Canonical typicality for translation-invariant quantum many-body systems

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Abstract

It is a basic fact of statistical physics that a system *S*, weakly coupled to a large bath *B*, will be in a thermal state, if the full system *SB* is described by a microcanonical ensemble. In the quantum case, it has been suggested in Ref. [1] that a much stronger statement is true, which has been coined *"canonical typicality"*: ***almost all individual pure states in the microcanonical subspace are locally close to a thermal state.***
However, the exact content of that statement has remained somewhat unclear so far. In [2], it was proven that most pure states are locally close to *some fixed state*, which is however in general not of the thermal (Gibbs) form. Significant progress has been made in Ref. [3], where canonical typicality was rigorously proven for high temperatures (resp. small interactions between *S* and *B*), under the assumption that the bath has exponential spectral density.
In this work, we **prove canonical typicality for translation-invariant quantum many-body systems with finite-range interaction**. This removes the assumptions of high temperature and exponential spectral density (the latter being satisfied automatically in the many-body context). For the case of small interaction, we give a sharp bound on the finite bath size necessary for thermalization, and we show that canonical typicality implies the **finite de Finetti Theorem** as a special case.

A taste of the proof: Gibbs states on infinite lattices

The proof relies heavily on mathematical physics classifications of Gibbs states on infinite quantum lattice systems, described by quasilocal algebras:

Given any translation-invariant state ω, its entropy density s(ω) and energy density *u*(ω) are defined by

$$s(\omega) := -\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\mathrm{tr}\left(\omega^{(\Lambda)}\log\omega^{(\Lambda)}\right),$$

$$u(\omega) := \lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\mathrm{tr}\left(\omega^{(\Lambda)}H(\Lambda)\right),$$

where *H*(Λ) contains all interaction terms of the Hamiltonian that are fully contained in region Λ, and $\omega^{(\Lambda)}$ is the reduced density matrix of ω on Λ.

- **Variational principle.** A translation-invariant state ω is a Gibbs state at inverse temperature β if and only if it maximizes the functional s(ω)-βu(ω).
- **Existence of a limit state.** We prove that there exists at least one limit point
$$\tau^{(m)} := \lim_{n\to\infty}\mathrm{Tr}_{\Lambda_m^C}|\psi_n\rangle\langle\psi_n|$$
and that the resulting state τ on the infinite lattice satisfies the variational principle, hence is a Gibbs state.

References

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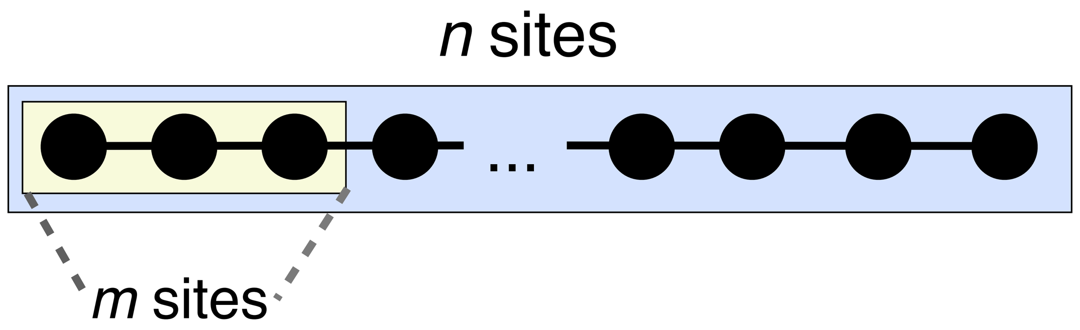
Result in a nutshell

Consider a 1D quantum system (theorem below: *d* dimensions), described by some translation-invariant Hamiltonian *H* with finite-range interaction.



From the global energy window subspace, corresponding to energy density *u*,

$$T_u^{(n)} := \mathrm{span}\left\{|E\rangle \;\middle|\; \frac{E}{n}\in(u-\delta_n,u+\delta_n)\right\}$$

draw a **global pure state** $|\psi\rangle\in T_u^{(n)}$ **at random.**

Then its reduced state on the first *m* sites will look **as if the whole system was in the corresponding Gibbs state** $\rho_\beta^{(n)} := \exp(-\beta H)/Z$ **:**

With probability very close to one,

$$\left\|\mathrm{Tr}_{[m+1,n]}|\psi\rangle\langle\psi| - \mathrm{Tr}_{[m+1,n]}\rho_\beta^{(n)}\right\|_1 < \varepsilon,$$

where ε tends to zero as *n* goes to infinity (while *m* remains fixed).

The main theorem

Theorem 1 *Let H be a translation-invariant Hamiltonian with finite-range interaction on a d-dimensional quantum lattice system, and* $\beta\geq 0$ *some inverse temperature such that there is a* unique *Gibbs state in the thermodynamic limit (if* $d=1$*, this is always satisfied), with energy and entropy rates* $u=u(\beta)$ *and* $s=s(\beta)$*, respectively. Let* $(\Lambda_n)_{n\in\mathbb{N}}$ *be a sequence of regions with* $\Lambda_n\stackrel{n\to\infty}{\longrightarrow}\infty$ *in the sense of van Hove. Choose some sequence of subspaces* $T_u^{(n)}$ *on* Λ_n *such that*

- $\dim T_u^{(n)} \geq e^{|\Lambda_n|s+o(|\Lambda_n|)}$ *and*
- $\mathrm{tr}\left(\tau_u^{(n)}H(\Lambda_n)\right)\leq|\Lambda_n|u+o(|\Lambda_n|)$,

for $\tau_u^{(n)}$ *the maximally mixed state on* $T_u^{(n)}$*. Such a sequence of subspaces always exists; if* $d=1$*, we may choose the microcanonical (energy window) subspaces*

$$T_u^{(n)} := \mathrm{span}\left\{|E_n\rangle \;\middle|\; \frac{E}{|\Lambda_n|}\in(u-\delta_n,u+\delta_n)\right\},$$

where $|E_n\rangle$ *denote an energy eigenvector on* Λ_n *corresponding to energy E, and* $\delta_n\searrow 0$ *slowly enough. For every n, draw a pure state* $|\psi_n\rangle\in T_u^{(n)}$ *at random, and determine the reduced state* $\mathrm{Tr}_{\Lambda_m^C}|\psi_n\rangle\langle\psi_n|$ *on some smaller region* $\Lambda_m\subset\Lambda_n$*, where* $m<n$ *is fixed. Then we have*

$$\left\|\mathrm{Tr}_{\Lambda_m^C}|\psi_n\rangle\langle\psi_n| - \mathrm{Tr}_{\Lambda_m^C}\rho_\beta^{(n)}\right\|_1 \stackrel{n\to\infty}{\longrightarrow} 0$$

with probability one, where

$$\rho_\beta^{(n)} = e^{-\beta H(\Lambda_n)}/\mathrm{tr}\left(e^{-\beta H(\Lambda_n)}\right)$$

is the Gibbs state corresponding to all interaction terms that are fully contained in the region Λ_n.

Note that the theorem becomes *wrong* if the reduction of the global Gibbs state, $\mathrm{Tr}_{\Lambda_m^C}\rho_\beta^{(n)}$ is replaced by the local Gibbs state $\rho_\beta^{(m)}$.

Finite size scaling for high temperature

- **No interaction and qubits**. In this case, that has been briefly addressed in [2], we can give a sharp bound on how large the "bath" (*n-m* sites) has to be in order to thermalize the "system" (*m* sites). This is based on the results in Ref. [4]
- In this case, the reduction of the global Gibbs state is
$$\mathrm{Tr}_{[m+1,n]}\rho_\beta^{(n)} = \rho_\beta^{(m)} = \rho_\beta^{\otimes m},$$
that is, the *m*-fold tensor product of the single-site Gibbs state.
- FIx some energy density *u*, and draw $|\psi\rangle$ from
$$T_u^{(n)} := \mathrm{span}\left\{|E\rangle \;\middle|\; \frac{E}{n}=u\right\}$$
at random (if that subspace is not empty). Then, with probability at least
$$1-2\exp\left(-\frac{\varepsilon^2\,2^{nc}}{(n+1)^2 18\pi^3}\right)$$
we have
$$\left\|\mathrm{Tr}_{[m+1,n]}|\psi\rangle\langle\psi| - \rho_\beta^{\otimes m}\right\|_1 < \frac{4m}{n}+\frac{n+1}{2^{nc-m}}+\varepsilon,$$
where *c* is a constant that only depends on *u* and the two energy levels of the single-site Hamiltonian.
- Thus, **the size of the full system, *n*, must grow linearly with the size of the subsystem *m*** in order to obtain a fixed trace distance to the Gibbs state locally.
- Due to the perturbation theorem in [3], this scaling remains valid in the case of **very high temperature, resp. very small interaction.**
- In the case of **local Hilbert space dimension larger than 2**, the results in [4] cannot be used directly, and scaling inequalities can be more difficult [5].

Relation to de Finetti Theorem

- **No interaction and qubits.** Suppose we are instead drawing from a finite energy shell, of width ε. Instead of a single Gibbs state, we obtain a convex combination of Gibbs states of different temperatures: there is a probability measure $\mu^{(n)}$ on **R**, converging to a δ-distribution for large *n*, such that the one-norm bound above still holds with high prob.:
$$\mathrm{Tr}_{[m+1,n]}|\psi\rangle\langle\psi| \approx \int_{\mathbb{R}}\rho_\beta^{\otimes m}\,d\mu^{(n)}(\beta).$$
- A characteristic feature of microcanonical subspaces in this case is their *permutation invariance*. Therefore, the result is in close conceptual analogy to the finite **de Finetti theorem**: permutation invariant states can locally be well approximated by convex combinations of product states.

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