Concentration of measure and the mean energy ensemble

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see also **arXiv:1003.4982**

Outline of the talk

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1. Motivation from statistical mechanics
   - Problem: single instances vs. ensembles?
   - Concentration of measure

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2. Typicality in mean energy ensemble
   - Main result: Concentration of measure
   - Typical reduced density matrix
   - No concentration in Ising model

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   - Problem: single instances vs. ensembles?
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2. Typicality in mean energy ensemble
   - Main result: Concentration of measure
   - Typical reduced density matrix
   - No concentration in Ising model
3. Conclusions

1. Motivation from statistical mechanics
Foundational questions on statistical physics

Two kinds of missing information:

- Observer's lack of knowledge: knows only volume, temperature, ...
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"Postulate of equal apriori probabilities":


=

=...

Why does it work?

1. Motivation from statistical mechanics
What about ergodicity?

Idea: Time evolution explores all accessible phase space uniformly.

Problems:

- Proven only for some special systems.
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Is there another justification?

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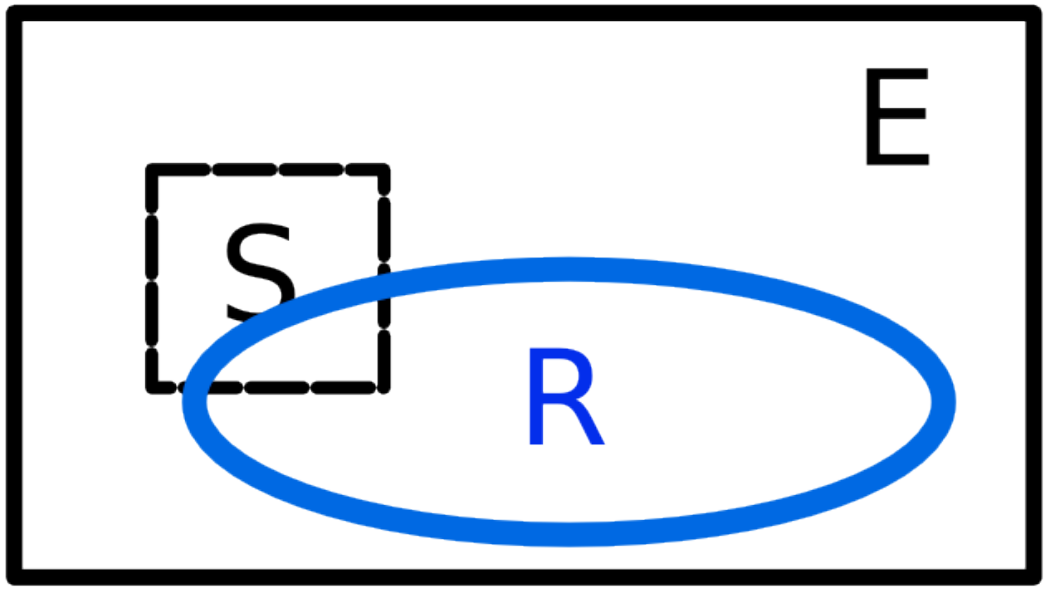
S. Popescu, A. J. Short, A. Winter, Nature Physics 2(11), 2006

$$\mathcal{H}_R \subset \mathcal{H}_S \otimes \mathcal{H}_E$$



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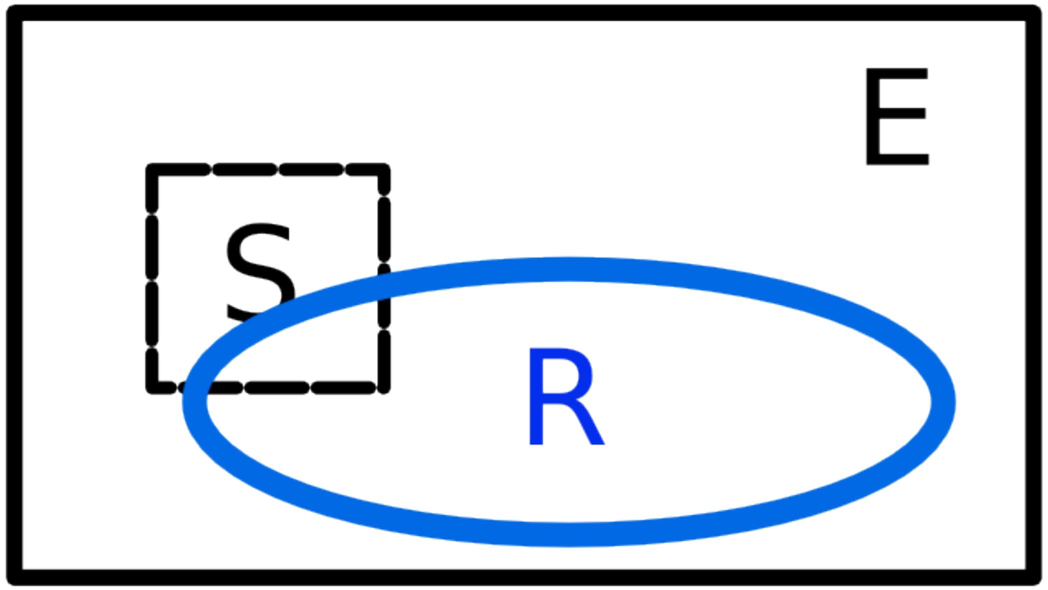

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$\mathcal{H}_S \otimes \mathcal{H}_E$: the "universe".

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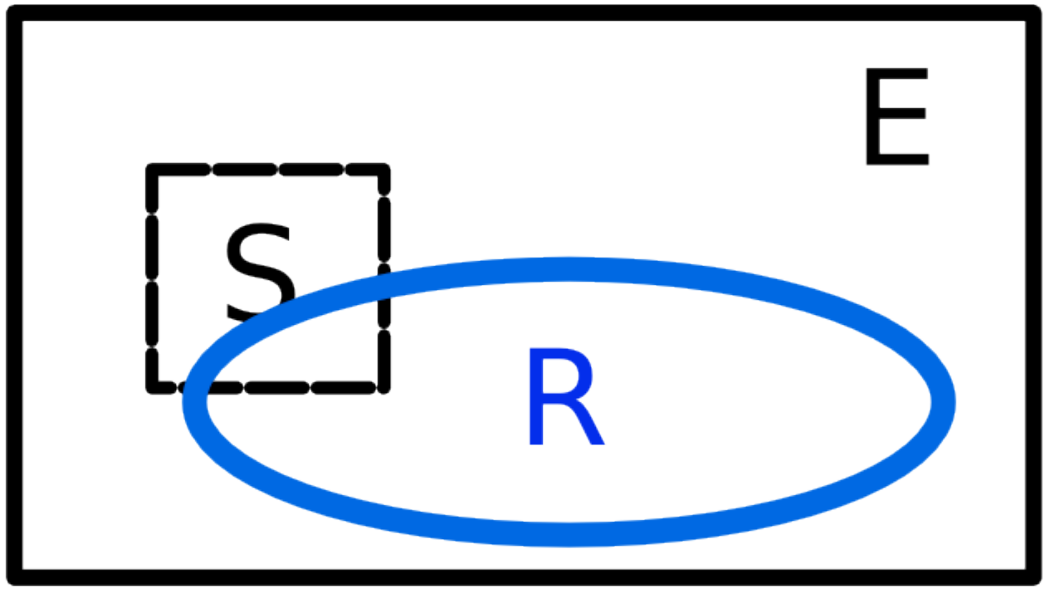

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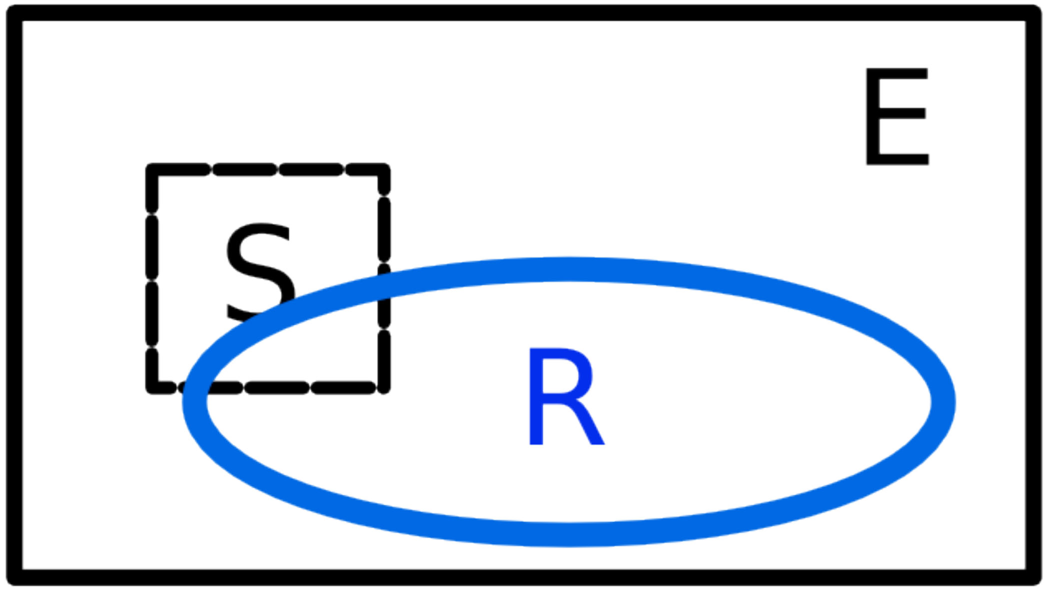

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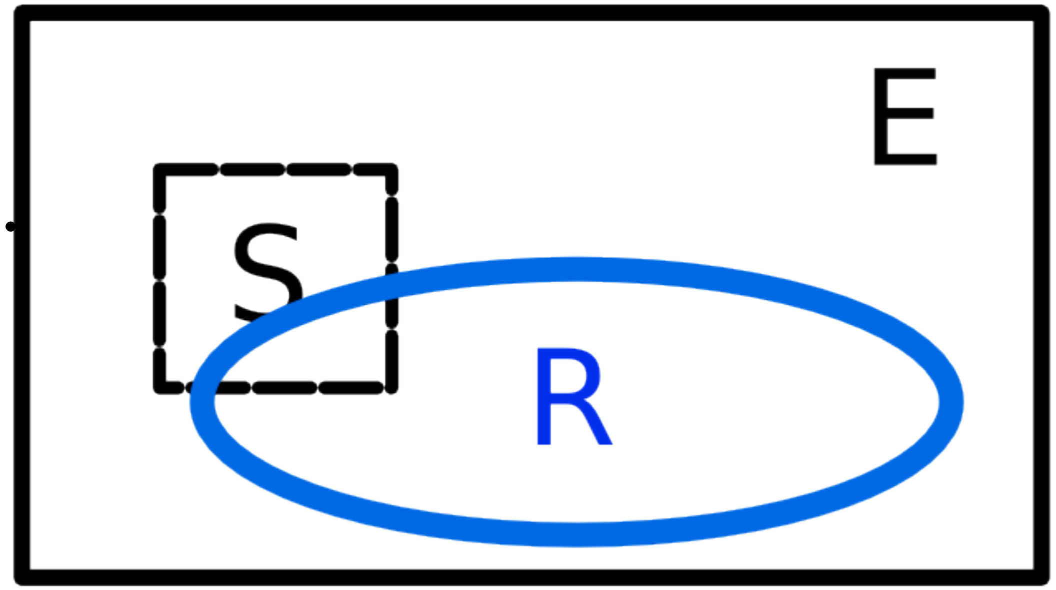

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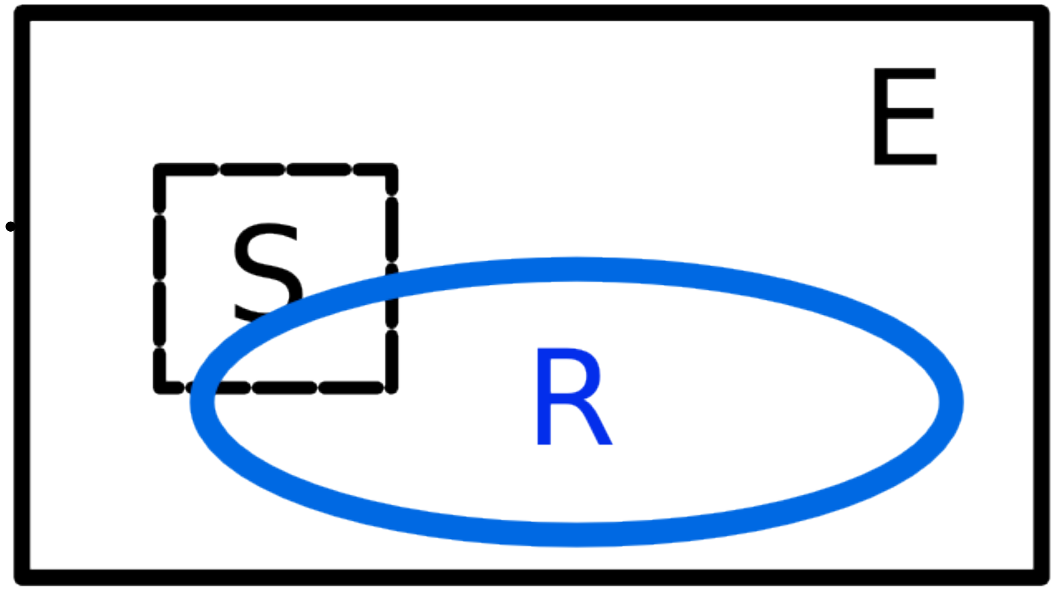

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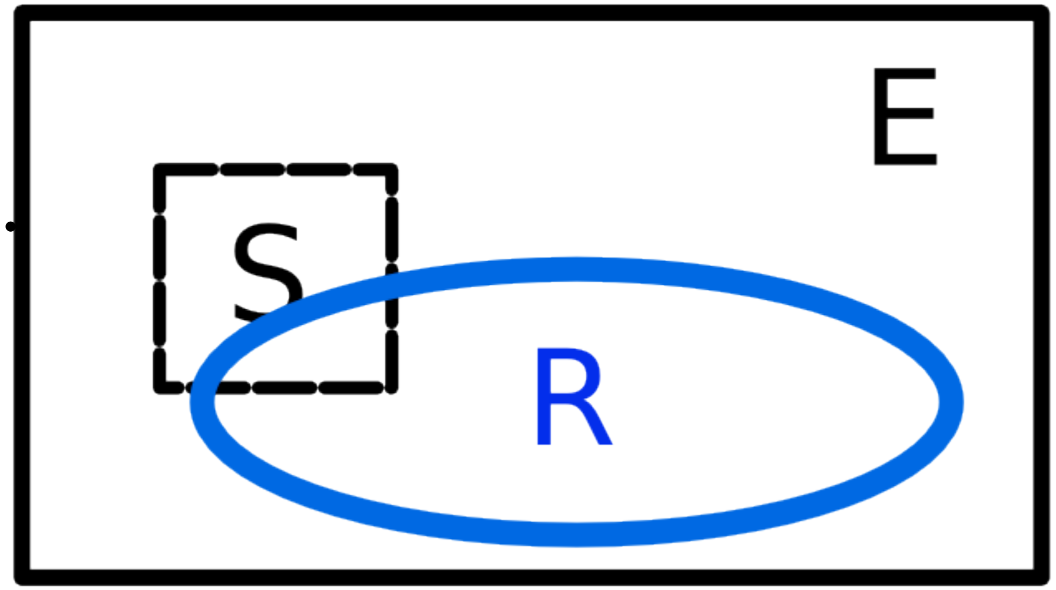

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Theorem (Concentration of measure): Draw $|\psi\rangle \in \mathcal{H}_R$ randomly acc. to unitarily invariant measure. Then,

$$\mathrm{Prob}\left[\|\rho_S - \Omega_S\|_1 \geq \varepsilon + \frac{d_S}{\sqrt{d_R}}\right] \leq 2\exp\left(-C d_R \varepsilon^2\right),$$

where $C = 1/18\pi^3$, $d_R = \dim \mathcal{H}_R$, $d_S = \dim \mathcal{H}_S$, $\Omega_S = \mathrm{Tr}_E\left(\mathbf{1}_S/d_S\right)$.

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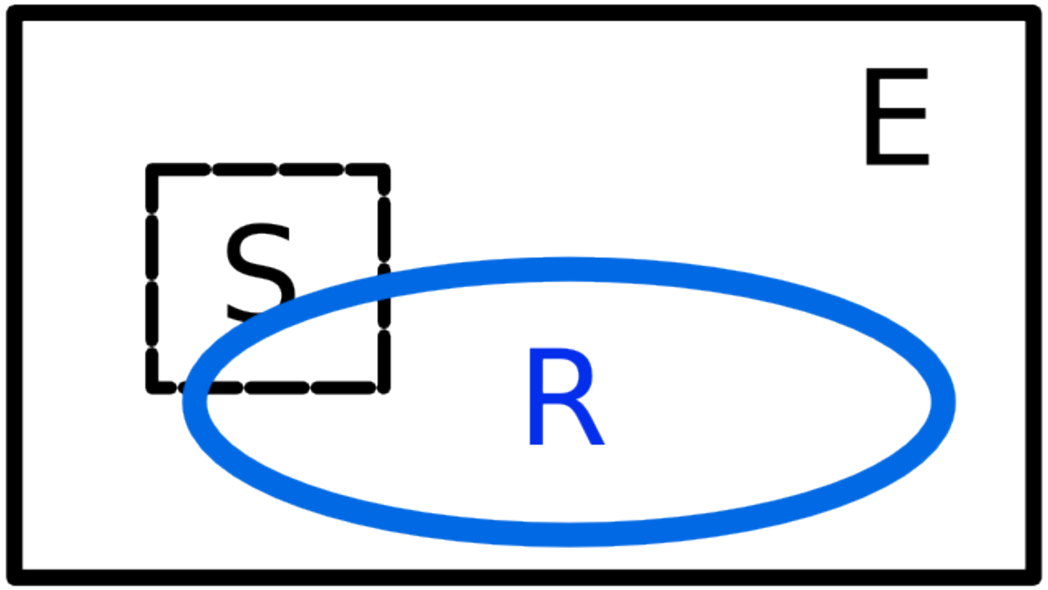

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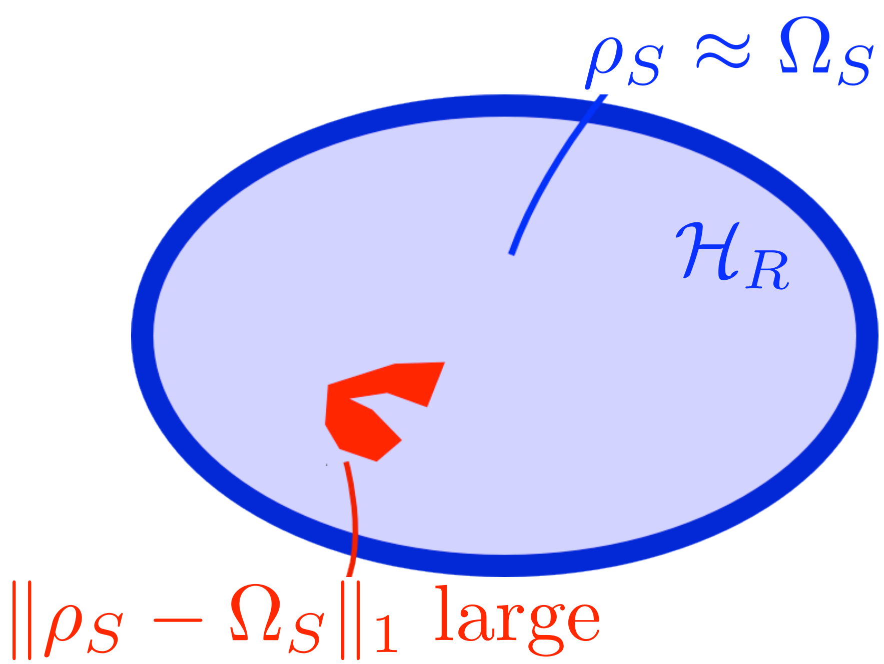

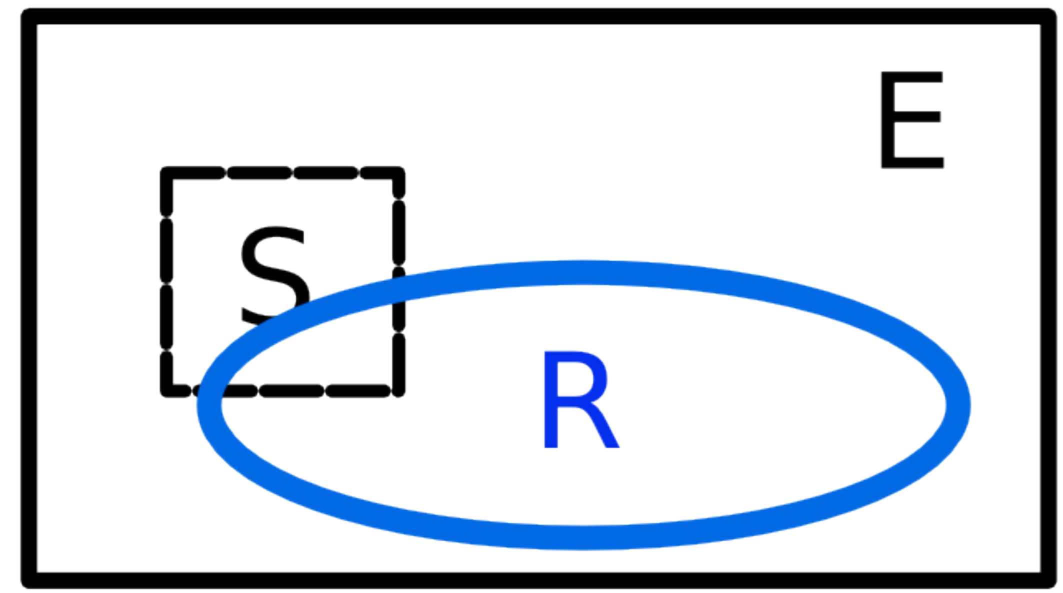

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1. Motivation from statistical mechanics

The perfect coffee machine

$|\psi\rangle$

$n = 1$

$|\psi\rangle$

$n = 2$

$|\psi\rangle$

$n = 3$

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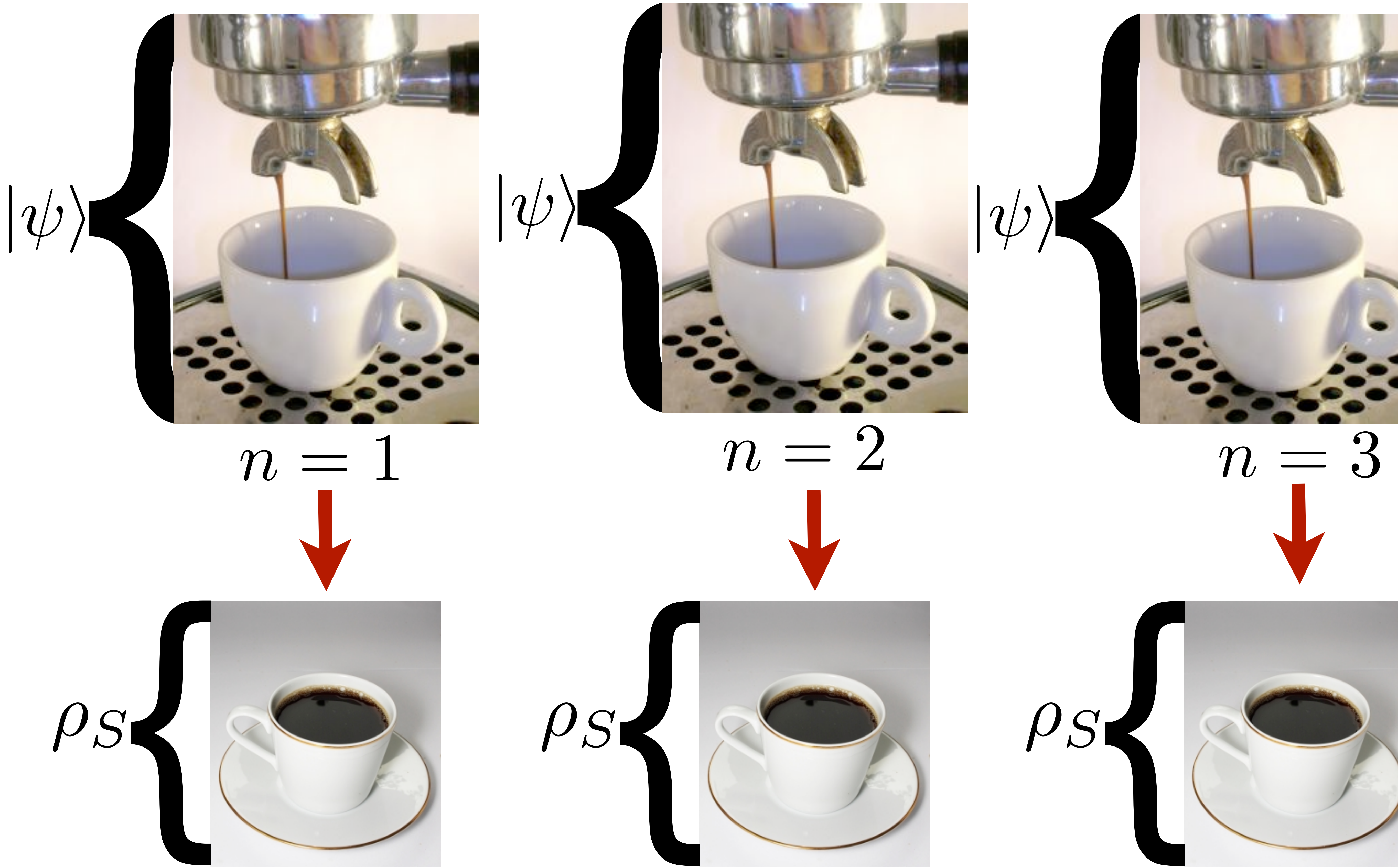

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ρ_S 

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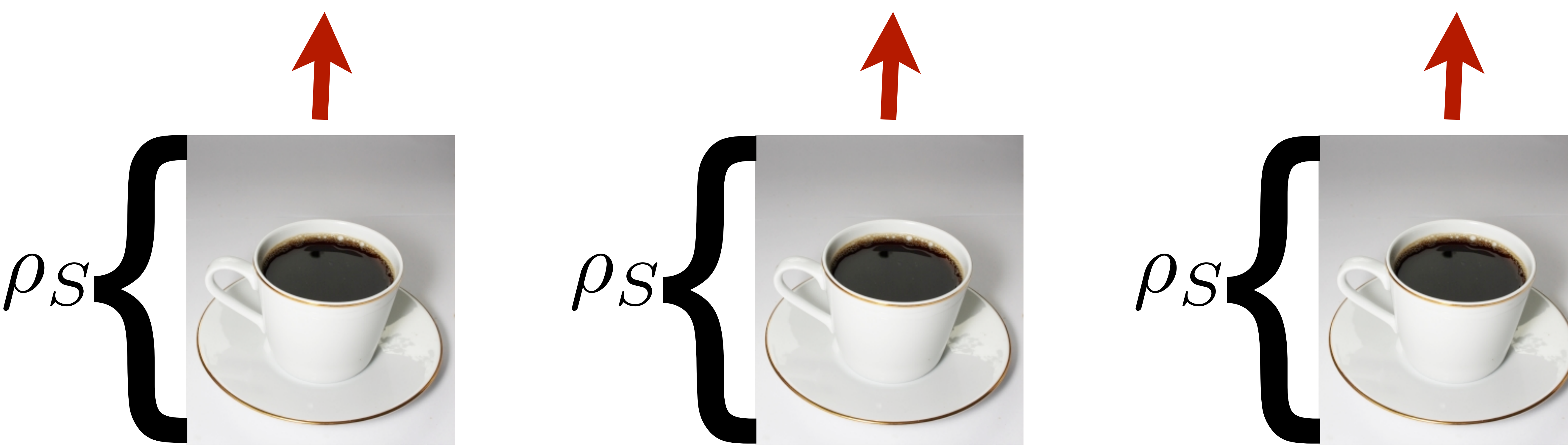
measurements ("coffee tomography")



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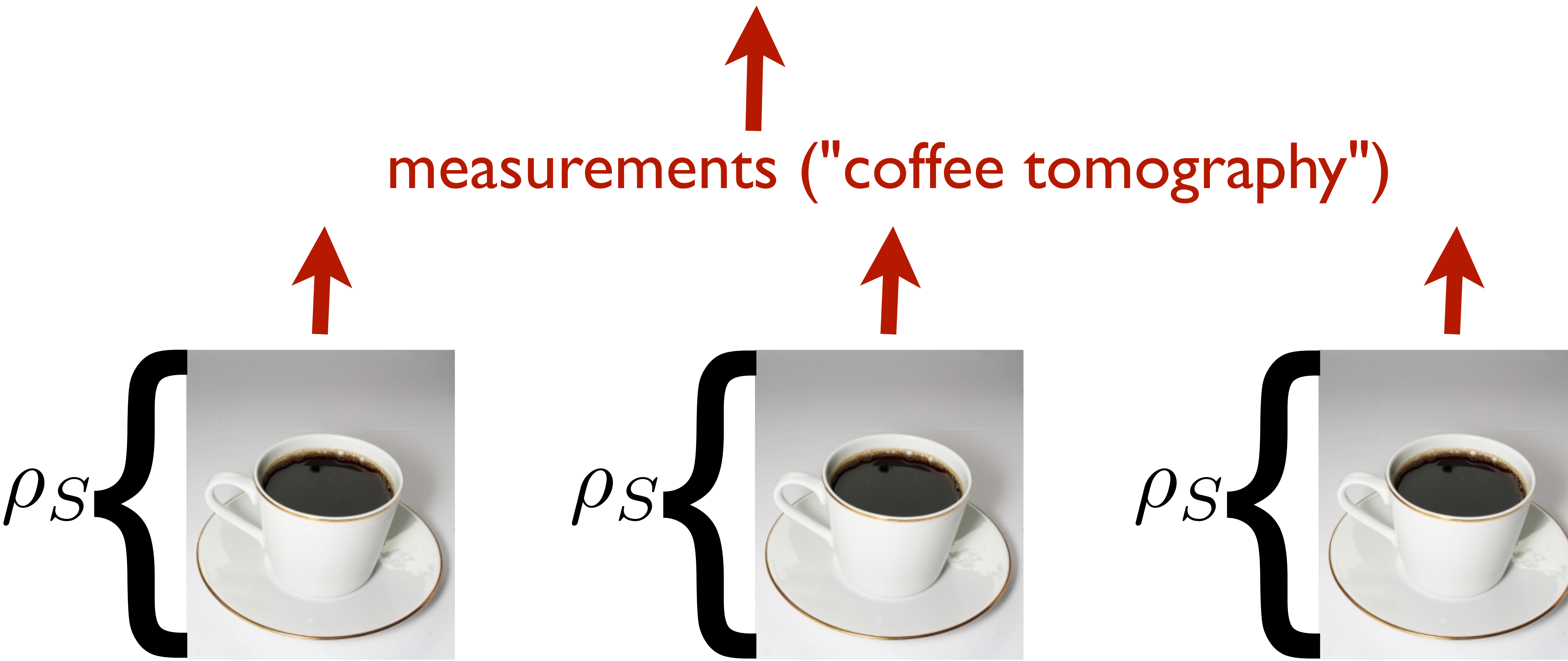
The perfect coffee machine

Reveals ρ_S. But $\rho_S \approx \Omega_S$ (microcanonical ensemble) for "almost all" $|\psi\rangle \in \mathcal{H}_R$.
Hence, almost all coffee machines (compatible with restrictions) prepare the microcanonical ensemble.



1. Motivation from statistical mechanics

Form of the reduced density matrix

- Exact form of Ω_S is not given by Popescu et al. (generality!).

- Goldstein, Lebowitz, Tumulka, Zanghi, PRL **96** (2006):
no interaction $H = H_S + H_{env}$, fixed energy E,
subspace $\mathcal{H}_R$ spanned by spectral window $[E - \Delta, E + \Delta]$,
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What if the constraint is *not* given by a *subspace*?

2. Typicality in mean energy ensemble

Going beyond subspaces

- Observers may have knowledge on systems that is different from "*being element of a subspace*".
- Example: given Hamiltonian H, the energy expectation value $\langle\psi|H|\psi\rangle = E$ might be known instead.

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$$M_E = \{|\psi\rangle \in \mathbb{C}^n \mid \langle\psi|H|\psi\rangle = E, \ \ \|\psi\| = 1\}$$

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Goal of our work:

- Prove typicality (=measure concentration) for *m*.e.e.,
- analyze its role in quantum statistical mechanics.

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Our result

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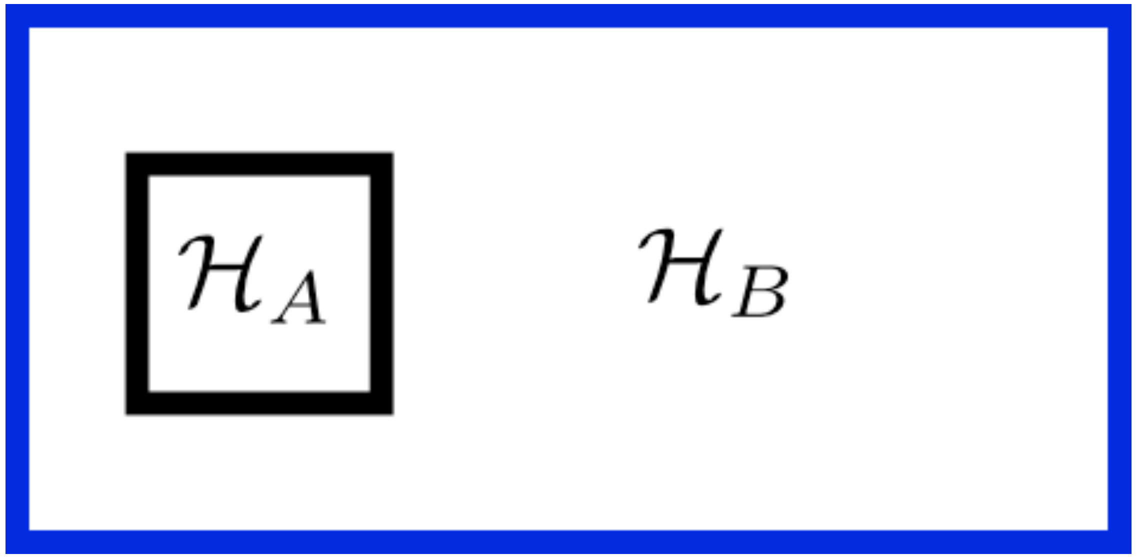
$$\dim \mathcal{H}_A = 3, \;\; \dim \mathcal{H}_B = n$$



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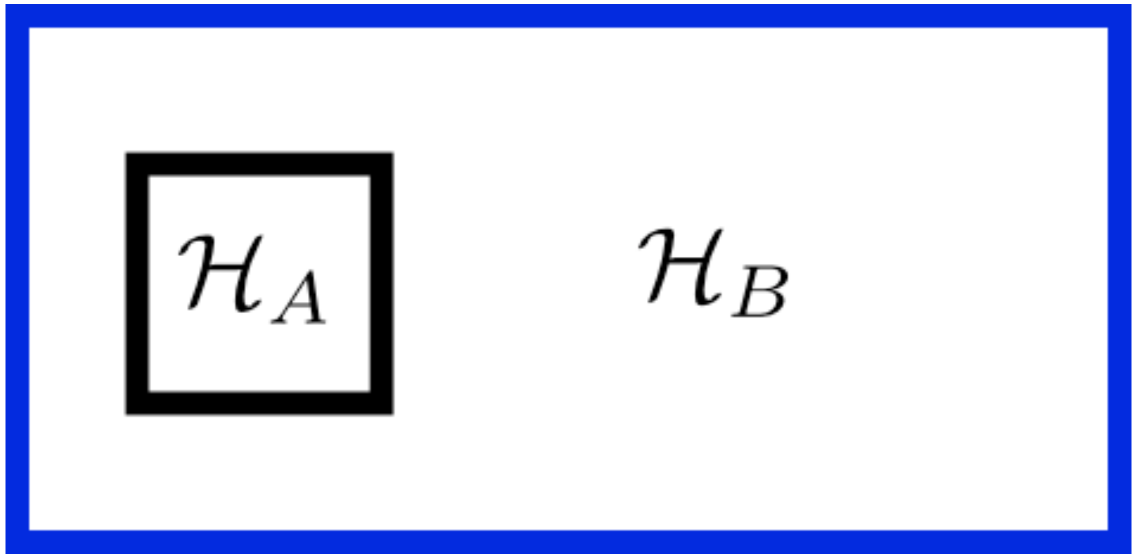

Hamiltonian $H = H_A + H_B$

$$H_A = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}, \ \ H_B = 0$$

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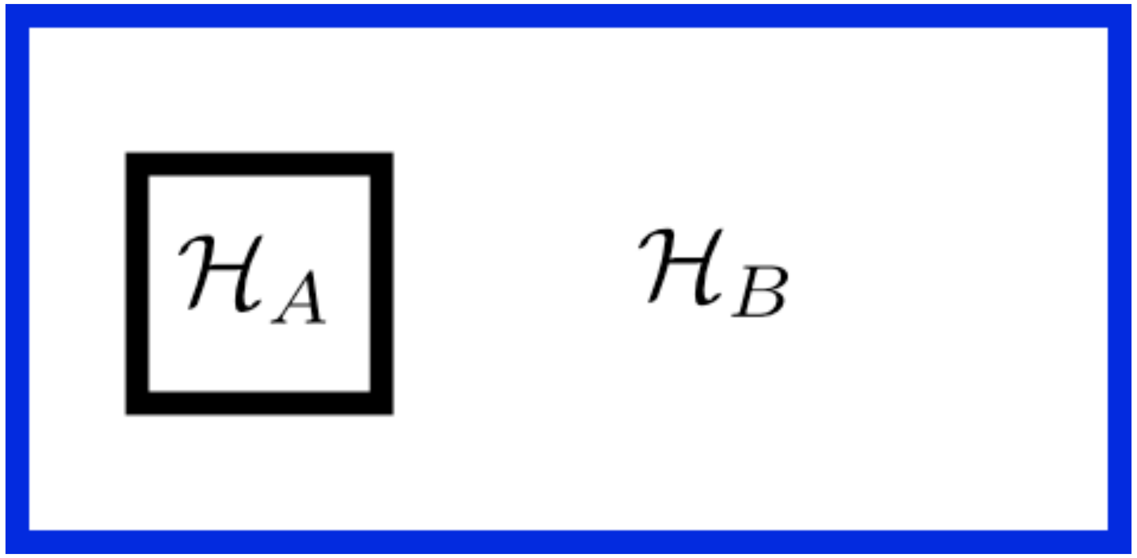

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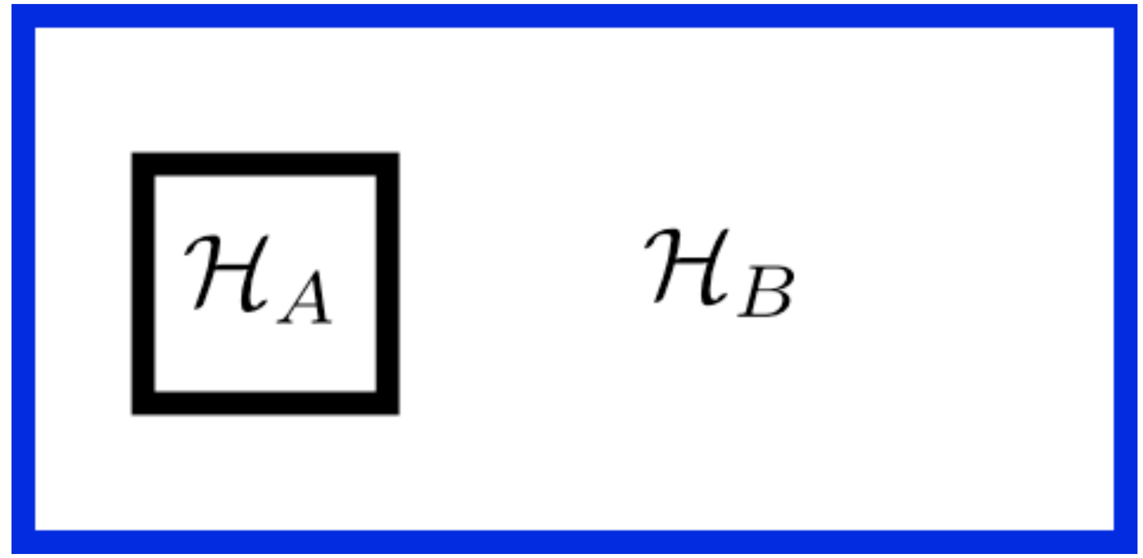

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Draw $|\psi\rangle \in \mathcal{H}$ randomly under $\|\psi\| = 1$ and $\langle\psi|H|\psi\rangle = 3/2$ and compute $\psi^A := \mathrm{Tr}_B|\psi\rangle\langle\psi|$. Then, with high probability,

$$\psi^A \approx \frac{1}{12}\begin{pmatrix} 5+\sqrt{7} & 0 & 0 \\ 0 & 2(4-\sqrt{7}) & 0 \\ 0 & 0 & -1+\sqrt{7} \end{pmatrix}$$

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More in detail,

$$\mathrm{Prob}\left\{\left\|\psi^A - \rho_c\right\|_2 > 3\sqrt{8}\left(\varepsilon + \frac{59}{\sqrt[4]{n}}\right)\right\} \leq 369960\, n^{\frac{3}{2}}\, e^{-\frac{3}{64}n\left(\varepsilon - \frac{1}{4n}\right)^2 + 4\sqrt{n}}.$$

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- Concentration of measure = typicality for energy ensemble
- Note that $[\psi^A, H_A] = 0$ but $\psi^A \neq \exp(-\beta H_A)$. Not Gibbs!

2. Typicality in mean energy ensemble

Our result

<u>General result</u> (arXiv:1003.4982): On a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with Hamiltonian $H = H_A + H_B$, draw a pure state $|\psi\rangle \in \mathcal{H}$ randomly under $\|\psi\| = 1$ and $\langle\psi|H|\psi\rangle = E$. Compute $\psi^A := \mathrm{Tr}_B |\psi\rangle\langle\psi|$. Then, with high prob. (made precise)

$$\psi^A \approx \rho_c \quad \text{where} \quad \rho_c = \frac{1}{\dim \mathcal{H}} \sum_{k=1}^{\dim \mathcal{H}_B} \frac{E+s}{H_A + E_k^B + s}$$

where $s \in \mathbb{R}$ is given by an algebraic equation, and E_k^B are the eigenvalues of H_B.

The amount of concentration and s depend on the spectrum!

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This follows from an even more general result:

2. Typicality in mean energy ensemble

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Main Theorem (arXiv:1003.4982): Let H be any observable on $\mathbb{C}^n$, and draw a pure normalized state $|\psi\rangle \in \mathbb{C}^n$ randomly under the constraint $\langle\psi|H|\psi\rangle = E$.

If f is any real function (on states) with $|f(x) - f(y)| \leq \lambda\|x - y\|$

then $\mathrm{Prob}\left\{|f(\psi) - \bar{f}| > \lambda\varepsilon\right\} \leq a \cdot n^{\frac{3}{2}} e^{-c\,n\left(\varepsilon - \frac{1}{4n}\right)^2 + 2\delta\sqrt{n}}$

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where the constants *a*, *c*, δ depend on the spectrum (with some freedom of choice), and $\bar{f}$ is the median of *f* on the mean energy ensemble.

The median $\bar{f}$ can be approximated by integration over a high-dimensional ellipsoid.

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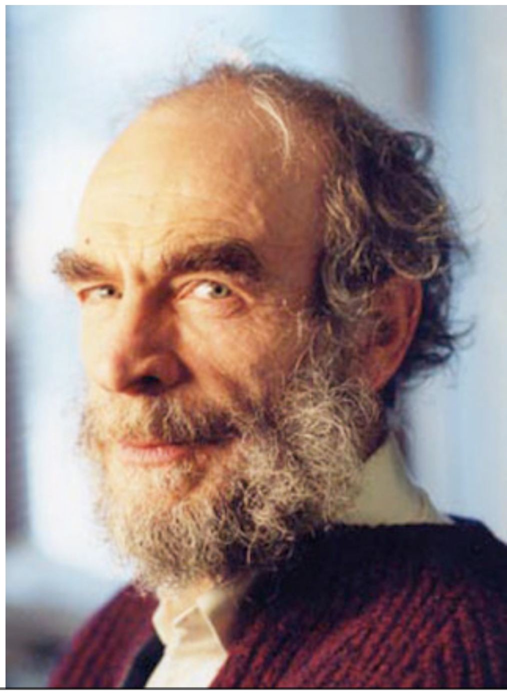
Typicality in mean energy ensemble!

For some spectra, this result can be trivial (e.g. $c \approx 0$)!

2. Typicality in mean energy ensemble

Idea of proof: integral geometry



M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces* (Birkhäuser '01).

GROMOV AWARDED 2009 ABEL PRIZE
The 2009 Abel Prize is awarded to **Mikhail Leonidovich Gromov**, Permanent
France, "for his revolutionary contributions to geometry." The award is 6 million

2. Typicality in mean energy ensemble

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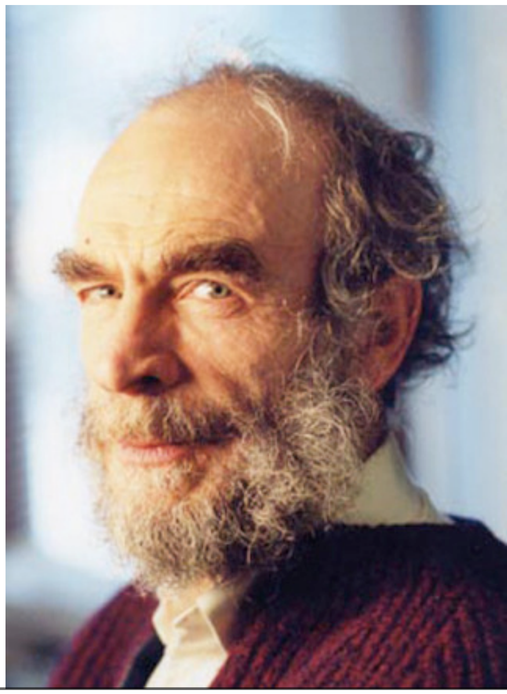

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GROMOV AWARDED 2009 ABEL PRIZE
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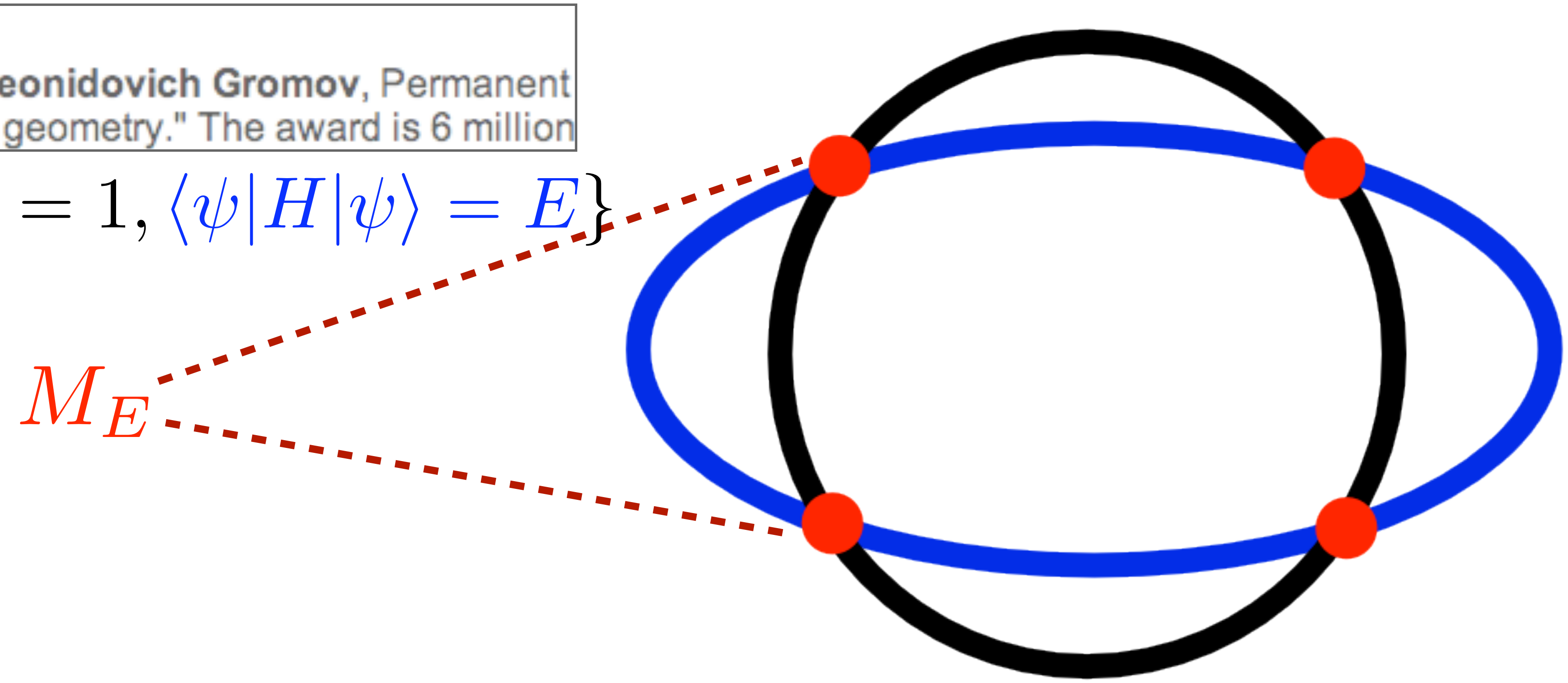

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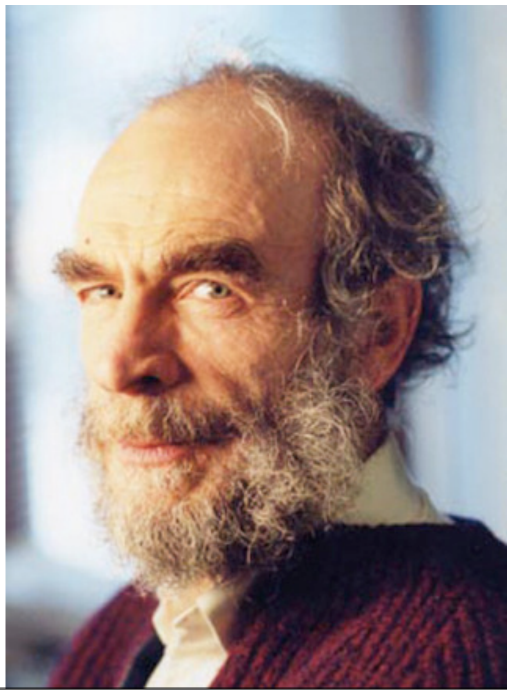

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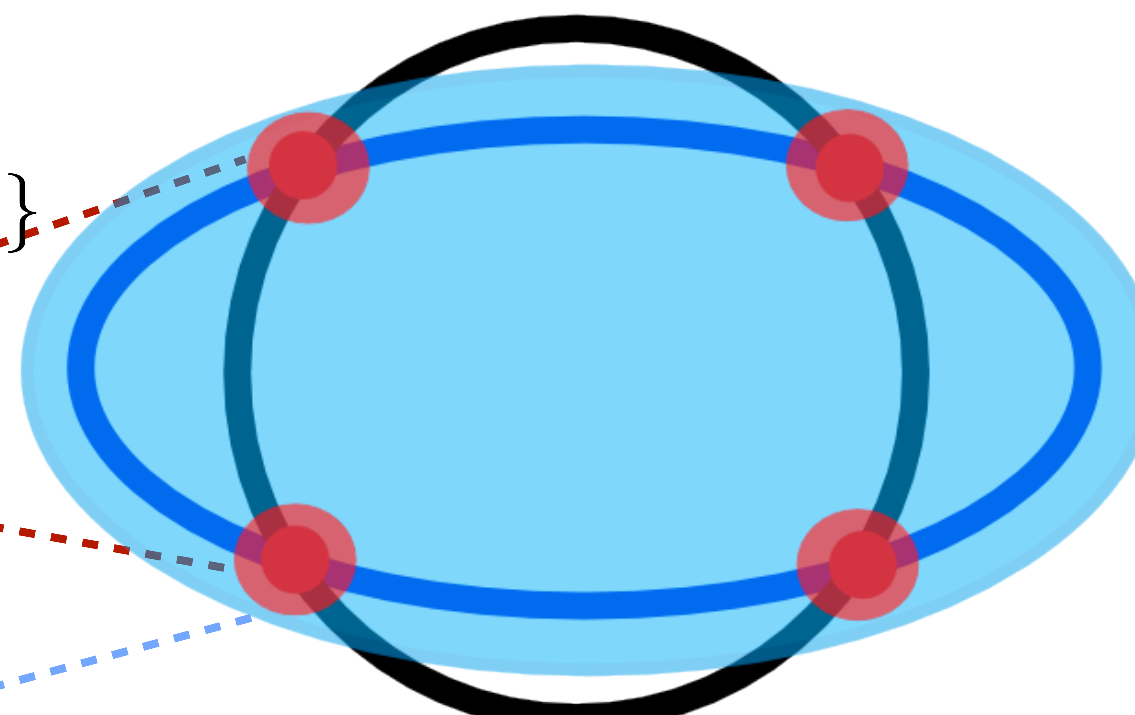

$$N = \{\psi : \langle\psi|H|\psi\rangle \le E(1 + 1/2n)\}$$

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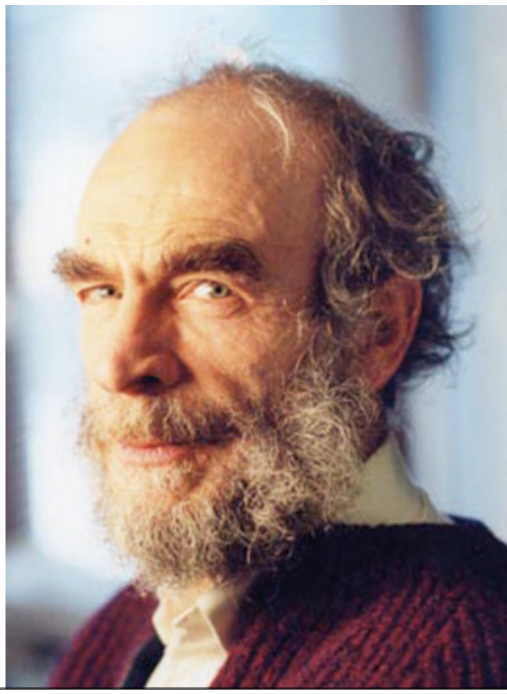

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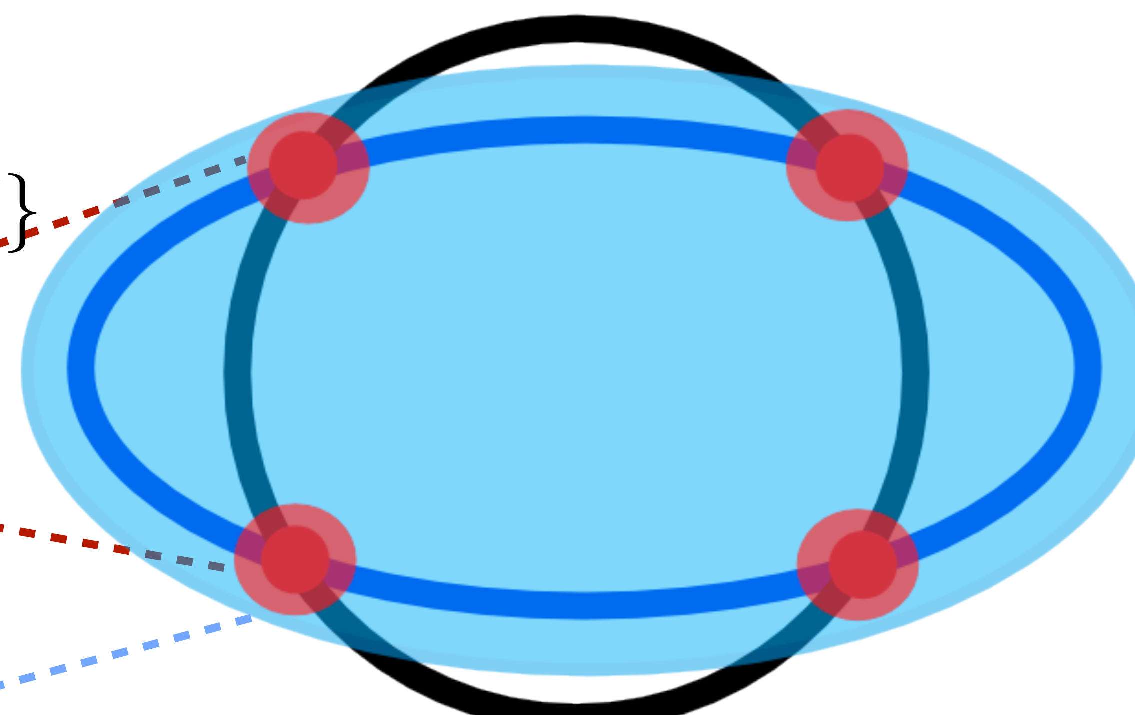

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covers a large part of *N* if energy offset chosen adequately.

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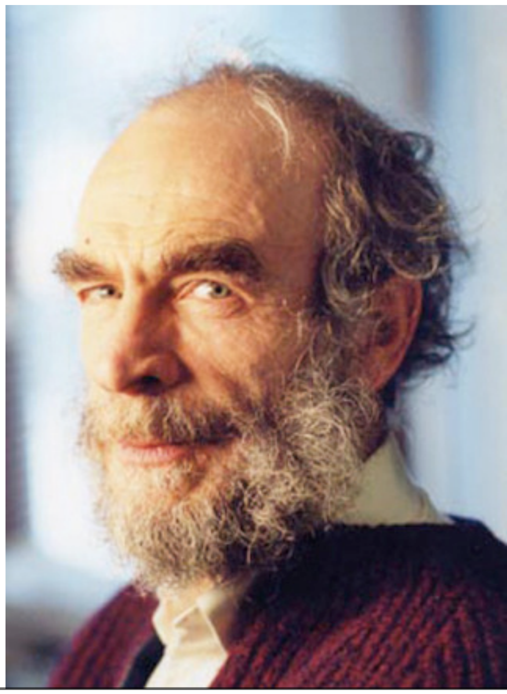

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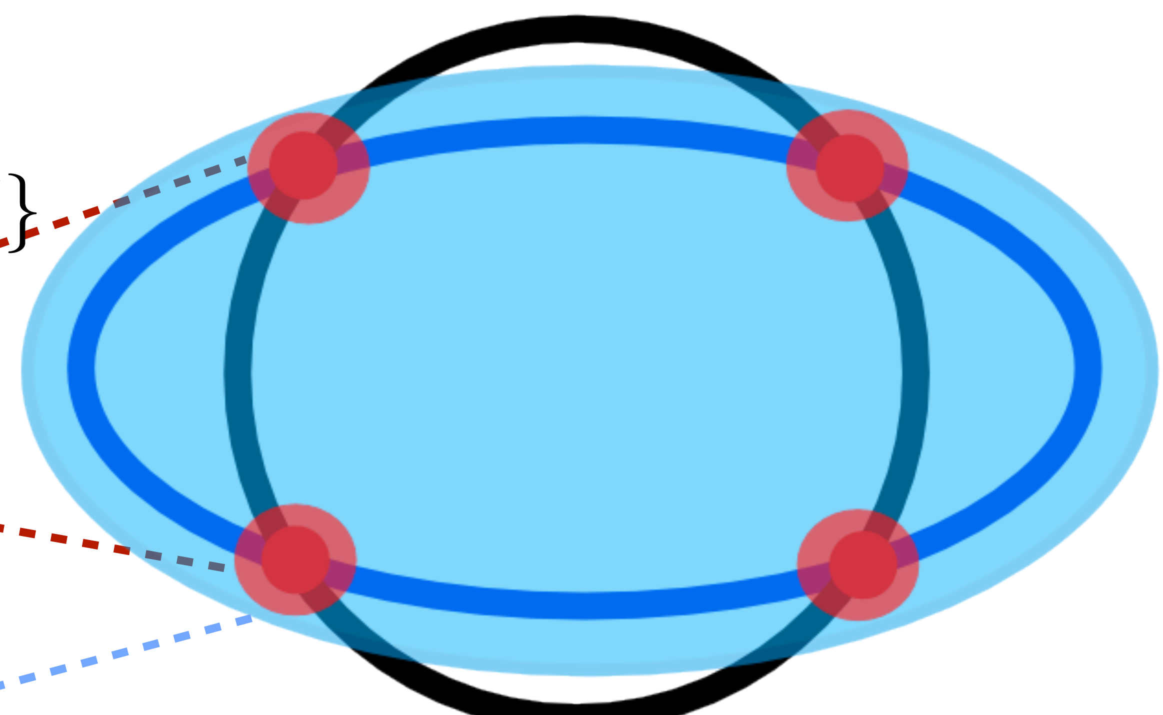

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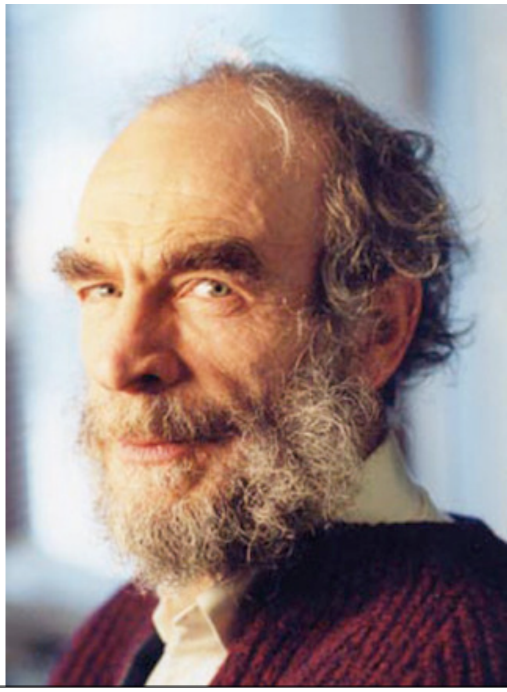

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Mean energy manifold inherits concentration of measure from surrounding ellipsoid.

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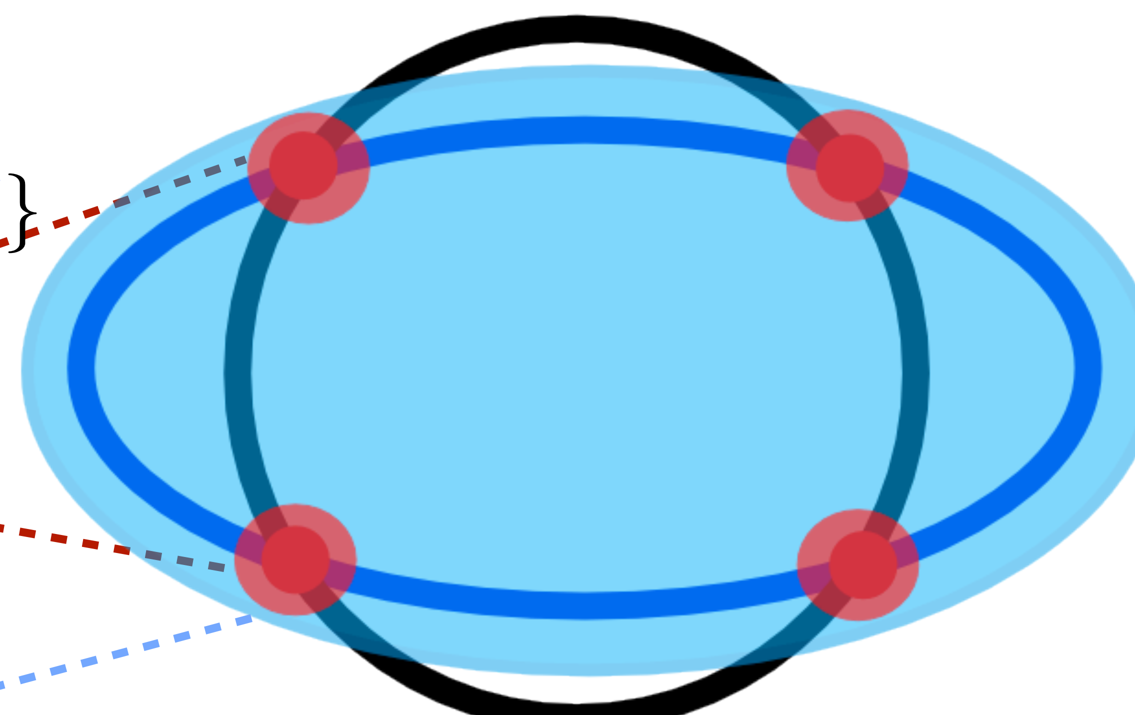

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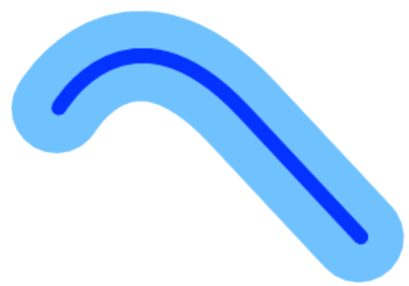
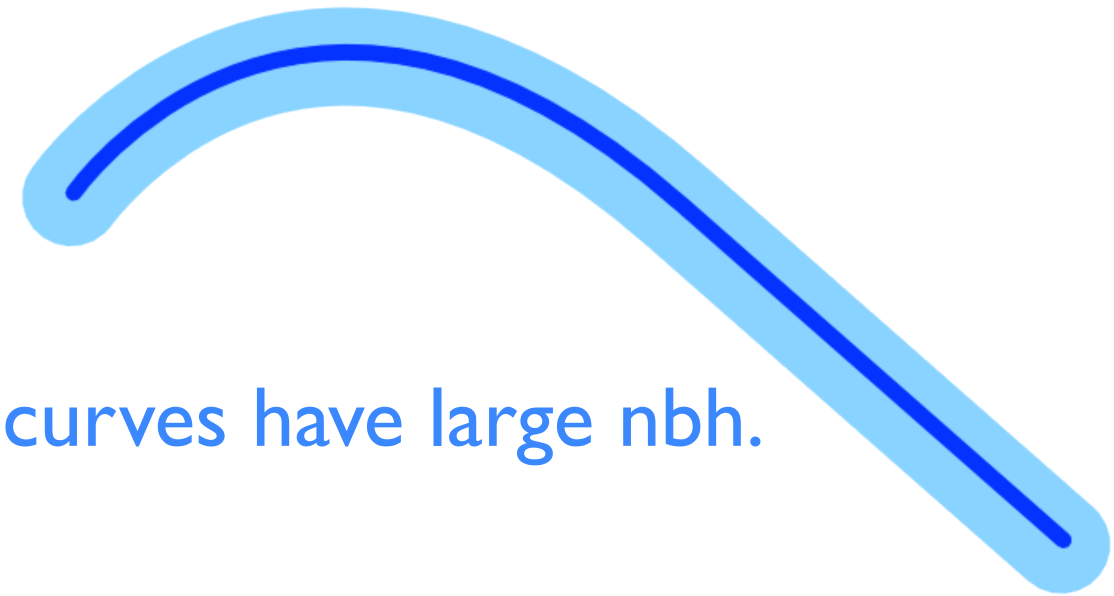
Proof: how to estimate neighborhood volume

Intuition:



short curves have small nbh...



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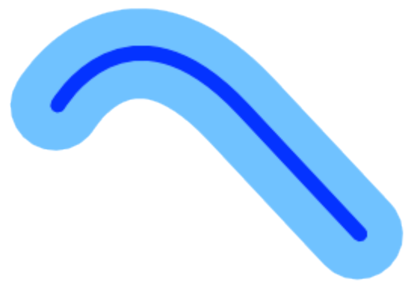

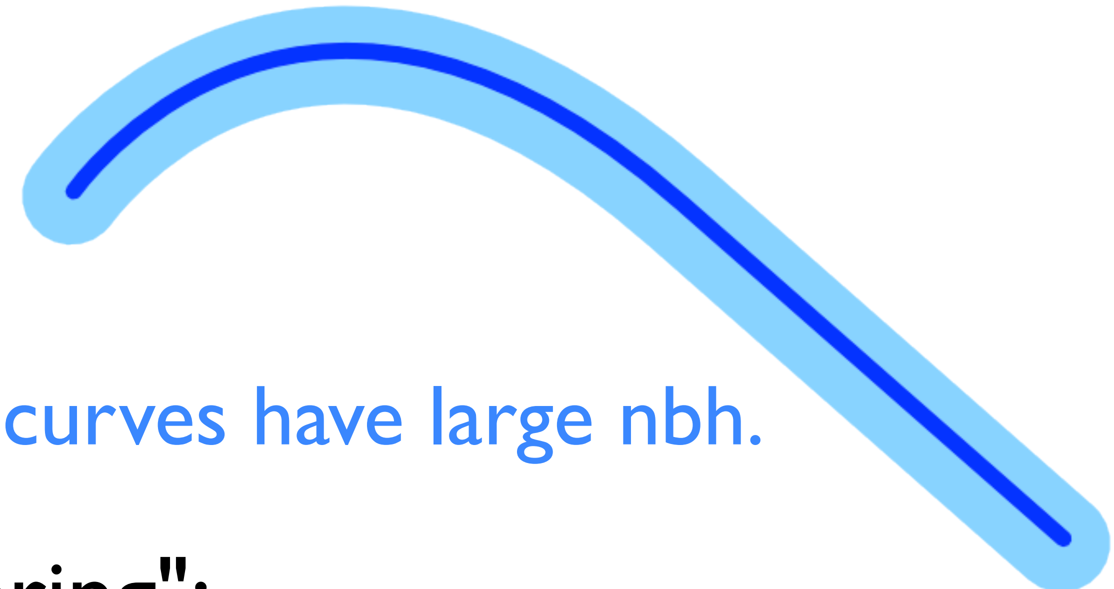

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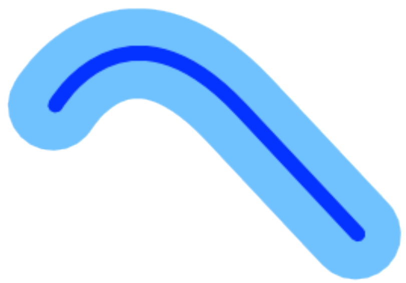

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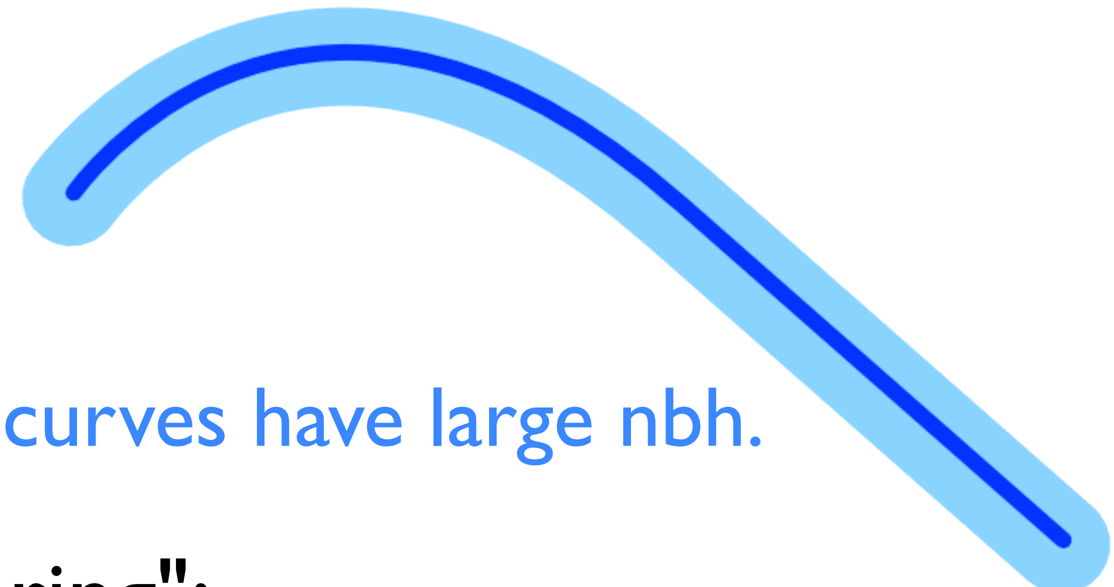

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Cauchy-Crofton formula ("Buffon's needle experiment"):

C: curve, D: domain (e.g. $D = U_\varepsilon(C)$)

$$\int_{\text{lines } L} \#(L \cap C)\, dL = 2 \cdot \text{length}(C)$$

$$\int_{\text{lines } L} \text{length}(L \cap D)\, dL = \pi \cdot \text{area}(D)$$

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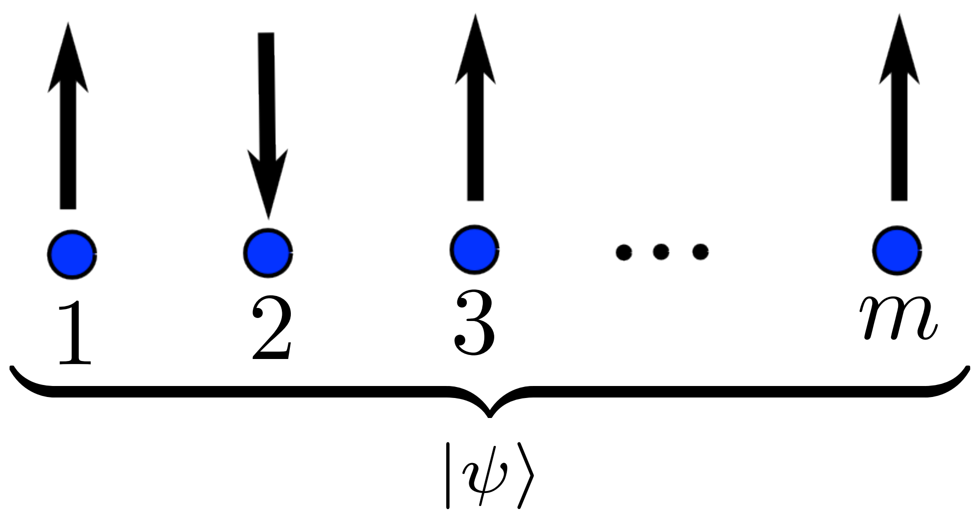

$$H = \frac{1}{2}\left(m\mathbf{1} + \sum_{i=1}^{m} Z_i \right)$$

Ground state energy 0, infinite temperature: energy $m/2$.

$\dim \mathcal{H} = 2^m =: n$. Draw $|\psi\rangle$ randomly under $\langle\psi|H|\psi\rangle \stackrel{!}{=} \alpha \cdot m$

where $0 \leq \alpha \leq 1/2$.

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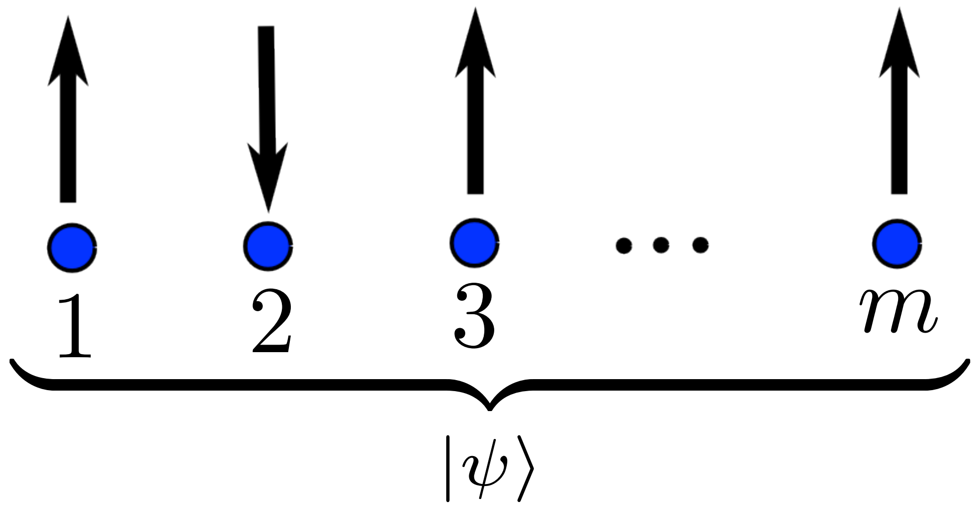

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Observation: Bound from our theorem gets useless:

$$\text{Prob}\left\{|f(\psi) - \bar{f}| > \lambda\varepsilon\right\} \lesssim \exp\left(-c\, n\varepsilon^2 + 2\delta\sqrt{n}\right)$$

For Ising spectrum, we get $c \approx 1/n.$ Why is that?

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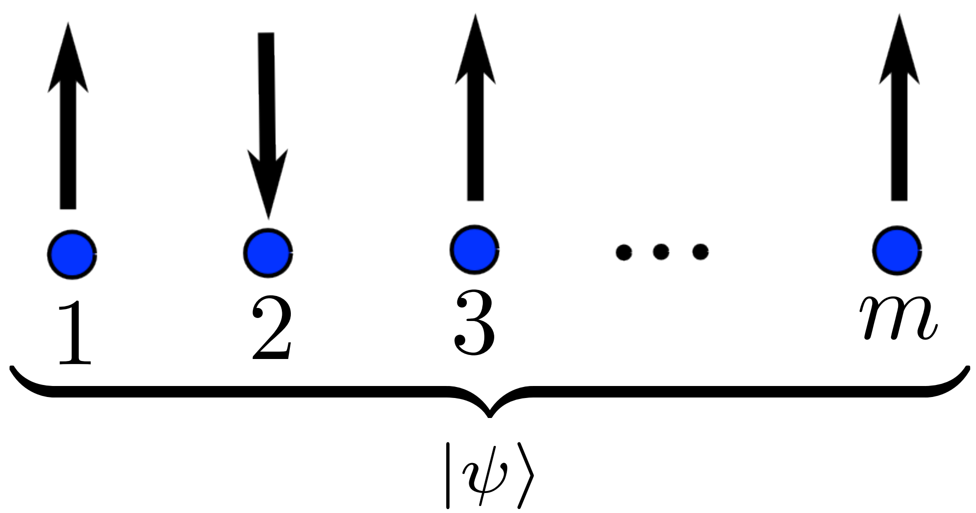

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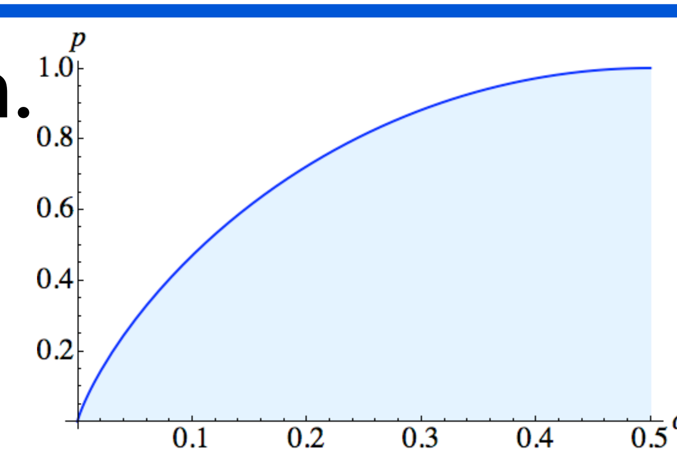
<u>Theorem</u>: There *is no* exponential concentration.
Best possible concentration bound is

$$\mathrm{Prob}\left\{|f - \bar{f}| > \lambda\varepsilon\right\} \lesssim \exp\left(-c\, n^{p}\varepsilon^2\right)$$

with $p \equiv p(\alpha) < 1$, see graph.



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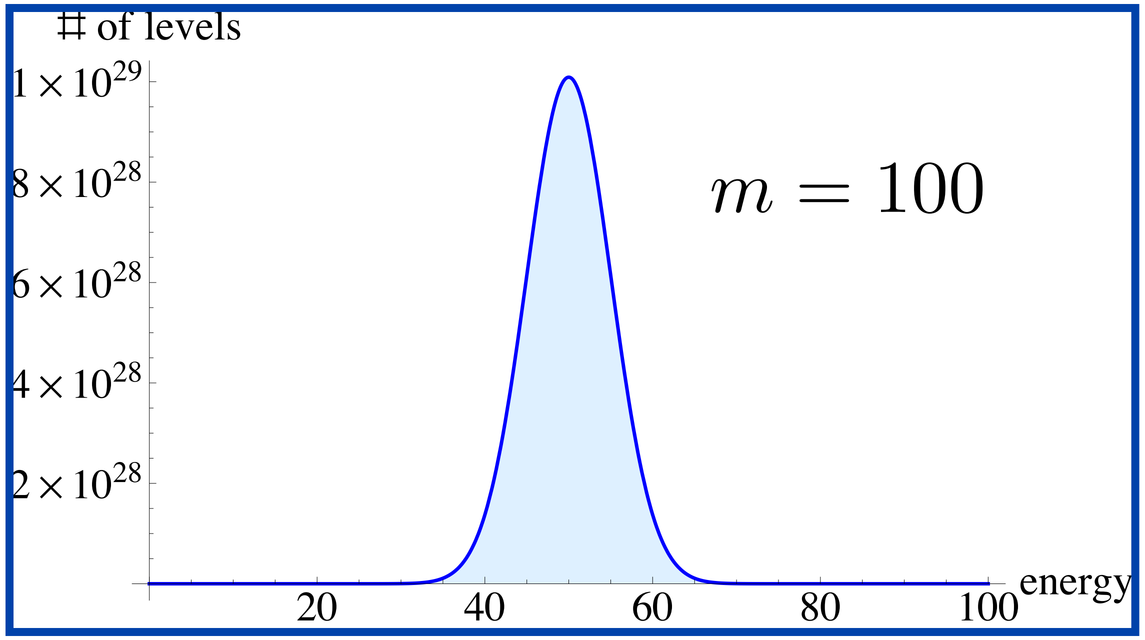

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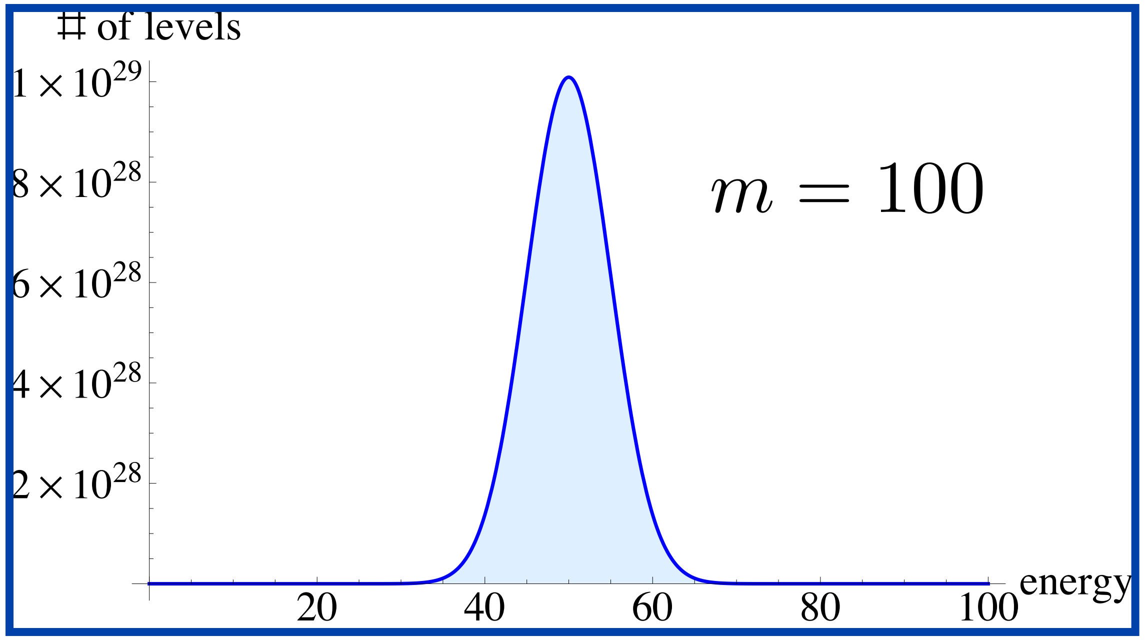
• Interpretation: "almost all" of the $n = 2^m$ energy levels are close to energy value $m/2$.
• If $|\psi\rangle$ is to have much smaller energy, then it "does not see" most of the levels $\Rightarrow$ effectively lives in smaller dim.



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- In those cases where *m.e.e.* concentrates, typical reduced density matrix is not of Gibbs form. Instead, a sum of terms $(H_A + s)^{-1}$ with some $s \in \mathbb{R}$.

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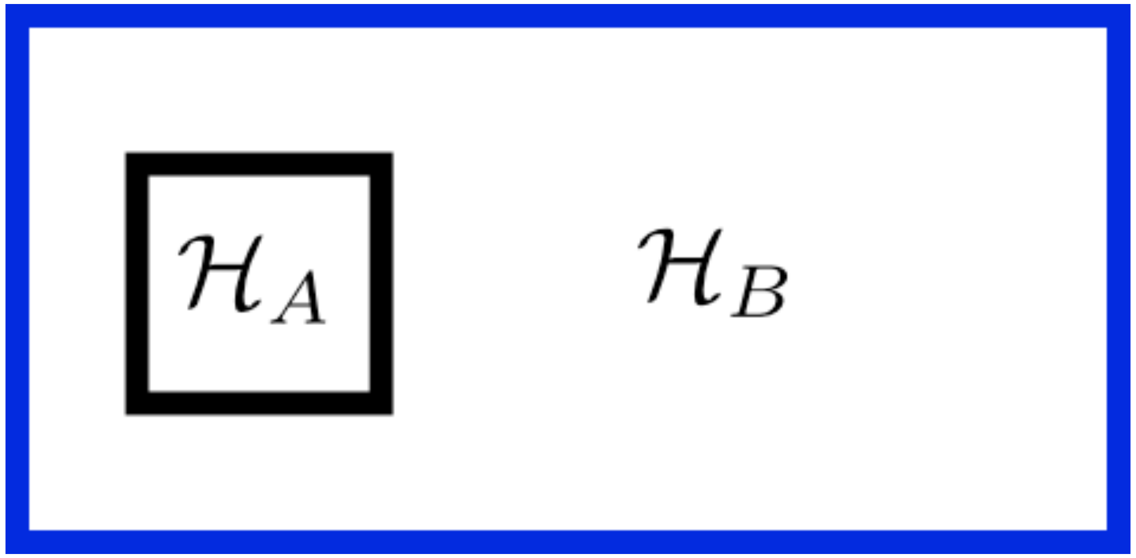

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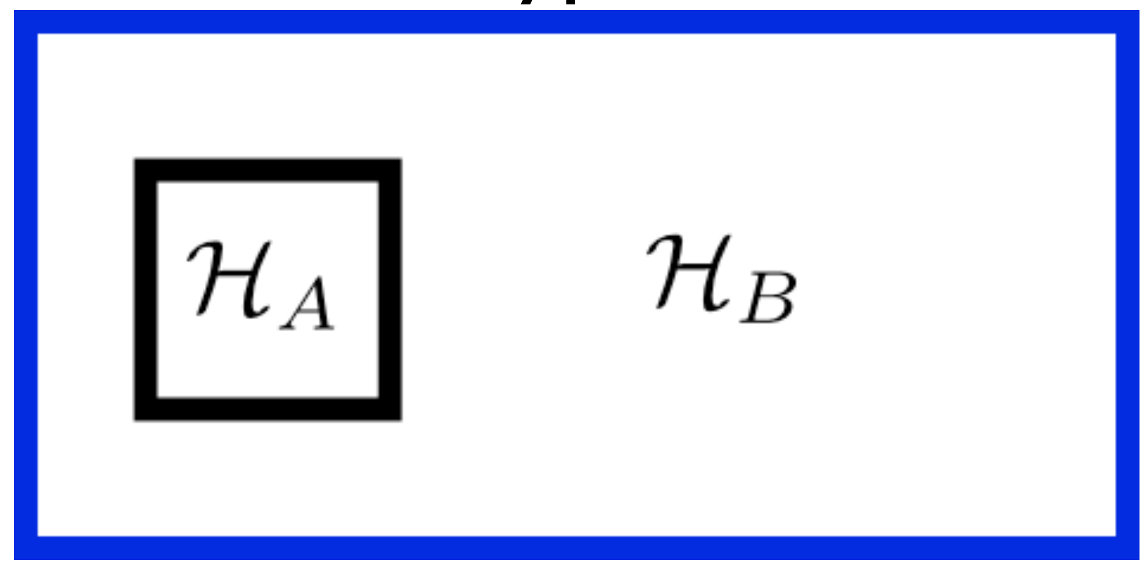
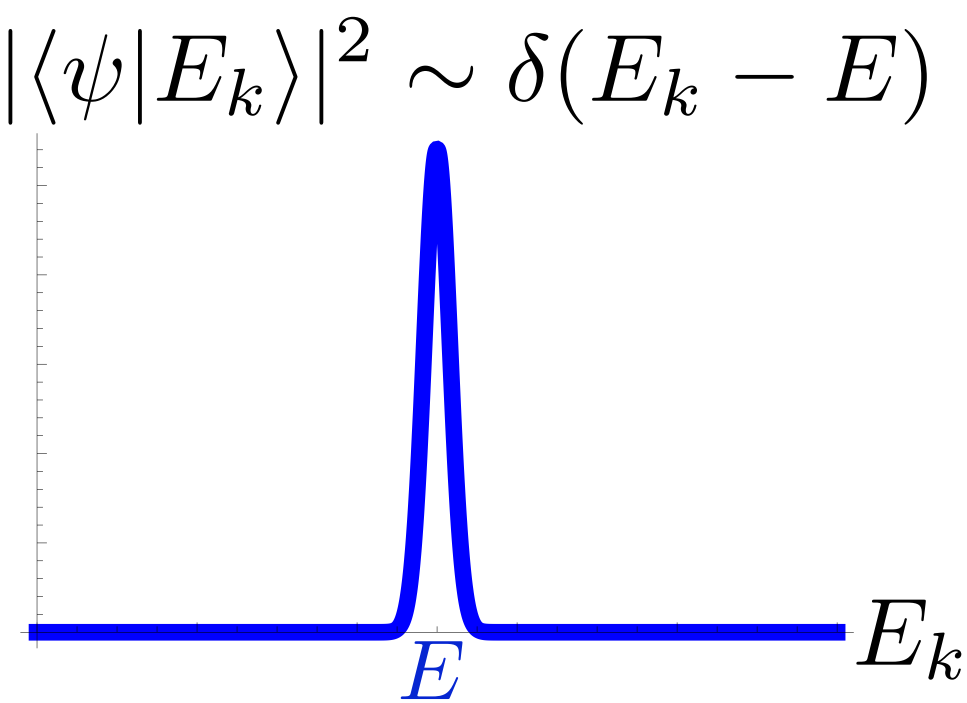
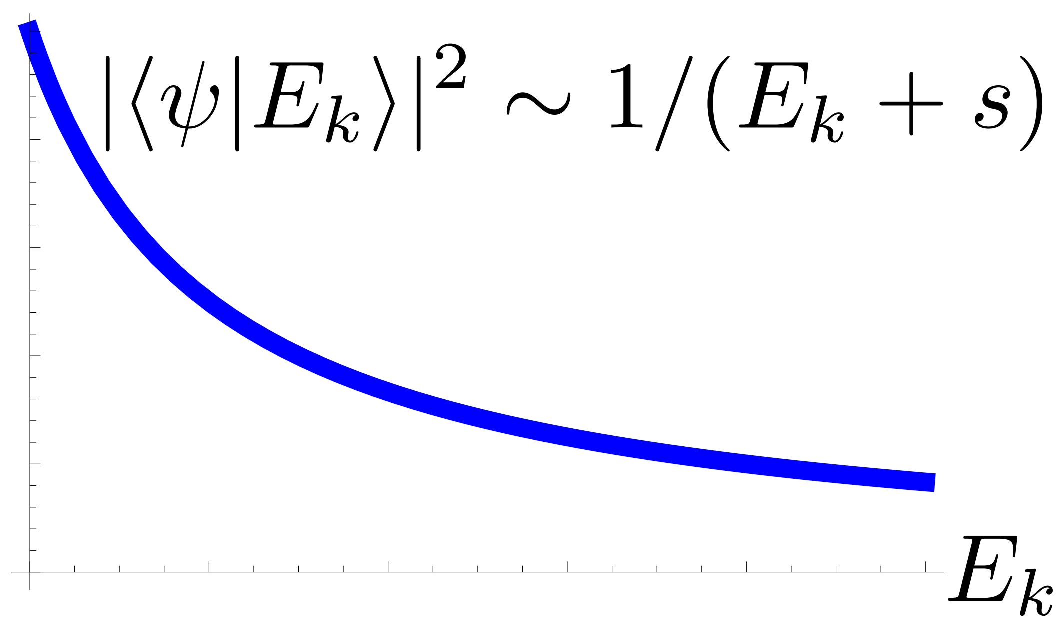




Gibbs situation:
Small spectral window.



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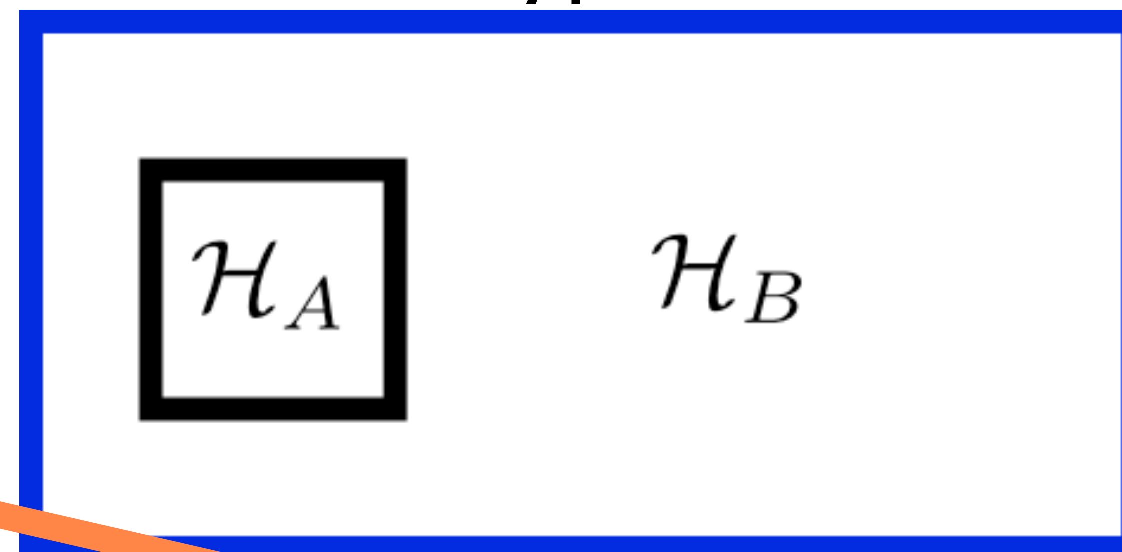
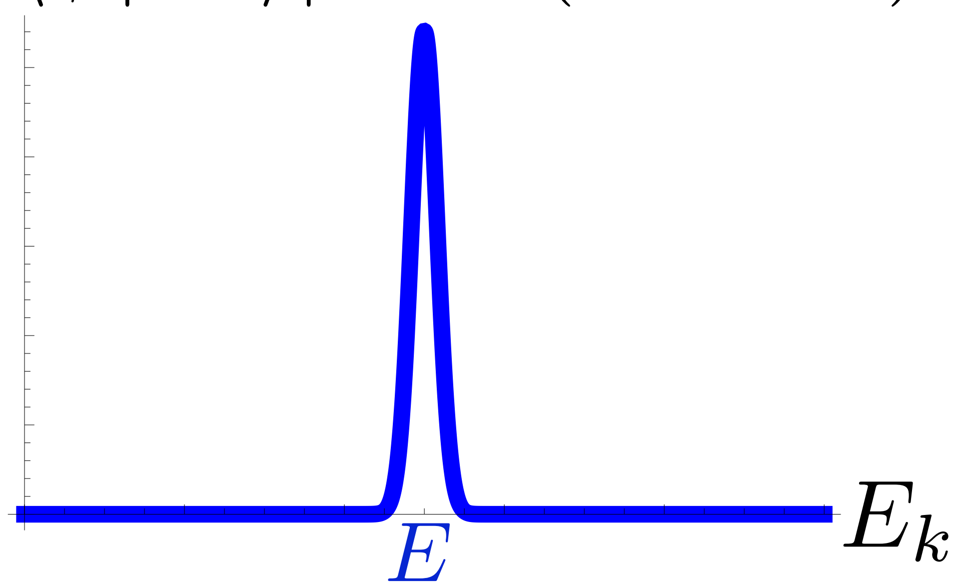
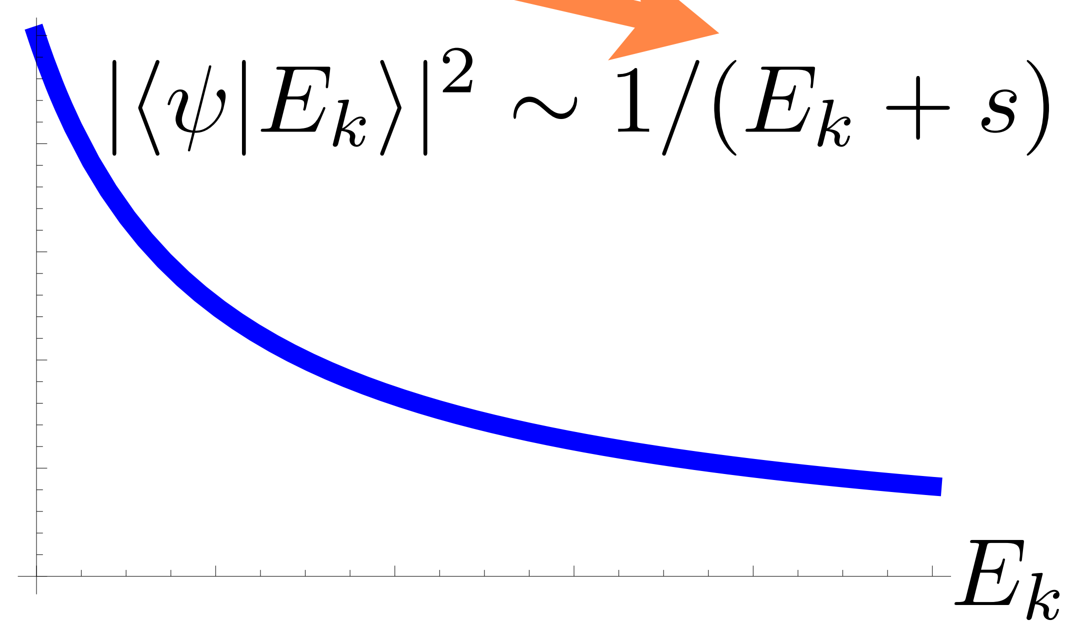


$|\langle\psi|E_k\rangle|^2 \sim \delta(E_k - E)$



$|\langle\psi|E_k\rangle|^2 \sim 1/(E_k + s)$



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M. P. Müller, D. Gross, J. Eisert, *Concentration of measure for quantum states with a fixed expectation value*, Commun. Math. Phys. 303/3, 785--824 (2011), arXiv:1003.4982