A Hamiltonian for the zeros of the Riemann zeta function

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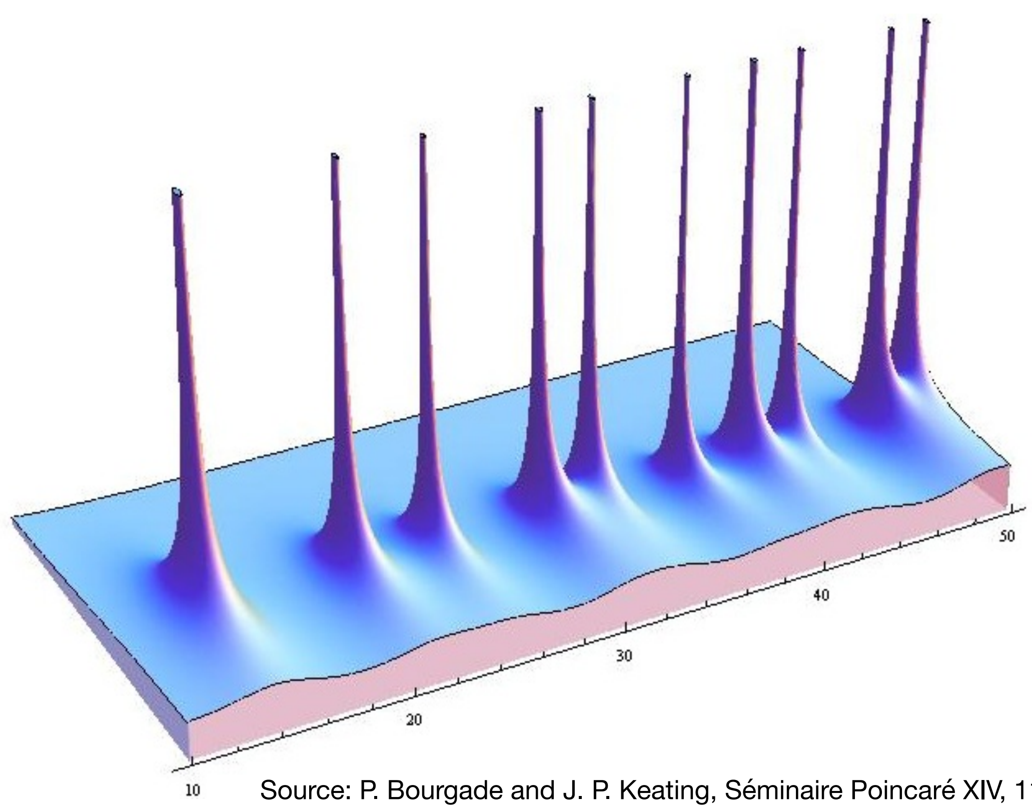
Joint work with Carl Bender and Dorje Brody

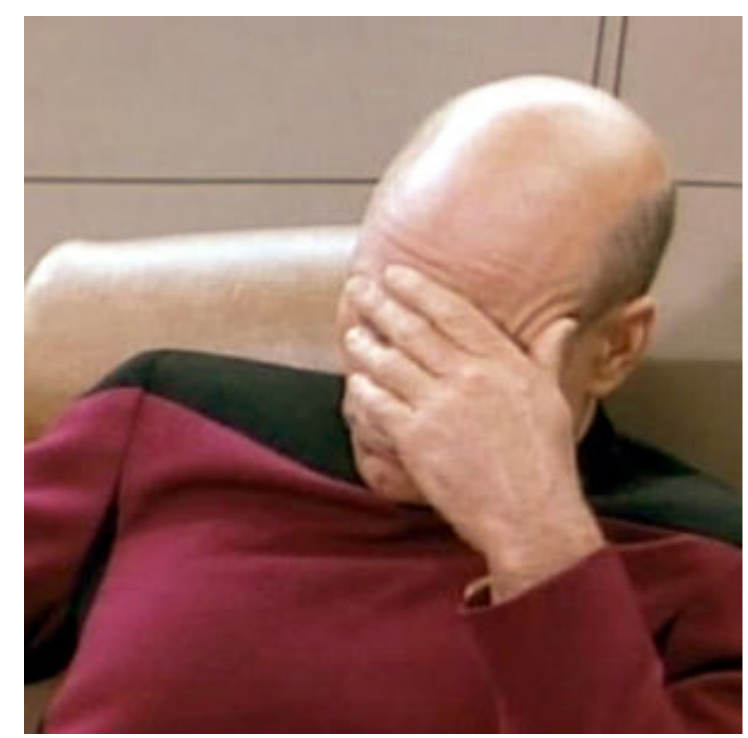


Source: P. Bourgade and J. P. Keating, Séminaire Poincaré XIV, 115-153 (2010).

alternative title: **How interesting but trivial results get terribly overhyped**


Physicists make major breakthrough towar
of Riemann hypothesis

By Sarah Cox - Senior Media Relations Officer
24 Mar 2017



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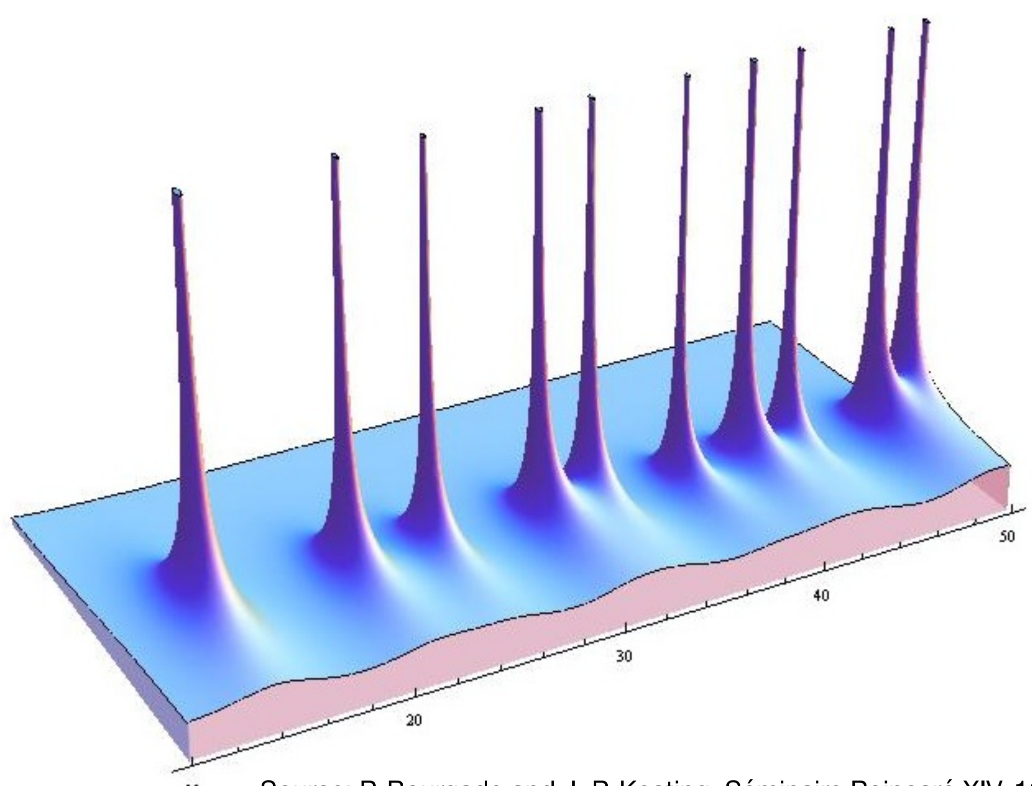

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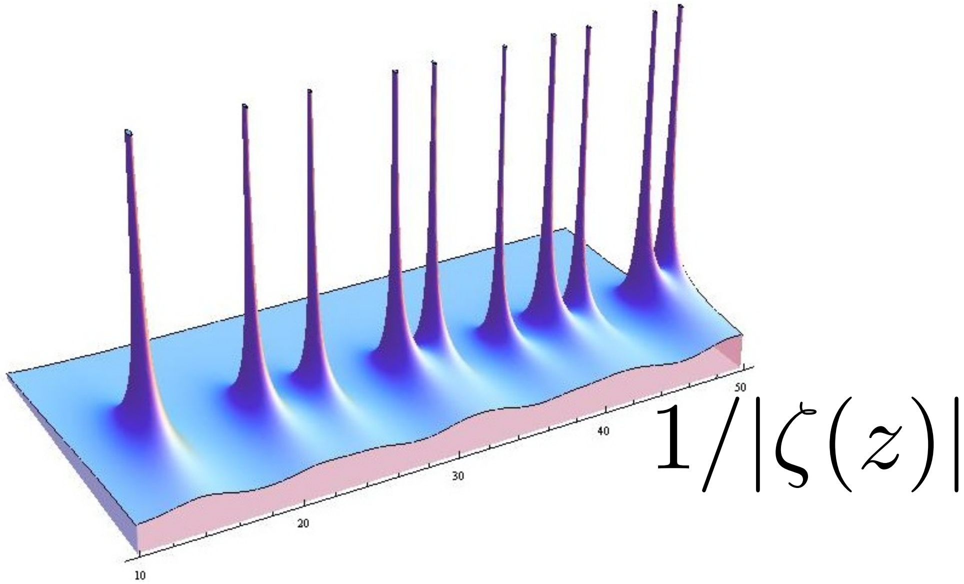
1. The Riemann hypothesis



2. How to add a non-integer number of terms

$$\sum_{n=1}^{-\frac{1}{2}} \frac{1}{n} = -2\log 2$$

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$$R = \hat{X}\hat{p} + \hat{p}\hat{X}$$

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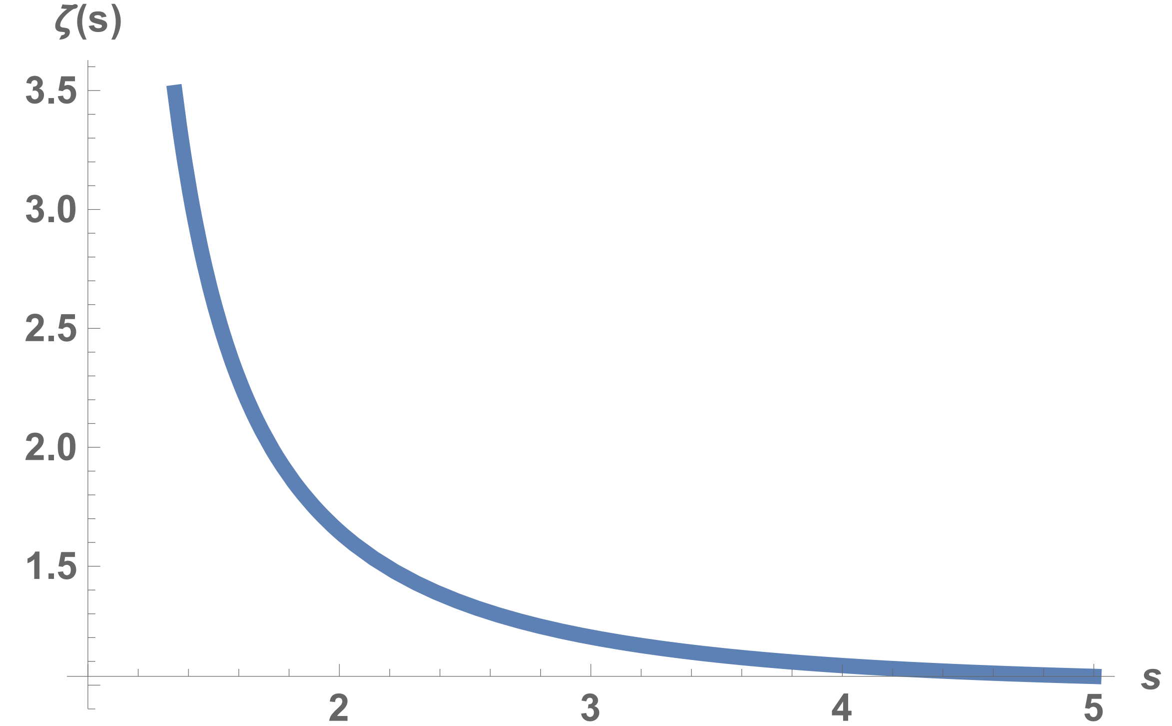

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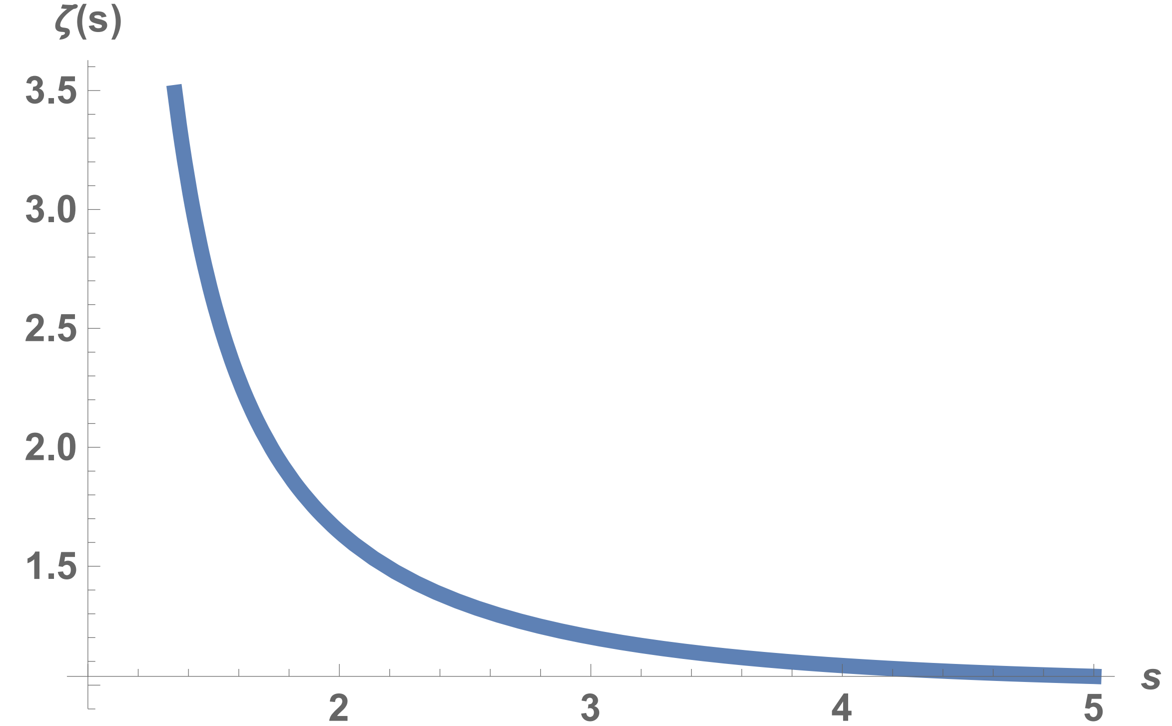

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Leonhard Euler evaluated this function at even integers:

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}.$$

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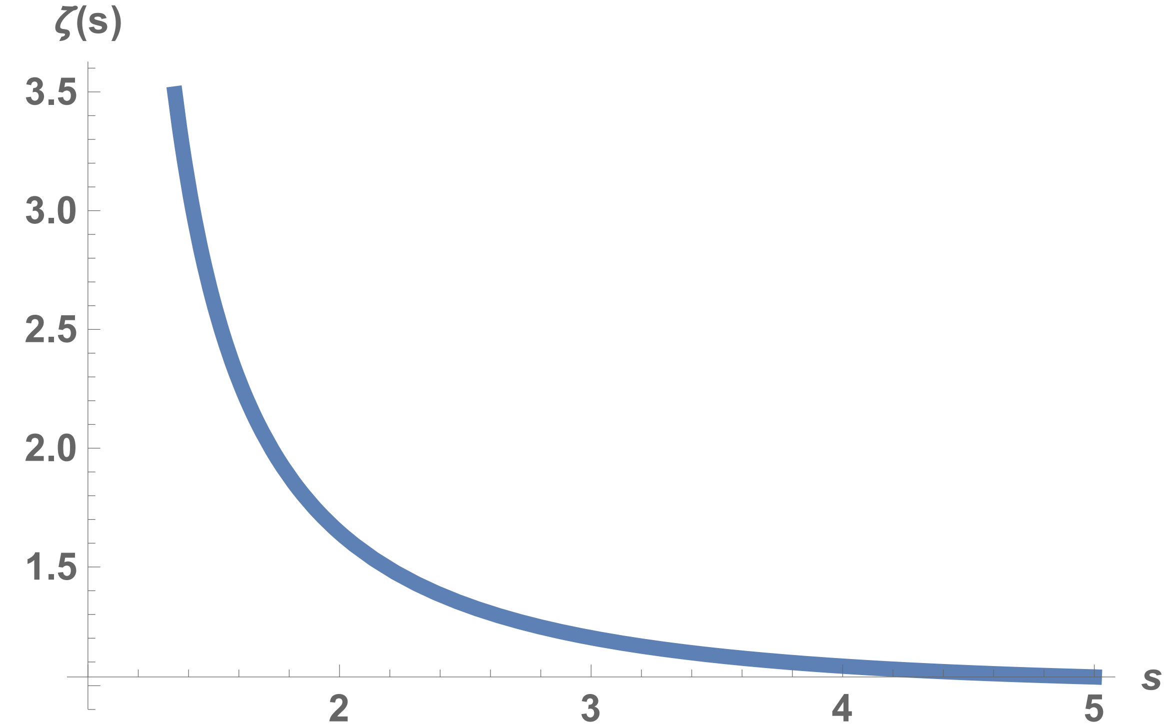

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Relation to prime numbers:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Easy proof: see Wikipedia.

Extending the definition of the zeta function

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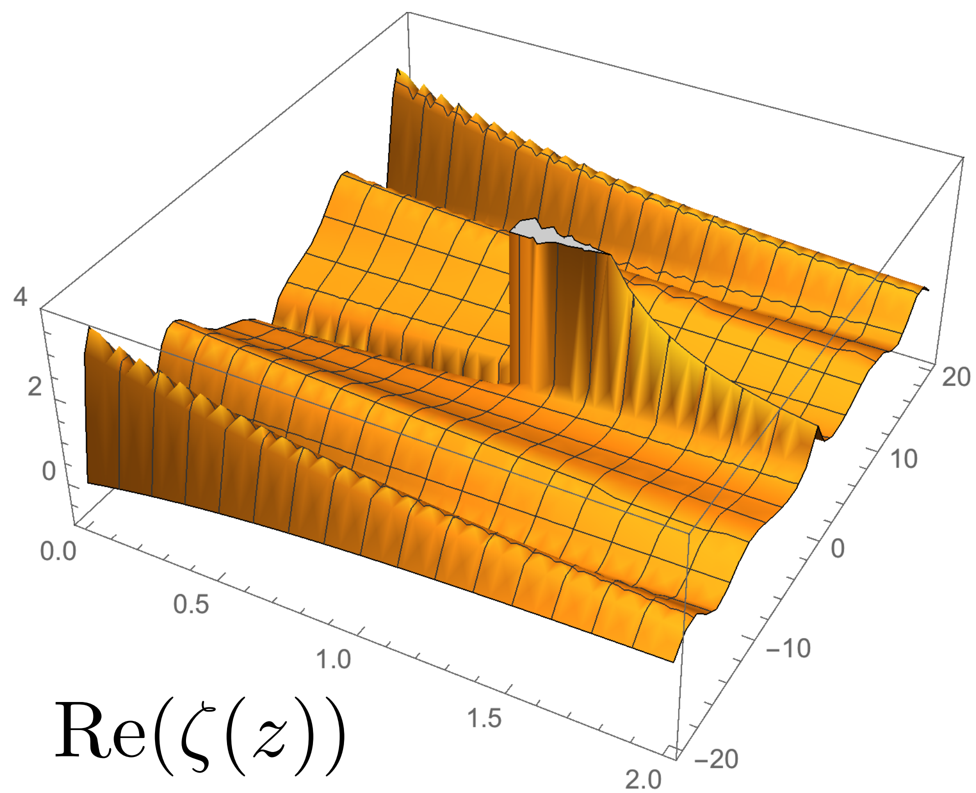

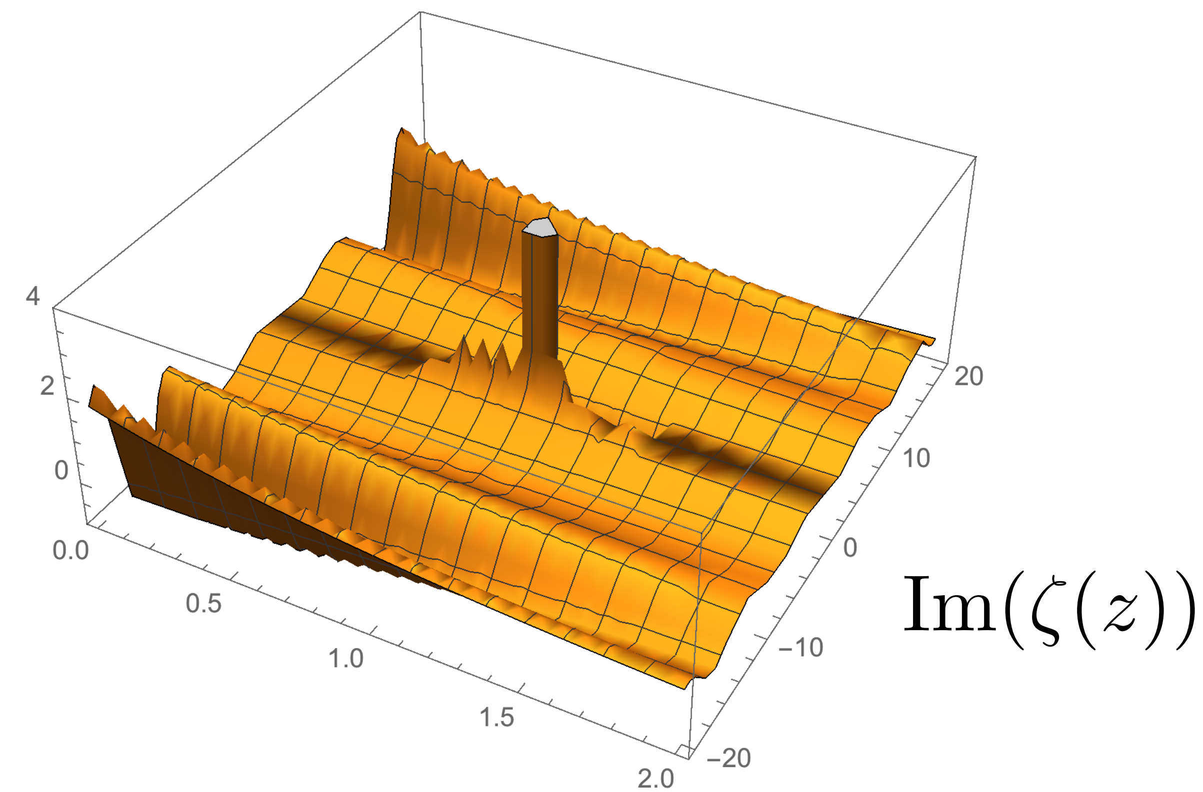

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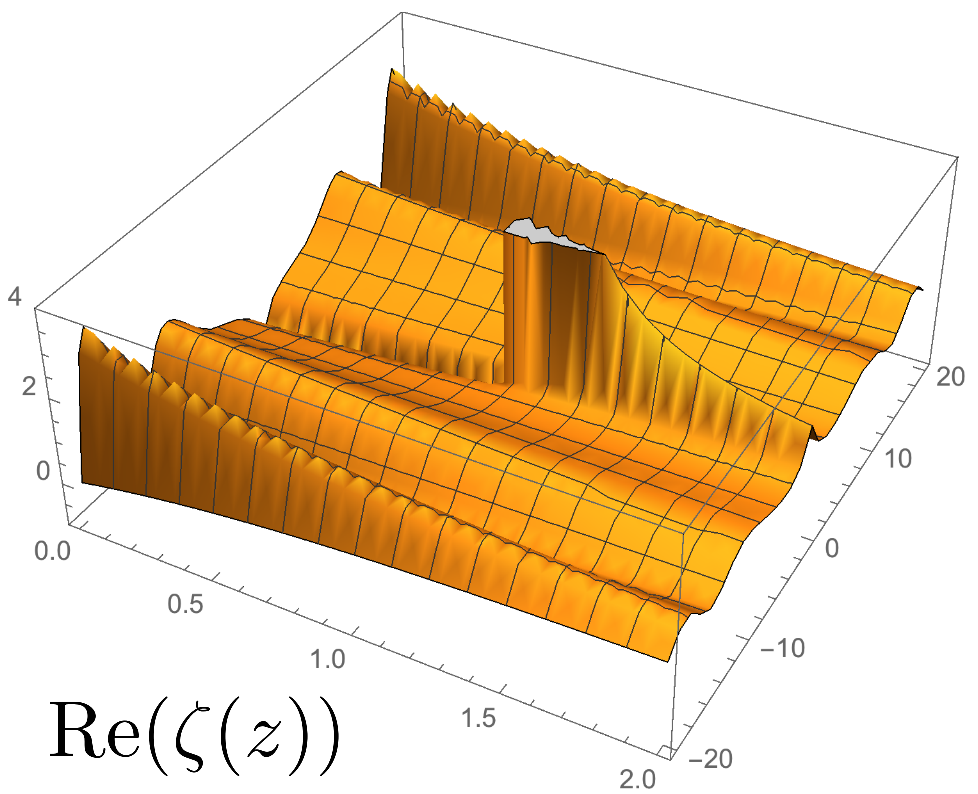

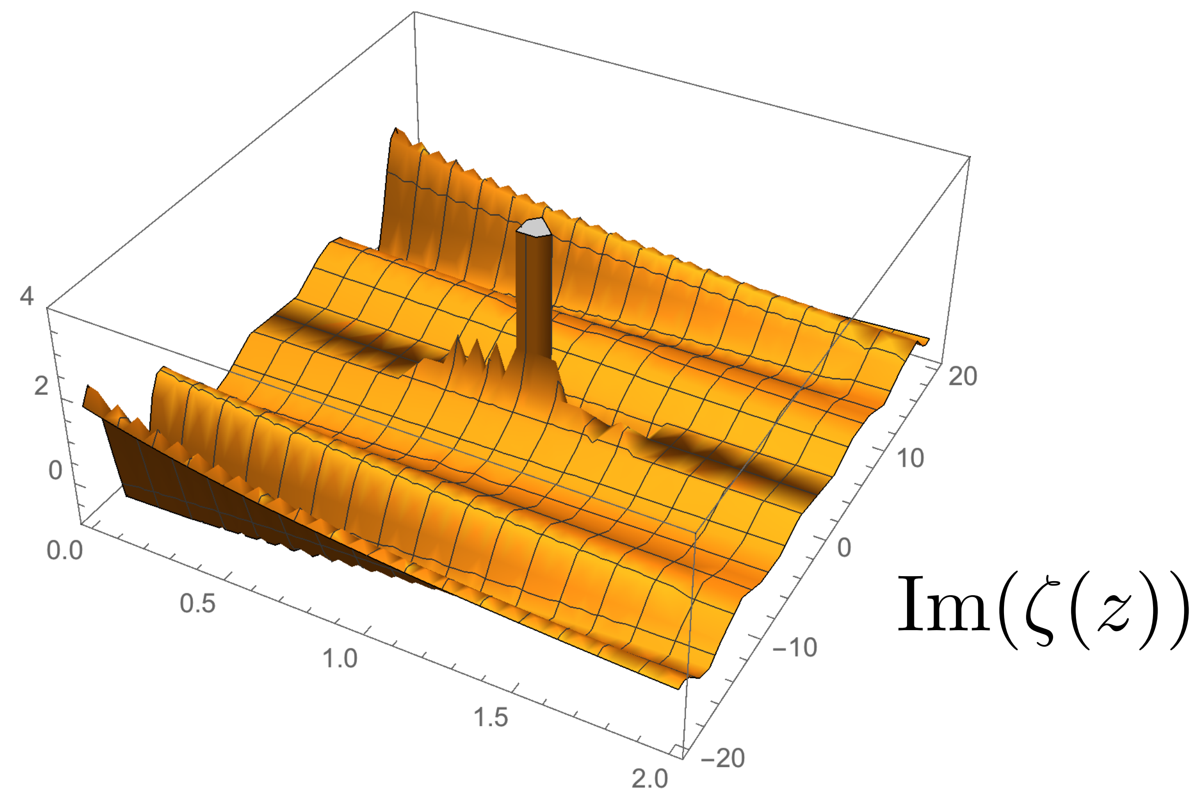

Functional equation $\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$

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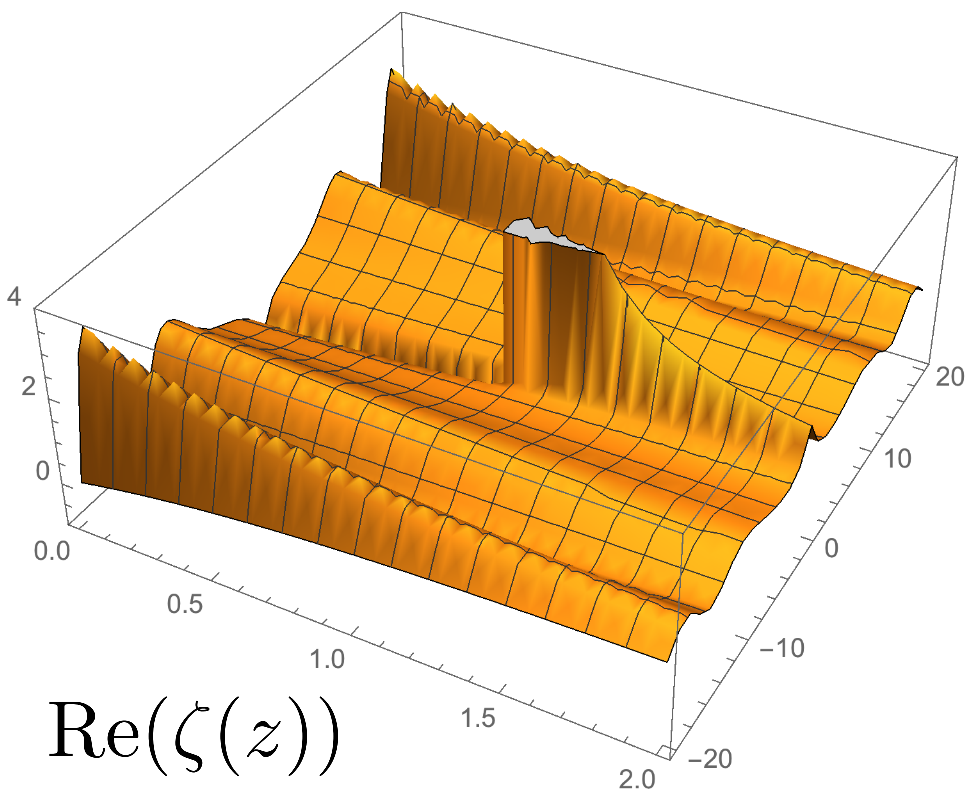

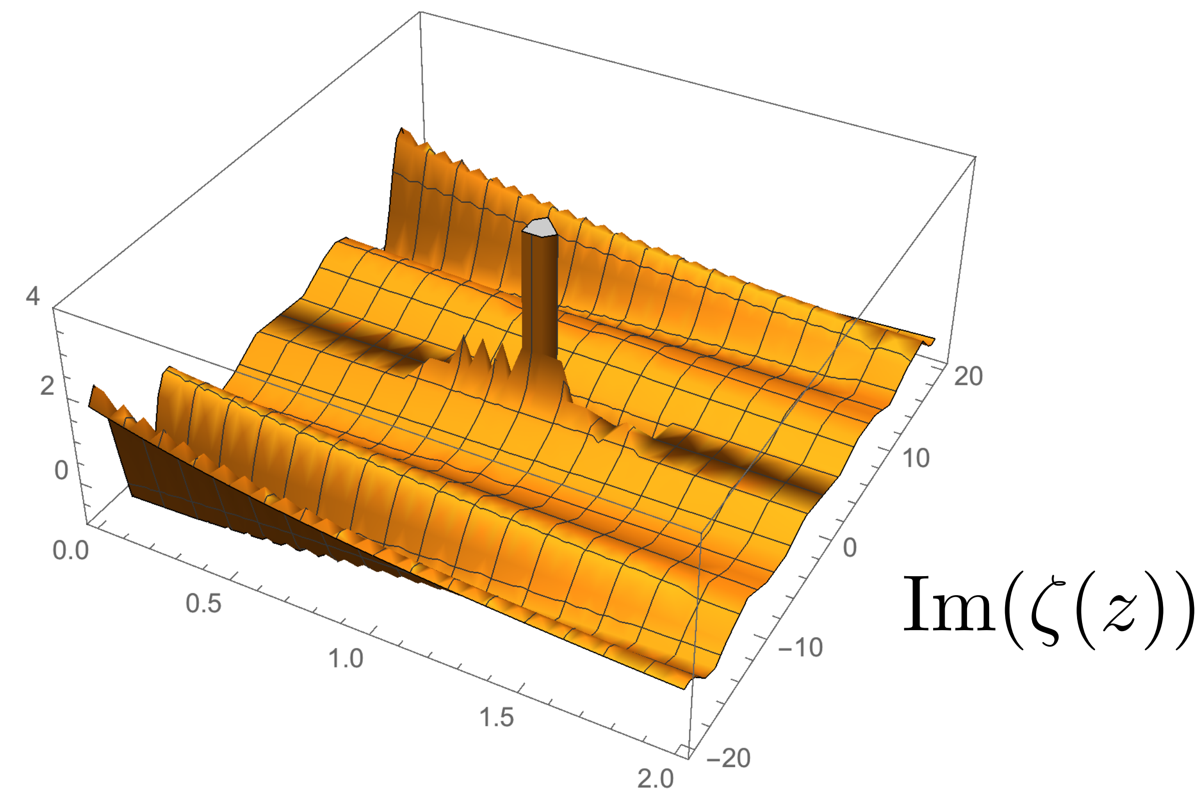

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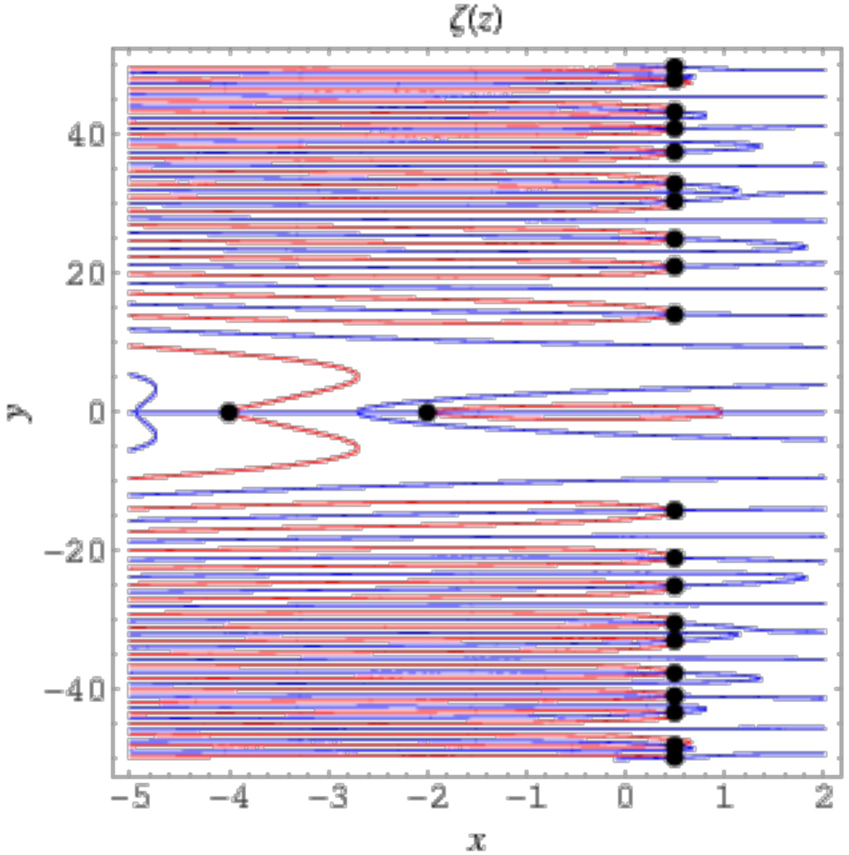
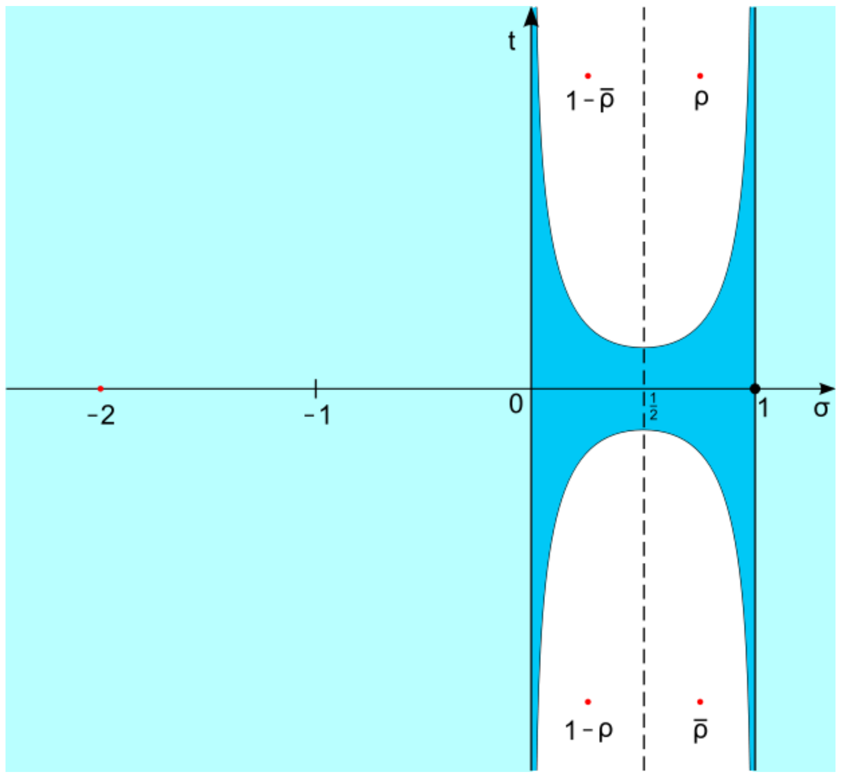


Source: mathworld.wolfram.com



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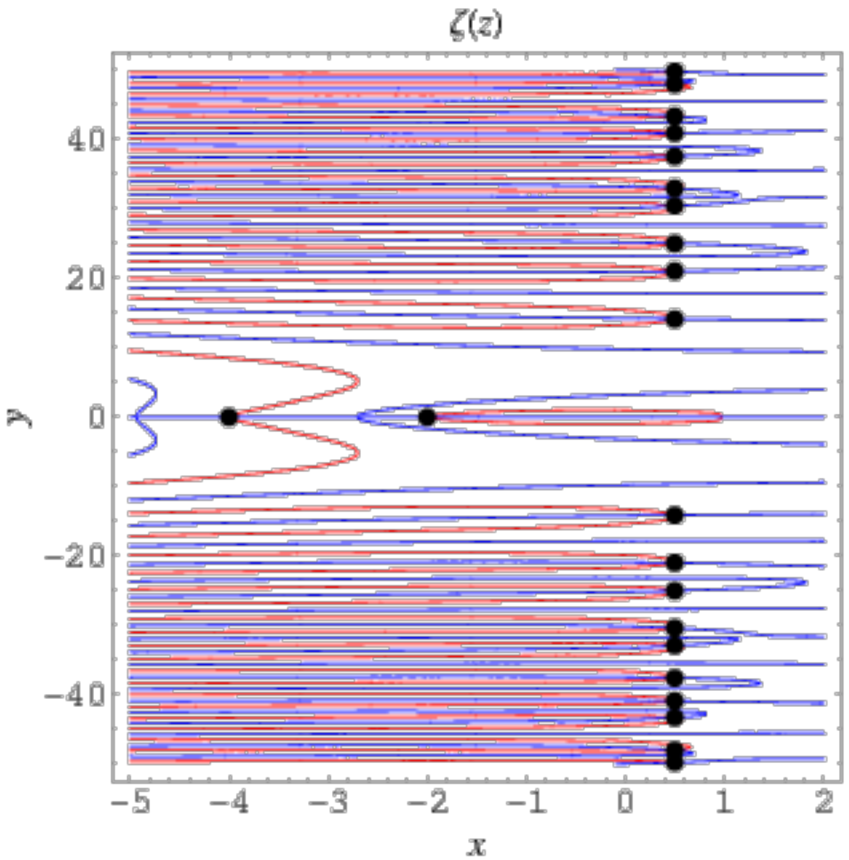

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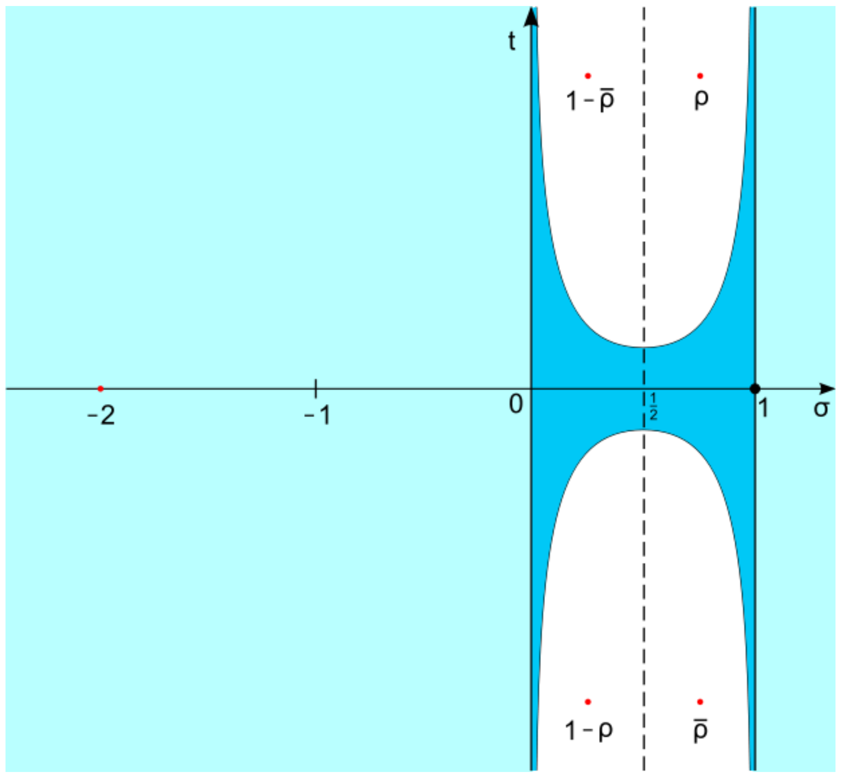

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If true, then many consequences for the distribution of prime numbers. For example,

$$|\pi(x) - \mathrm{Li}(x)| < \frac{1}{8\pi}\sqrt{x}\log x \qquad \text{for all } x \geq 2657,$$

where $\pi(x) = \#\text{ of primes } \leq x,$

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- Evidence

  Numerically, true for first 10^{13} zeros.

  Conrey 1989: at least 2/5 of all zeros lie on the critical line.

The Riemann zeros and operator theory?

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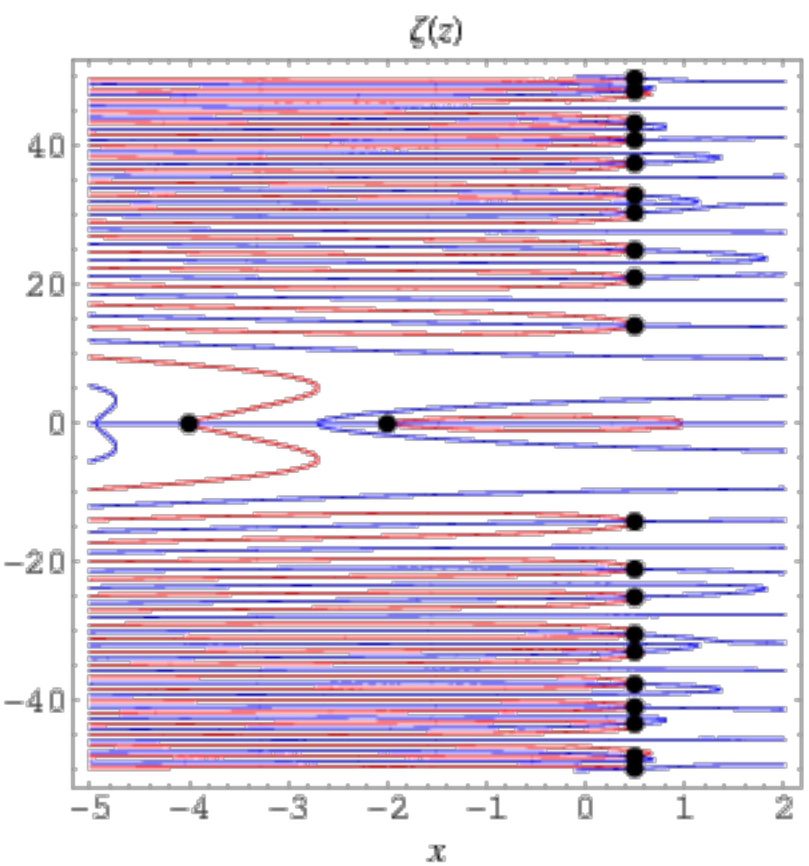

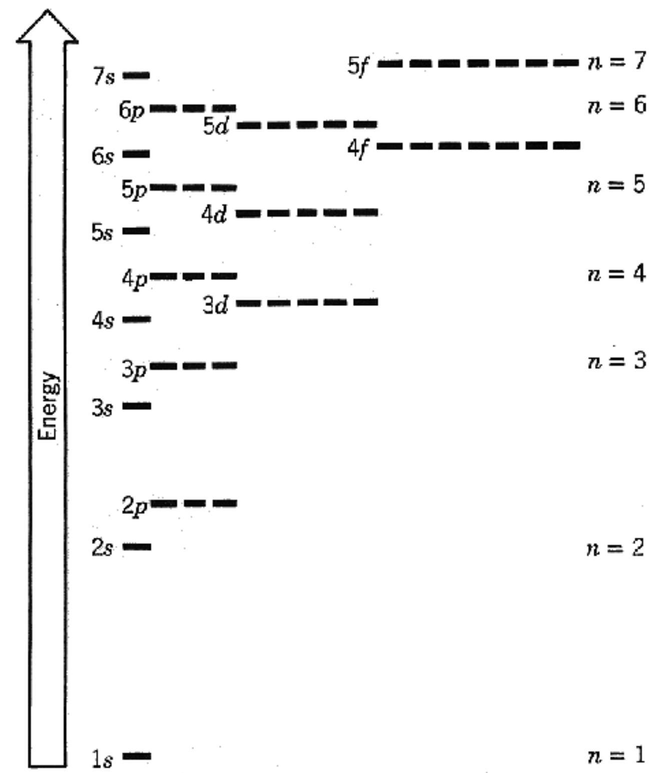

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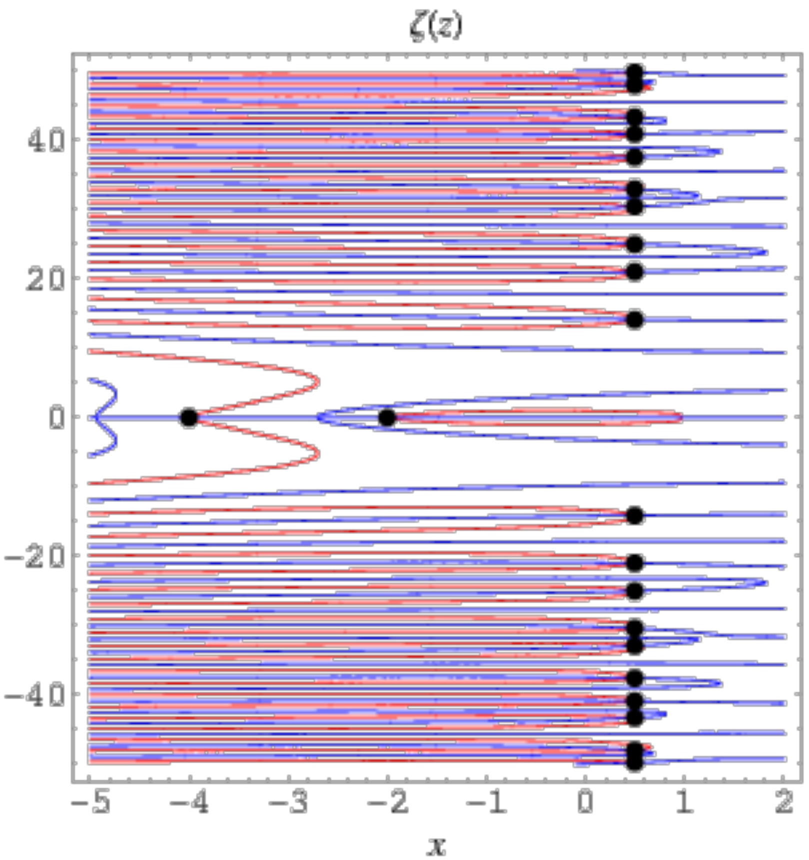

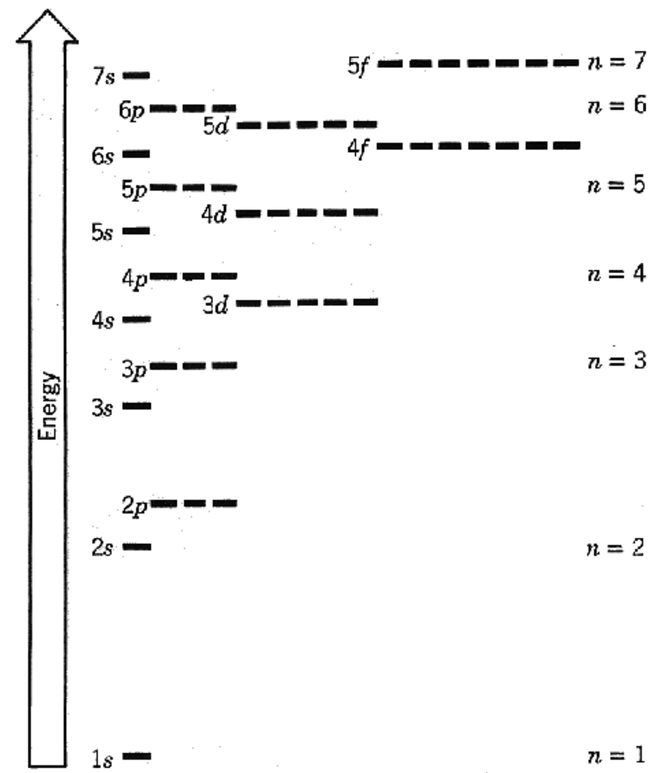

Hilbert-Pólya conjecture (early 20th century):

Are the Riemann zeros

$$s_n = \frac{1}{2} + i\,E_n$$

with E_n the eigenvalues of an unbounded self-adjoint operator?

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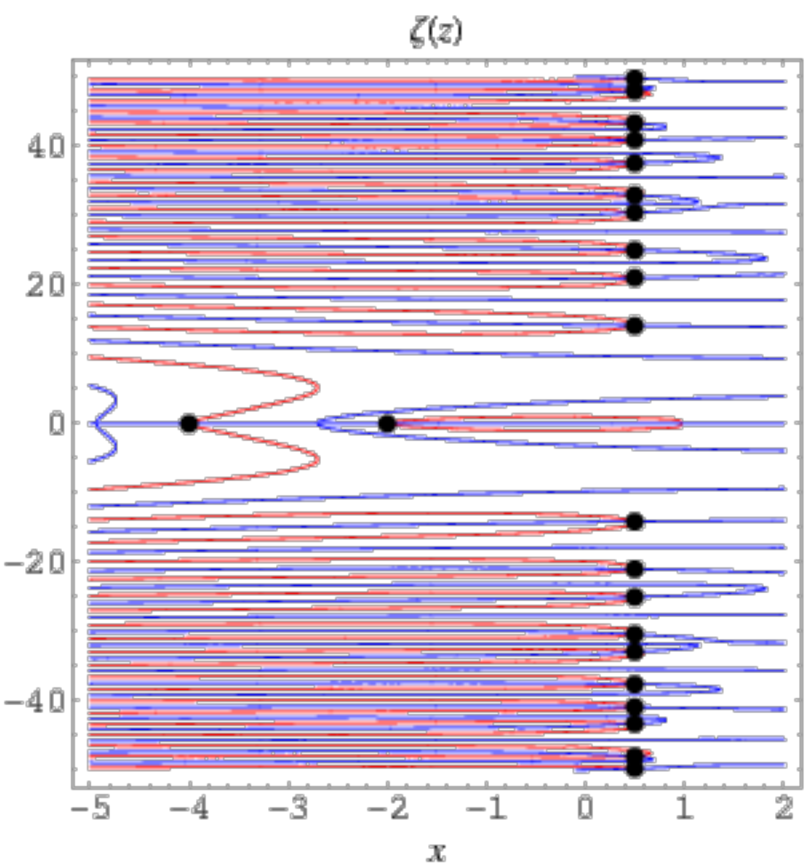

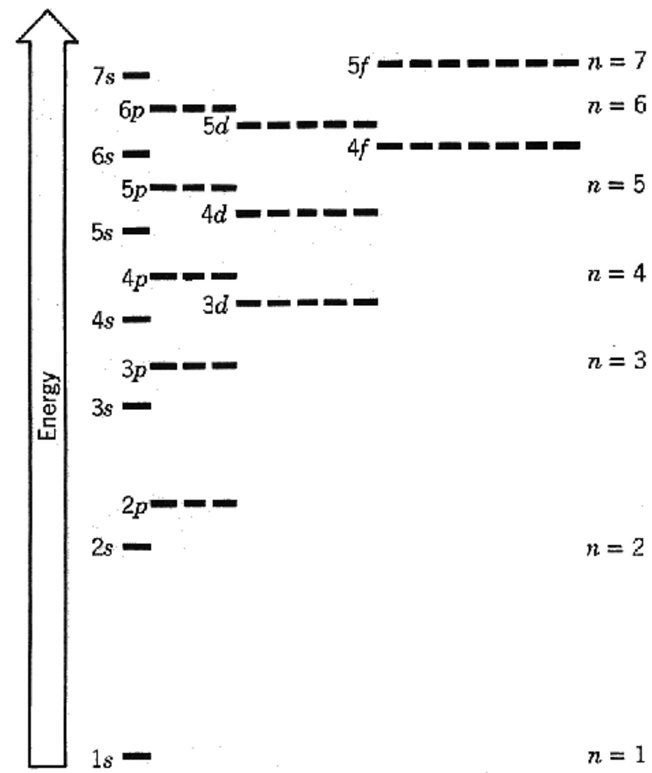

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Proof idea: Find an operator $H = H^\dagger$ that has eigenvalues $i(2s_n - 1)$ with s_n the non-trivial Riemann zeros.
Then the Riemann hypothesis follows.

The Riemann zeros and operator theory?

Montgomery ~1973: spacing statistics of the Riemann zeros corresponds to that of GUE random matrices

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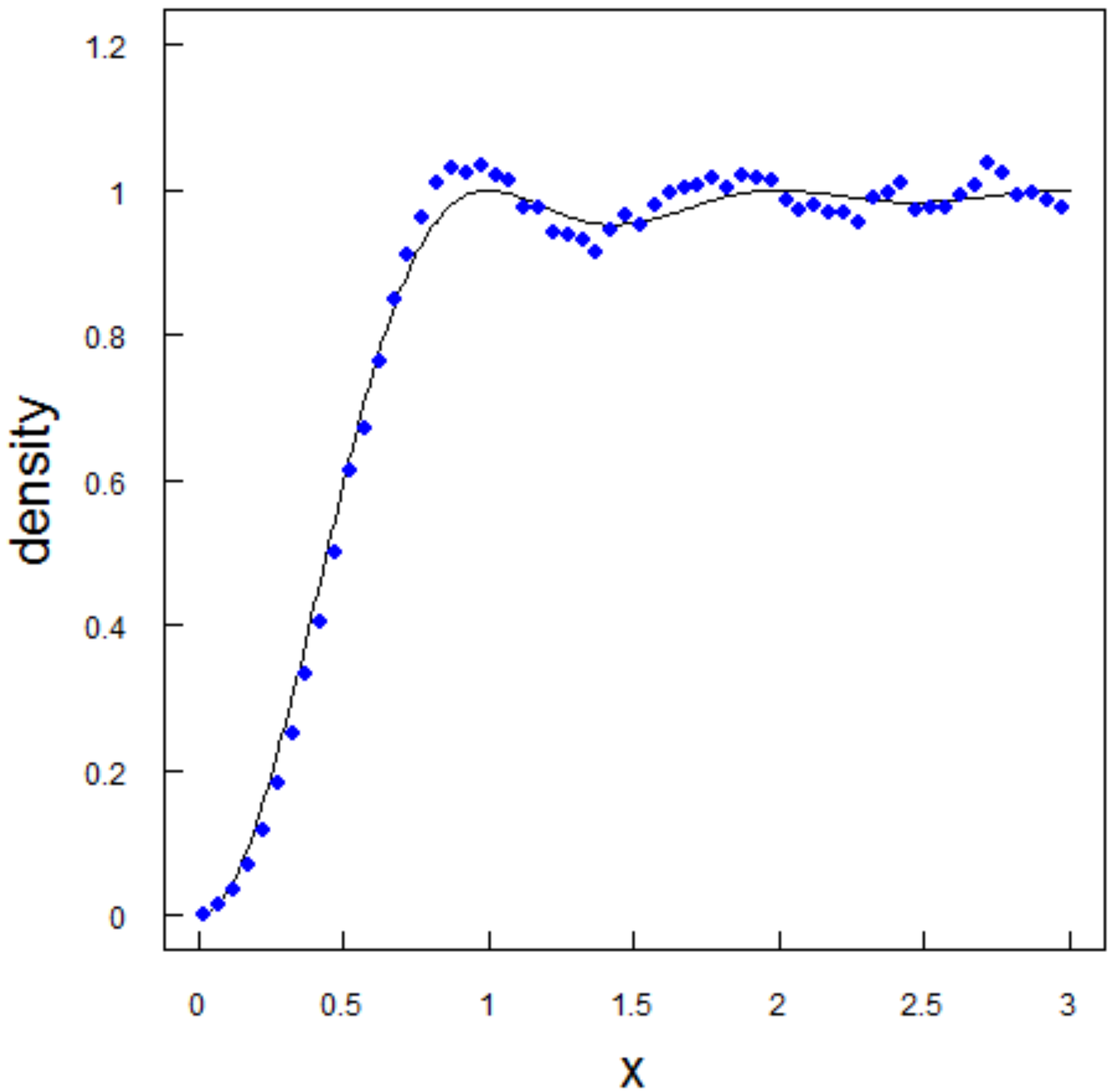

numerics by Odlyzko, 1987: normalized distribution of spacings.

blue=first 10^5 Riemann zeros,
black=eigenvalues of random GUE matrices.

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Simplest quantization: $H = \hat{x}\hat{p} + \hat{p}\hat{x}$.

$$\hat{x} f(x) = x \cdot f(x), \qquad \hat{p} f(x) = -i\partial_x f(x).$$

position operator   momentum operator

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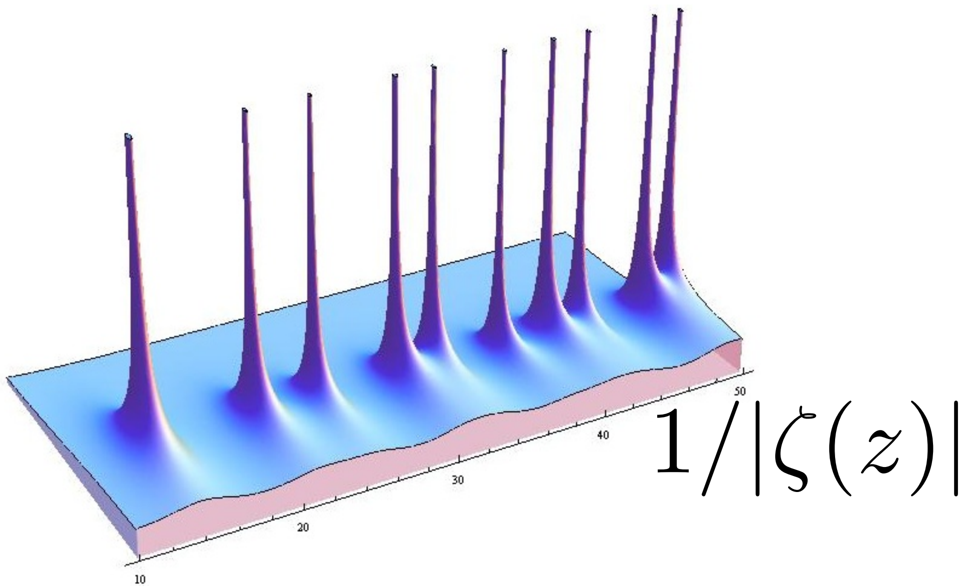

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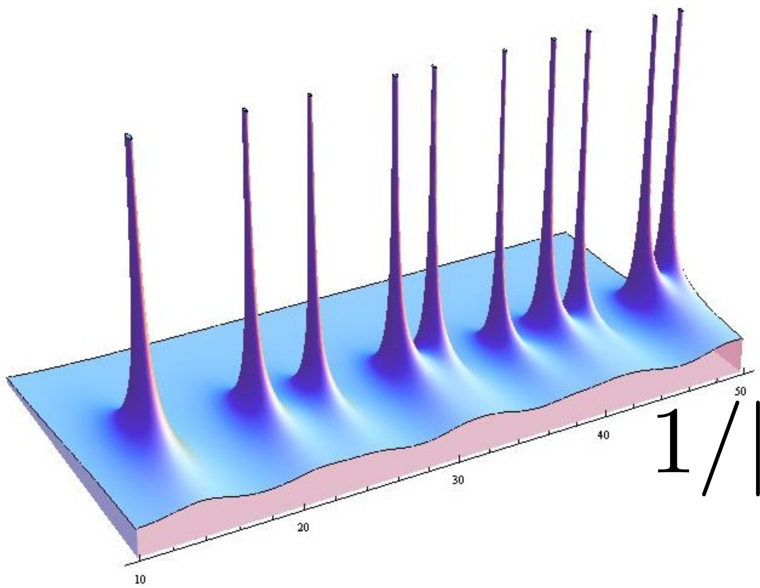

$1/|\zeta(z)|$

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How to add a non-integer number of terms **("Fractional Sums")**

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My home town as a teenager...

Morsbrunn
150 cows, 90 people, 3 dunghills,
divided by a big moat
-> boring
-> fractional sums

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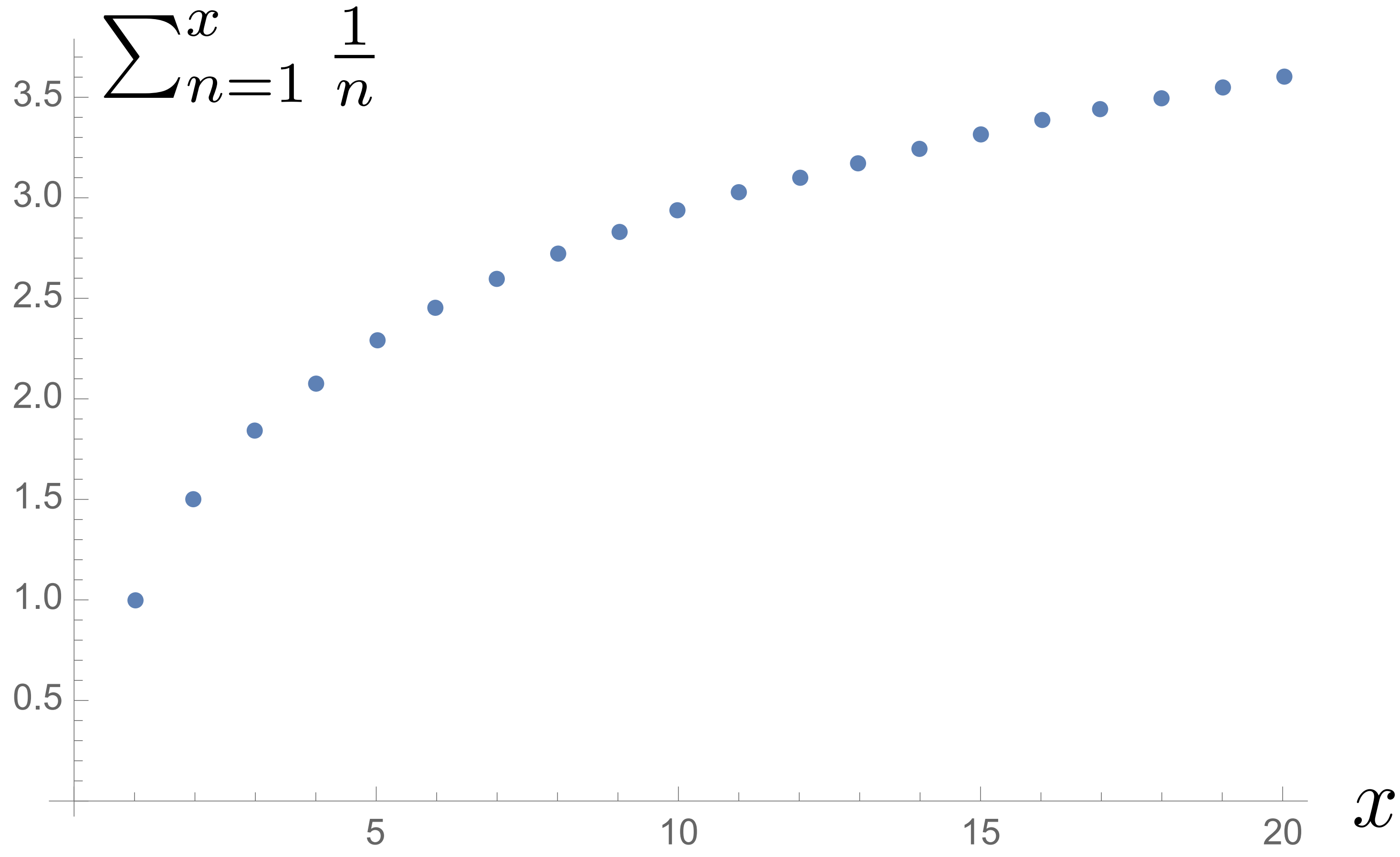

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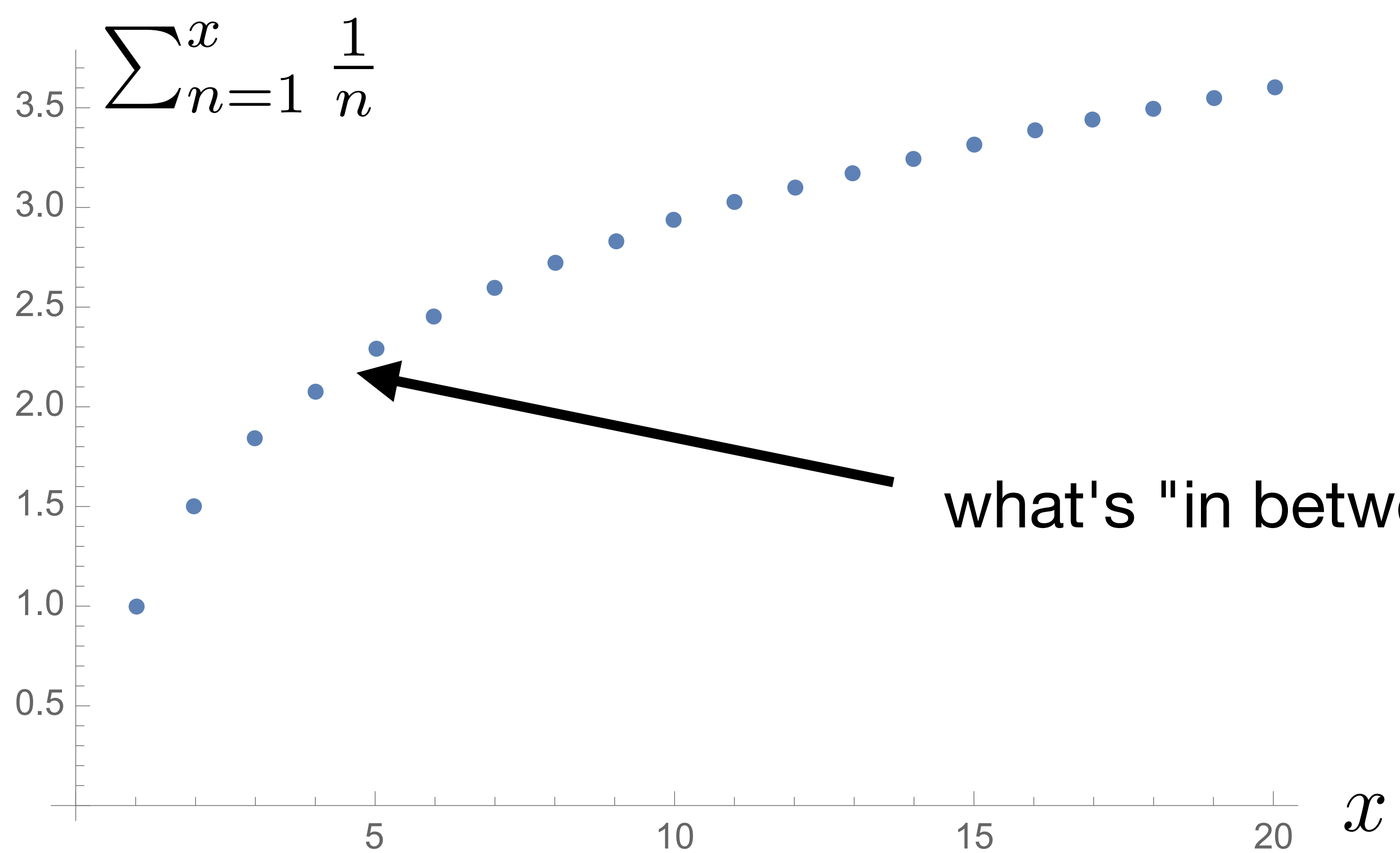

what's "in between"!?

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Idea: Whatever $\displaystyle\sum_{n=1}^{-1/2}\frac{1}{n}$ is, it should still respect $\displaystyle\sum_{a}^{b}+\sum_{b+1}^{c}=\sum_{a}^{c}$.

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$$\sum_{n=1}^{x} \frac{1}{n} + \sum_{n=x+1}^{x+N} \frac{1}{n} = \sum_{n=1}^{N} \frac{1}{n} + \sum_{n=N+1}^{N+x} \frac{1}{n}.$$

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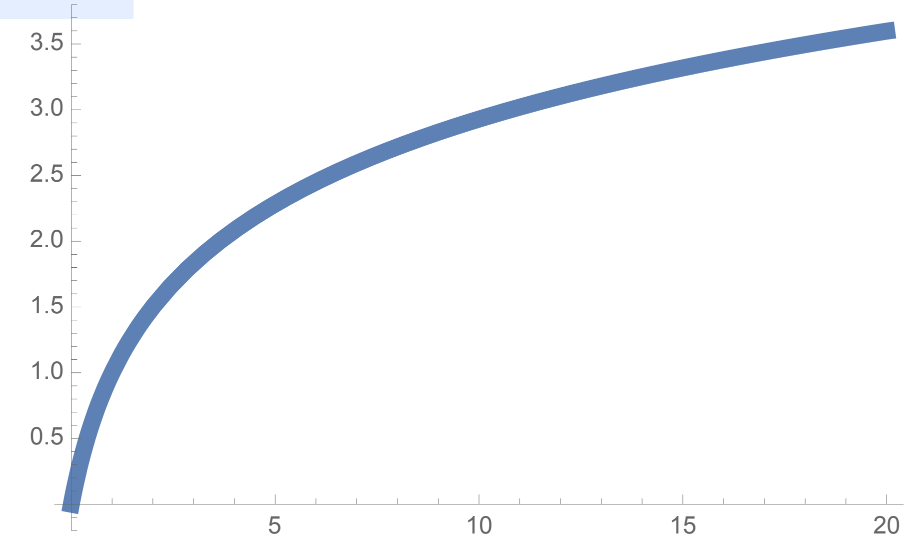

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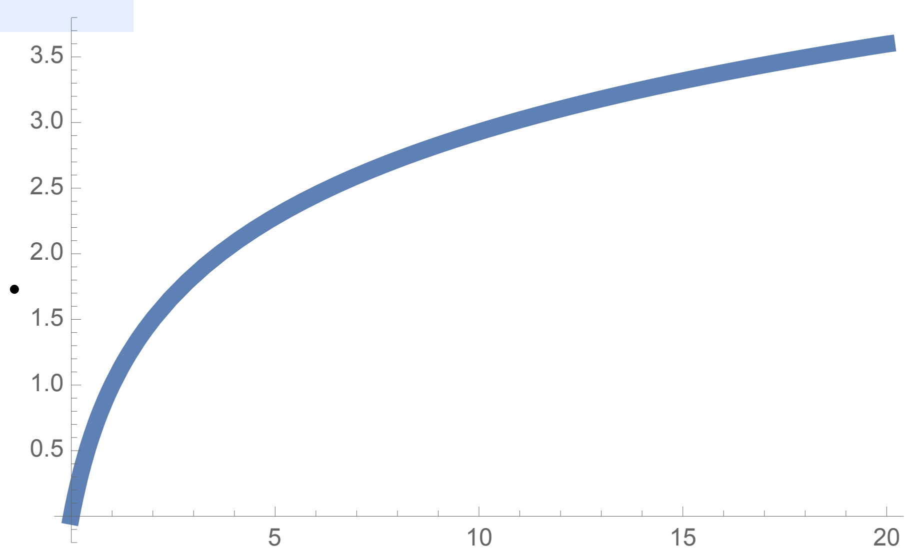
$$\sum_{n=1}^{-1/2} \frac{1}{n} = -2 \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \right) = -2 \log 2.$$



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$$\Sigma : \left(-\frac{1}{2}\right) = \Sigma : \frac{1}{2} - 2 = -2\, l\, 2,$$

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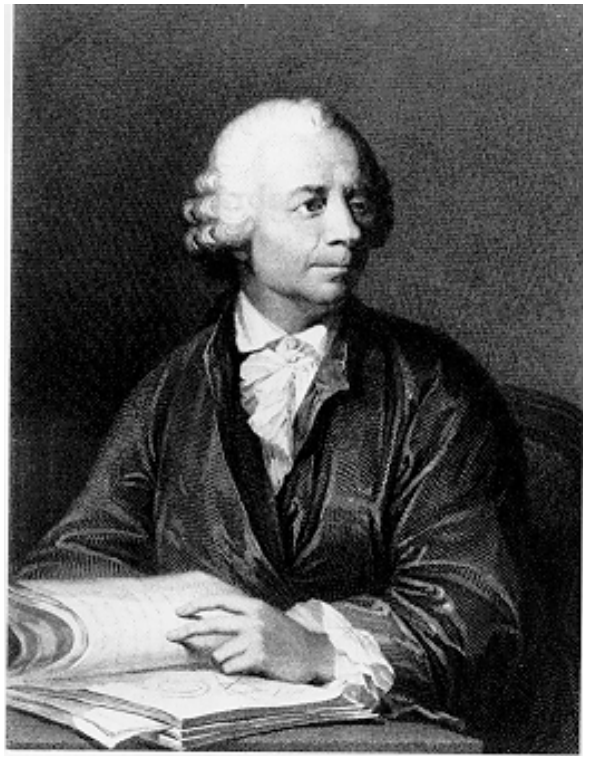
Leonhard Euler (1707-83)



General theory

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Start with the well-known identities for polynomials

$$\sum_{n=1}^{x} 1 = x, \quad \sum_{n=1}^{x} n = \frac{x(x+1)}{2}, \quad \sum_{n=1}^{x} n^2 = \frac{x(x+1)(2x+1)}{6}, \ldots$$

and postulate that they hold for all $x \in \mathbb{C}$.

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E.g. $\sqrt{x}$ and $\log x$ are asymptotically constant.

Some consequences

$x \in \mathbb{C}$

$$\prod_{n=1}^{x} n = x! = \Gamma(x+1).$$

$$\sum_{n=x}^{-x} \frac{1}{n} = \pi \cot(\pi x)$$

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MM and **Dierk Schleicher**, American Math. Monthly **118** (2011).

Summation operator

On the space of asymptotically polynomial functions, we get an operator Σ with

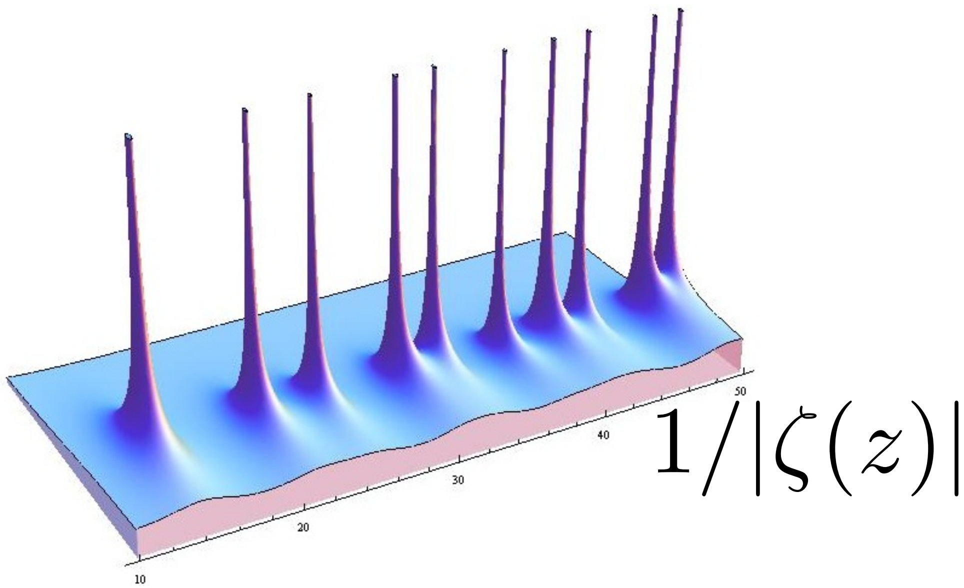
$$(\Sigma f)(x) := \sum_{n=1}^{x} f(n)$$

The difference operator $(\Delta f)(x) := f(x) - f(x-1)$ is an inverse:

$$\Delta\Sigma = \mathbf{1}, \qquad \Sigma\Delta f(x) = f(x) - f(0)$$

Outline

1. The Riemann hypothesis



2. How to add a non-integer number of terms

$$\sum_{n=1}^{-\frac{1}{2}} \frac{1}{n} = -2\log 2$$

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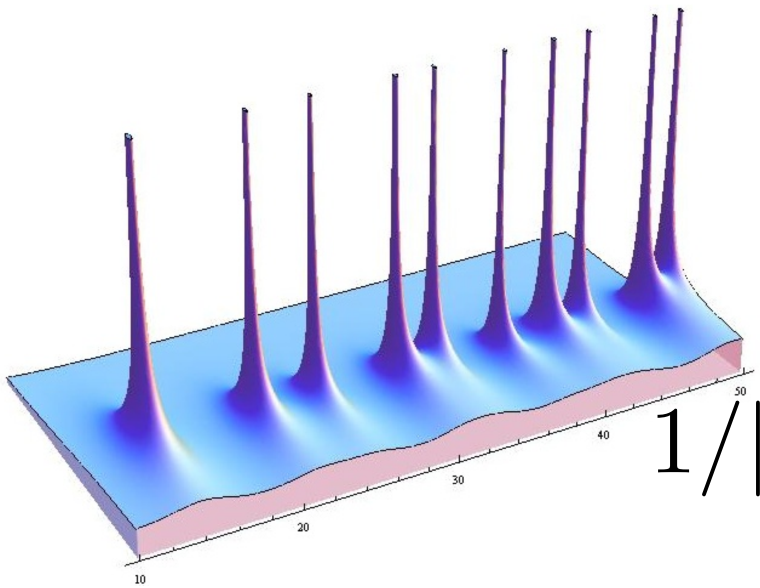

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⟶ restrict to subspace of sublinear functions. Then:

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Don't know if there's an inner product such that $R = R^\dagger$.

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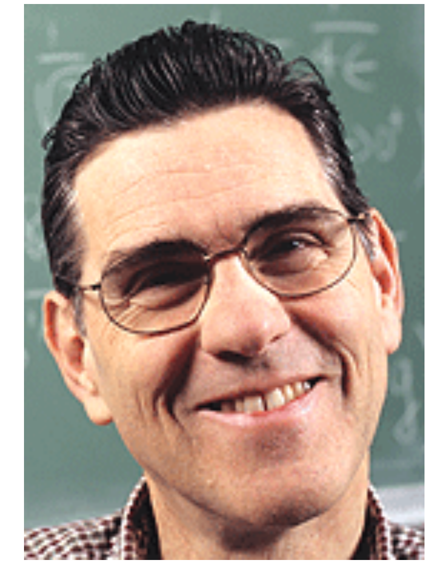

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