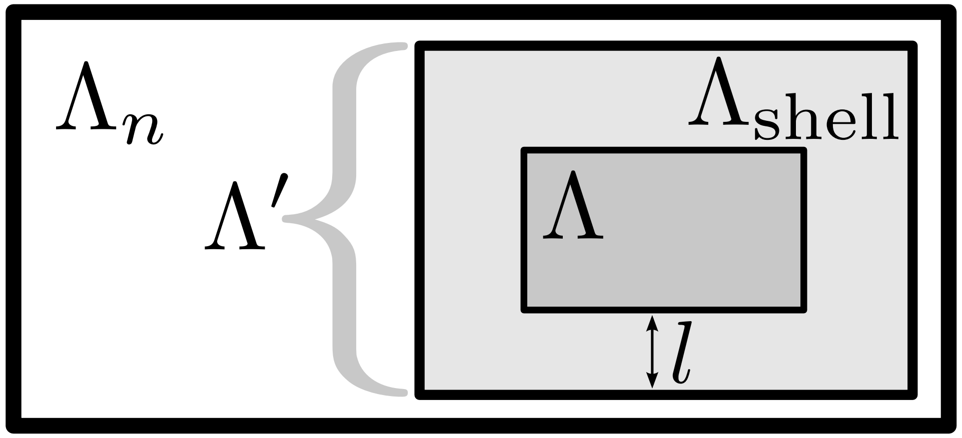
$$\left\|\mathrm{Tr}_{\Lambda_{\text{shell}}}(\omega_E) - \mathrm{Tr}_{\Lambda_n\setminus\Lambda}|E\rangle\langle E|\right\|_1 \leq \kappa \cdot e^{-c(l-r)/2},$$

where $\kappa = 2AJ(CA+2)\sqrt{\frac{l-r}{8cv^2}}$ *and* $J = \max_X \|h_X\|$, *which is weakly diagonal in the eigenbasis* $\{|e\rangle\}$ *of* $H_{\Lambda'}$, *i.e.*

$$|\langle e_1|\omega_E|e_2\rangle| \leq e^{-(l-r)(e_1-e_2)^2/(8cv^2)}.$$

MM, E. Adlam, Ll. Masanes, and N. Wiebe, arXiv:1312.7420.

Weak eigenstate thermalization



local terms in *H*

Theorem 4. *There is a state ω_E on Λ' such that*

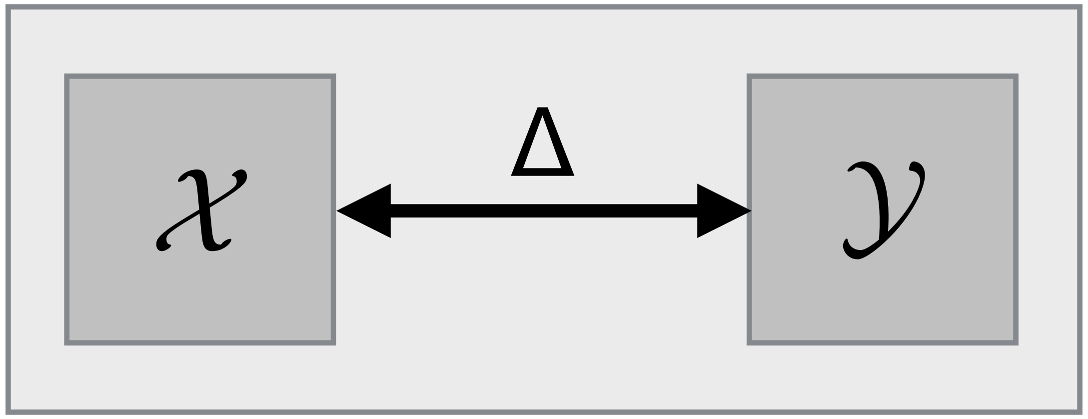
$$\left\|\mathrm{Tr}_{\Lambda_{\text{shell}}}(\omega_E) - \mathrm{Tr}_{\Lambda_n \setminus \Lambda} |E\rangle\langle E|\right\|_1 \leq \kappa \cdot e^{-c(l-r)/2},$$

where $\kappa = 2AJ(CA+2)\sqrt{\frac{l-r}{8cv^2}}$ *and* $J = \max_X \|h_X\|$, *which is weakly diagonal in the eigenbasis* $\{|e\rangle\}$ *of* $H_{\Lambda'}$, *i.e.*

$$|\langle e_1|\omega_E|e_2\rangle| \leq e^{-(l-r)(e_1-e_2)^2/(8cv^2)}.$$

The other constants come from the Lieb-Robinson bound.

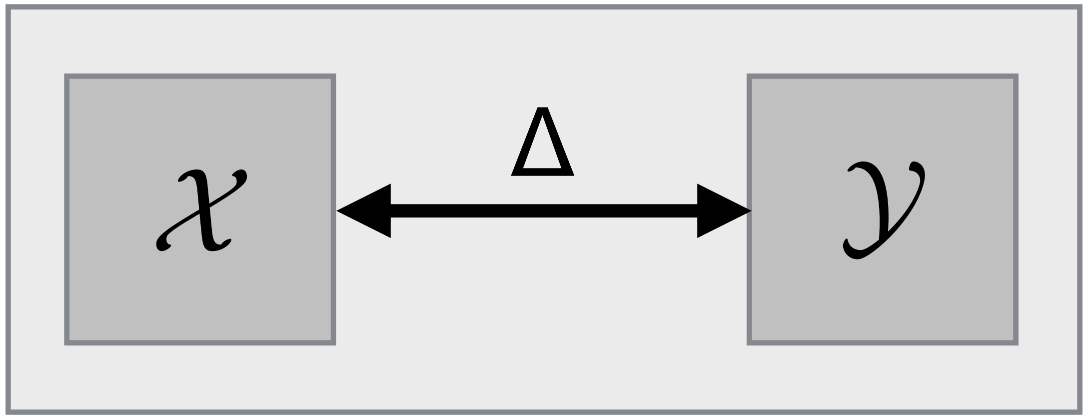
Weak eigenstate thermalization



X, Y observables on $\mathcal{X}, \mathcal{Y}$.

$$X(t) = e^{iHt} X e^{-iHt}.$$

Weak eigenstate thermalization



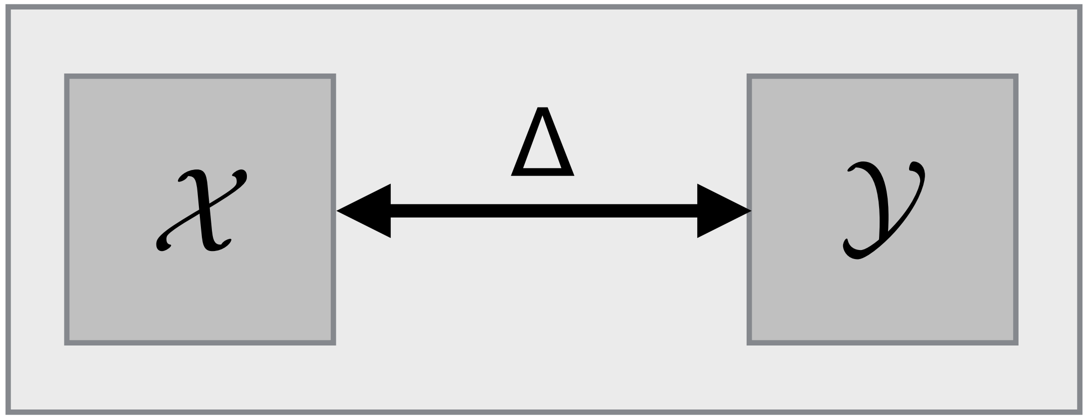
X, Y observables on $\mathcal{X}, \mathcal{Y}$.

$$X(t) = e^{iHt} X e^{-iHt}.$$

Lieb-Robinson bound: there are constants *c,C,v>0* such that

$$\| \, [X(t), Y] \, \| \leq C \, \|X\| \cdot \|Y\| \cdot \min\{|\mathcal{X}|, |\mathcal{Y}|\} e^{-c(\Delta - v|t|)}.$$

Weak eigenstate thermalization



X, Y observables on $\mathcal{X}, \mathcal{Y}$.

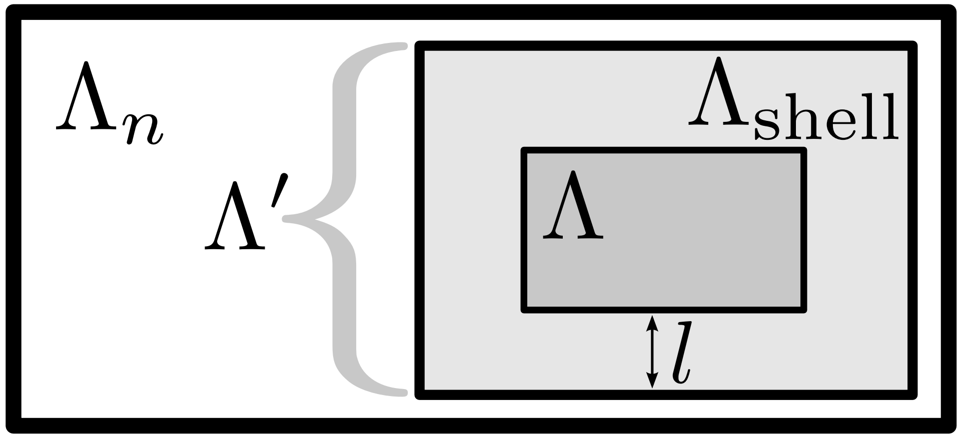
$$X(t) = e^{iHt} X e^{-iHt}.$$

Lieb-Robinson bound: there are constants *c,C,v>0* such that

$$\| \, [X(t), Y] \, \| \leq C \, \|X\| \cdot \|Y\| \cdot \min\{|\mathcal{X}|, |\mathcal{Y}|\} e^{-c(\Delta - v|t|)}.$$

→ finite speed of signal transmission
in quantum systems with finite interaction range.

Weak eigenstate thermalization



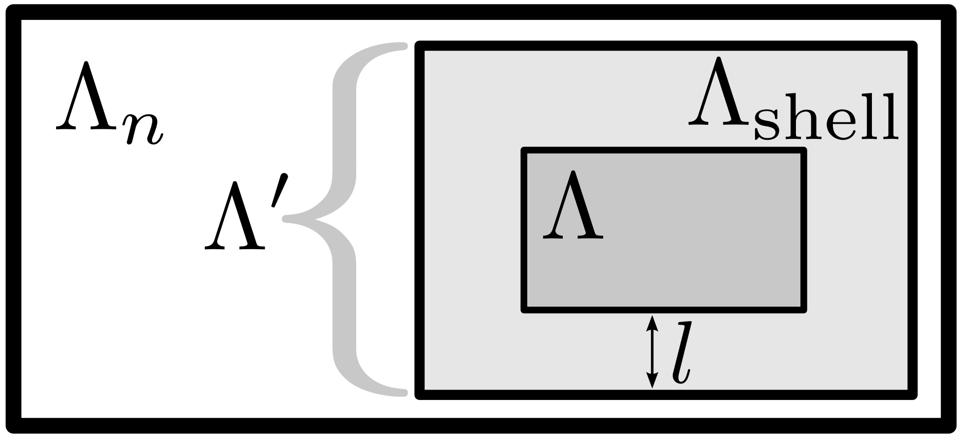
Theorem 4. *There is a state* ω_E *on* Λ' *such that*

$$\left\| \mathrm{Tr}_{\Lambda_{\text{shell}}}(\omega_E) - \mathrm{Tr}_{\Lambda_n \setminus \Lambda} |E\rangle\langle E| \right\|_1 \leq \kappa \cdot e^{-c(l-r)/2},$$

where $\kappa = 2AJ(CA+2)\sqrt{\frac{l-r}{8cv^2}}$ *and* $J = \max_X \|h_X\|$, *which is weakly diagonal in the eigenbasis* $\{|e\rangle\}$ *of* $H_{\Lambda'}$, *i.e.*

$$|\langle e_1|\omega_E|e_2\rangle| \leq e^{-(l-r)(e_1-e_2)^2/(8cv^2)}.$$

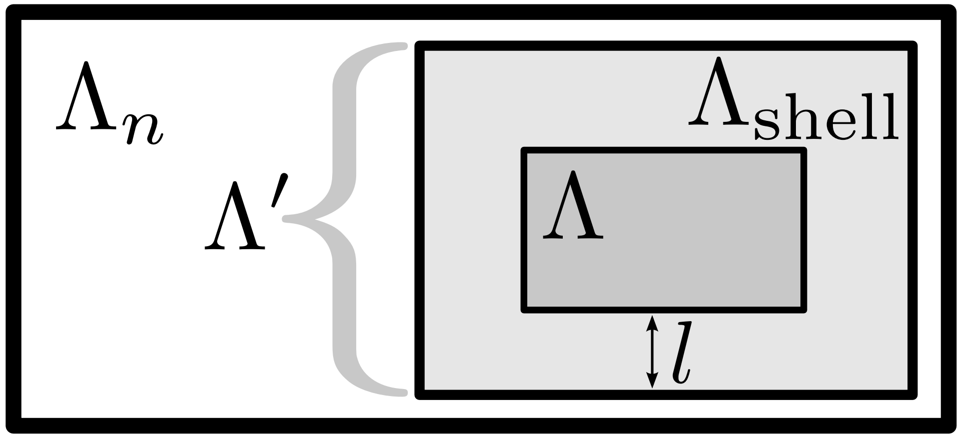
Weak eigenstate thermalization: physical interpretation



$$\omega_E := \int_{-\infty}^{\infty} dt\, g(t) e^{-iH_{\Lambda'}t} \left(\mathrm{Tr}_{\Lambda_n \setminus \Lambda'} |E\rangle\langle E|\right) e^{iH_{\Lambda'}t}.$$

Gaussian, centered at 0

Weak eigenstate thermalization: physical interpretation

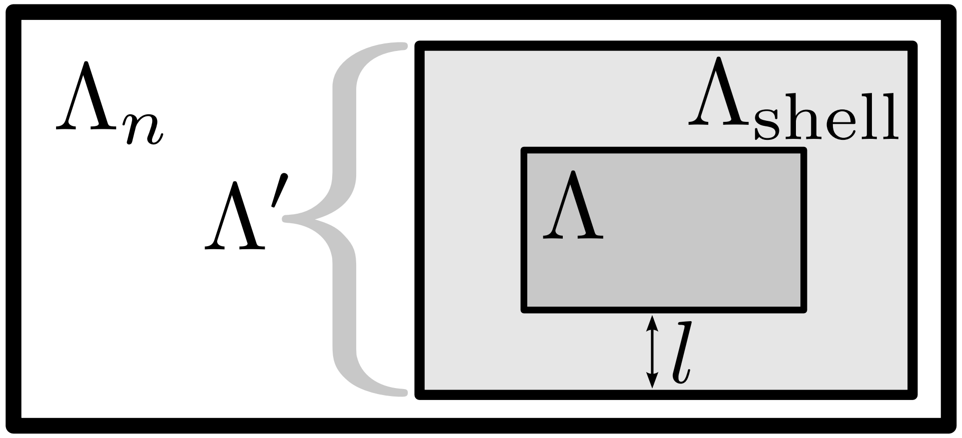


$$\omega_E := \int_{-\infty}^{\infty} dt\, g(t) e^{-iH_{\Lambda'}t} \left(\mathrm{Tr}_{\Lambda_n\setminus\Lambda'}|E\rangle\langle E|\right) e^{iH_{\Lambda'}t}.$$

Gaussian, centered at 0

- Lieb-Robinson: result will in Λ still look very much as if $|E\rangle\langle E|$ evolved under the full Hamiltonian H_{Λ_n}
  $\Rightarrow \mathrm{Tr}_{\Lambda_{\mathrm{shell}}}\omega_E \approx \mathrm{Tr}_{\Lambda_n\setminus\Lambda}|E\rangle\langle E|.$

Weak eigenstate thermalization: physical interpretation



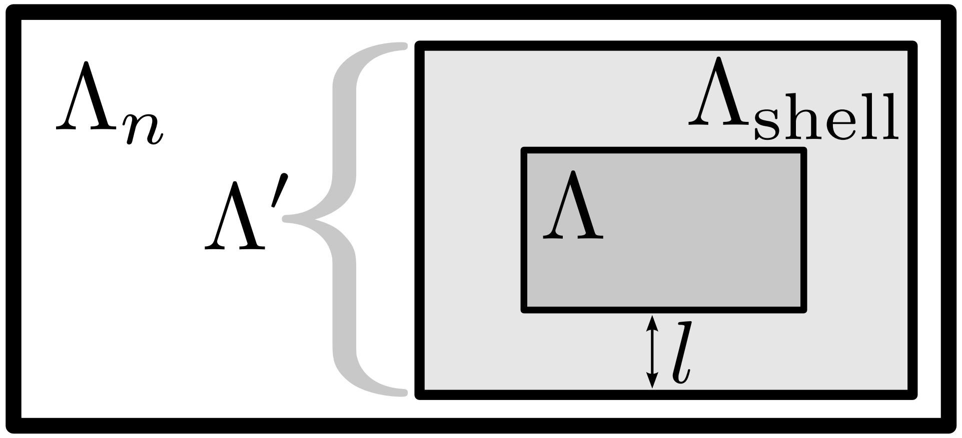
$$\omega_E := \int_{-\infty}^{\infty} dt\, g(t) e^{-iH_{\Lambda'}t} \left(\mathrm{Tr}_{\Lambda_n\setminus\Lambda'}|E\rangle\langle E|\right) e^{iH_{\Lambda'}t}.$$

Gaussian, centered at 0

- Lieb-Robinson: result will in Λ still look very much as if $|E\rangle\langle E|$ evolved under the full Hamiltonian H_{Λ_n}
  $\Rightarrow \mathrm{Tr}_{\Lambda_{\text{shell}}}\omega_E \approx \mathrm{Tr}_{\Lambda_n\setminus\Lambda}|E\rangle\langle E|.$

- Decoherence across boundary of Λ' suppresses off-diag.:
  $|e_1 - e_2| \gg 1 \Rightarrow |\langle e_1|\omega_E|e_2\rangle| \approx 0.$

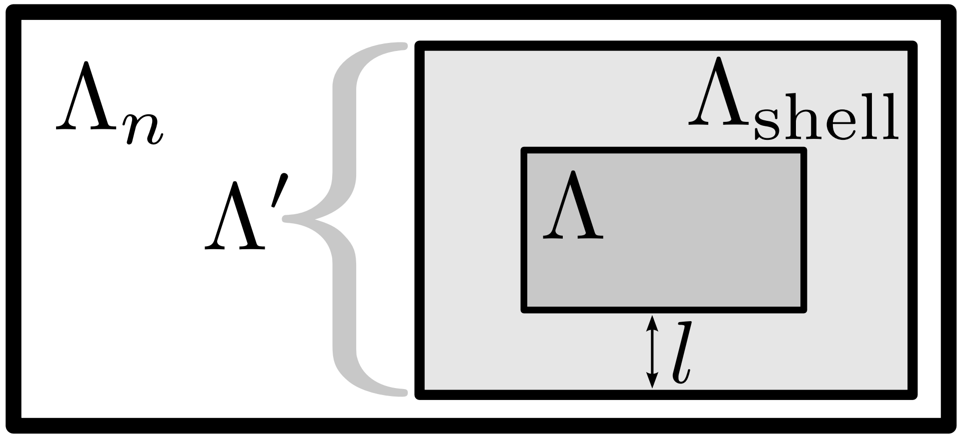
Weak eigenstate thermalization: proof sketch



$$\omega_E := \int_{-\infty}^{\infty} dt\, g(t) e^{-iH_{\Lambda'}t} \left(\mathrm{Tr}_{\Lambda_n\setminus\Lambda'}|E\rangle\langle E|\right) e^{iH_{\Lambda'}t}.$$

Gaussian, centered at 0

Weak eigenstate thermalization: proof sketch



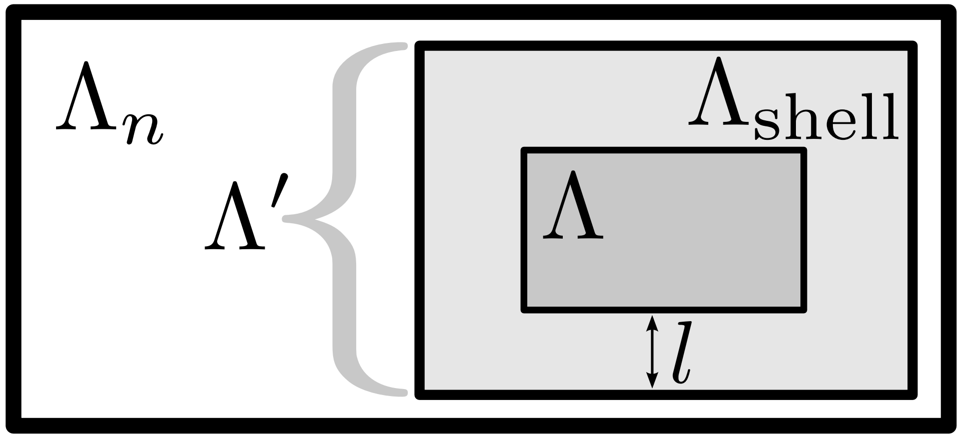
$$\omega_E := \int_{-\infty}^{\infty} dt\, g(t) e^{-iH_{\Lambda'}t} \left(\mathrm{Tr}_{\Lambda_n \setminus \Lambda'} |E\rangle\langle E|\right) e^{iH_{\Lambda'}t}.$$

Gaussian, centered at 0

Decoherence across boundary:

$$\begin{aligned}\langle e_1|\omega_E|e_2\rangle &= \int dt\, g(t)\, e^{-i(e_1-e_2)t} \langle e_1|\mathrm{Tr}_{\bar{\Lambda}'}(|E\rangle\langle E|)|e_2\rangle \\ &= e^{-(e_1-e_2)^2\sigma^2/2}\langle e_1|\mathrm{Tr}_{\bar{\Lambda}'}(|E\rangle\langle E|)|e_2\rangle.\end{aligned}$$

Weak eigenstate thermalization: proof sketch



$$\omega_E := \int_{-\infty}^{\infty} dt\, g(t) e^{-iH_{\Lambda'}t} \left(\mathrm{Tr}_{\Lambda_n \setminus \Lambda'} |E\rangle\langle E|\right) e^{iH_{\Lambda'}t}.$$

Gaussian, centered at 0

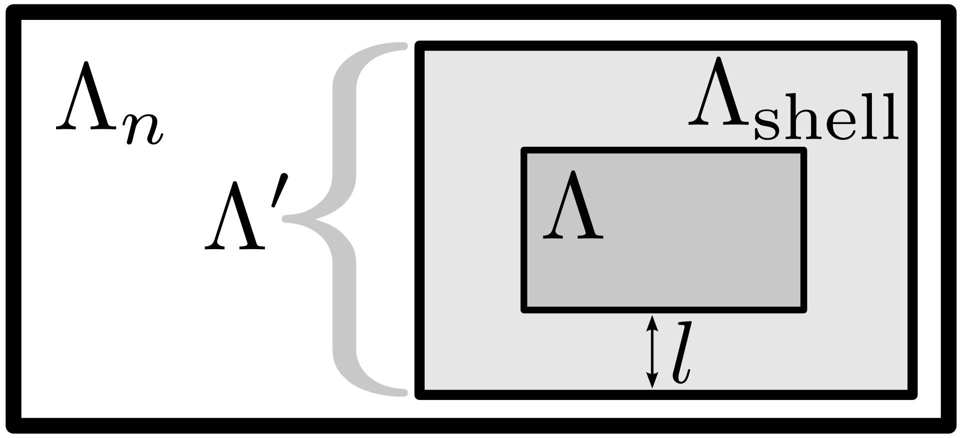
Local similarity to eigenstate: long chain of inequalities.

$$\left\| e^{-iHt} e^{i(H-H_A)t} X e^{-i(H-H_A)t} e^{iHt} - X \right\|_\infty = \left\| \int_0^t dt_1 \frac{\partial}{\partial t_1} \left(e^{-iHt_1} e^{i(H-H_A)t_1} X e^{-i(H-H_A)t_1} e^{iHt_1} \right) \right\|_\infty$$

$$\leq \int_0^{|t|} dt_1 \left\| \left[H_A\,, e^{iH_{\Lambda'}t_1} X e^{-iH_{\Lambda'}t_1} \right] \right\|_\infty .$$

Use Lieb-Robinson bound.

Weak eigenstate thermalization



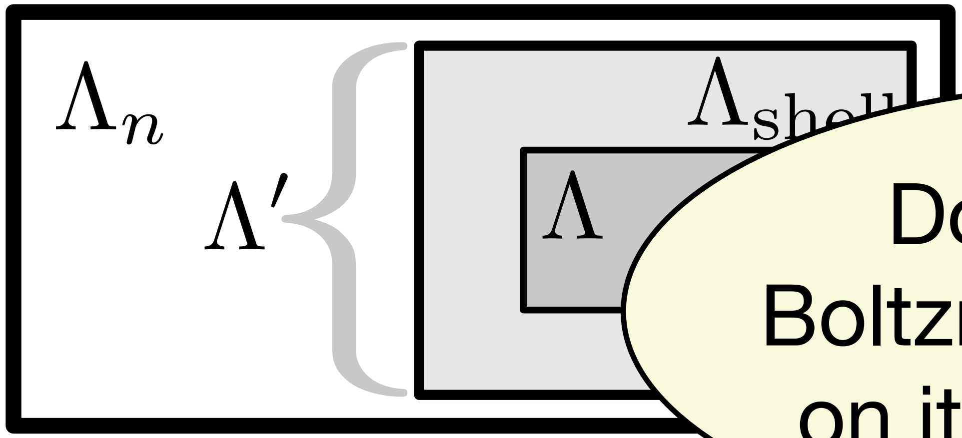
Theorem 4. *There is a state ω_E on Λ' such that*

$$\left\| \mathrm{Tr}_{\Lambda_{\text{shell}}}(\omega_E) - \mathrm{Tr}_{\Lambda_n \setminus \Lambda} |E\rangle\langle E| \right\|_1 \leq \kappa \cdot e^{-c(l-r)/2},$$

where $\kappa = 2AJ(CA+2)\sqrt{\frac{l-r}{8cv^2}}$ *and* $J = \max_X \|h_X\|$, *which is weakly diagonal in the eigenbasis* $\{|e\rangle\}$ *of* $H_{\Lambda'}$, *i.e.*

$$|\langle e_1|\omega_E|e_2\rangle| \leq e^{-(l-r)(e_1-e_2)^2/(8cv^2)}.$$

Weak eigenstate thermalization



Does it have Boltzmann weights on its diagonal??

Theorem 4. *There is a state* ω_E *on* Λ' *such that*

$$\left\|{\rm Tr}_{\Lambda_{\rm shell}}(\omega_E)-{\rm Tr}_{\Lambda_n\setminus\Lambda}|E\rangle\langle E|\right\|_1\leq\kappa\cdot e^{-c(l-r)/2},$$

where $\kappa = 2AJ(CA+2)\sqrt{\frac{l-r}{8cv^2}}$ *and* $J = \max_X\|h_X\|$, *which is weakly diagonal in the eigenbasis* $\{|e\rangle\}$ *of* $H_{\Lambda'}$, *i.e.*

$$|\langle e_1|\omega_E|e_2\rangle|\leq e^{-(l-r)(e_1-e_2)^2/(8cv^2)}.$$

Need more assumptions. (Translation-invariance! Non-integrability?)

Outline

1. How do quantum systems thermalize?

New approaches to old questions
Canonical typicality
Dynamical thermalization

2. Weak eigenstate thermalization

Lieb-Robinson bounds
Weak ETH: physical interpretation
Weak ETH: proof sketch

3. Some math. details on part 1

Detailed theorems and proof sketches
Finite-size bounds for non-interacting systems

Outline

1. How do quantum systems thermalize?

New approaches to old questions
Canonical typicality
Dynamical thermalization

2. Weak eigenstate thermalization

Lieb-Robinson bounds
Weak ETH: physical interpretation
Weak ETH: proof sketch

→ 3. Some math. details on part 1

Detailed theorems and proof sketches
Finite-size bounds for non-interacting systems

Outline

1. How do quantum systems thermalize?

New approaches to old questions
Canonical typicality
Dynamical thermalization

2. Weak eigenstate thermalization

Lieb-Robinson bounds
Weak ETH: physical interpretation
Weak ETH: proof sketch

3. Some math. details on part 1

Detailed theorems and proof sketches
Finite-size bounds for non-interacting systems

On blackboard.