An information-theoretic approach to space dimensionality and quantum theory

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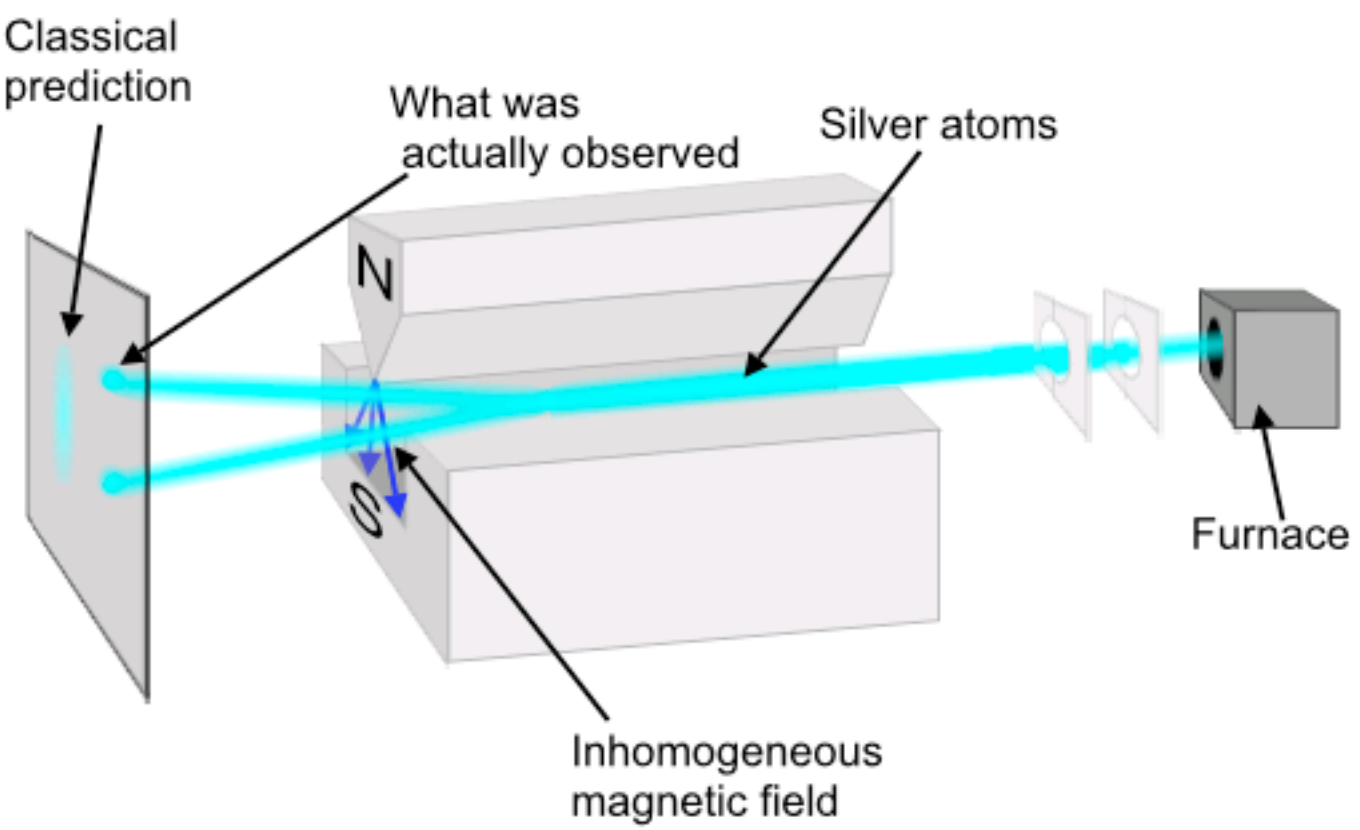

Joint work with Lluís Masanes

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Overview

- The motivation: a curious observation
  - geometry of quantum states vs. physical space; von Weizsäcker‘s idea
- The framework
  - *d*-dim. physical space; probabilistic events ⟶ convex state spaces
- Three information-theoretic postulates (A,B,C)
  - A+B: *d*-dim. Bloch ball; physical geometry from probability measurements
  - A+B+C: derive that *d*=3, quantum theory, unitary time evolution
  - an impossible generalization
- What does this tell us? Some speculation

1. Motivation: a curious observation

Quantum *n*-level state space: $S_n = \left\{ \rho \in \mathbb{C}^{n\times n}_{s.a.} \;\middle|\; \rho \geq 0,\;\; \mathrm{tr}(\rho) = 1 \right\}.$

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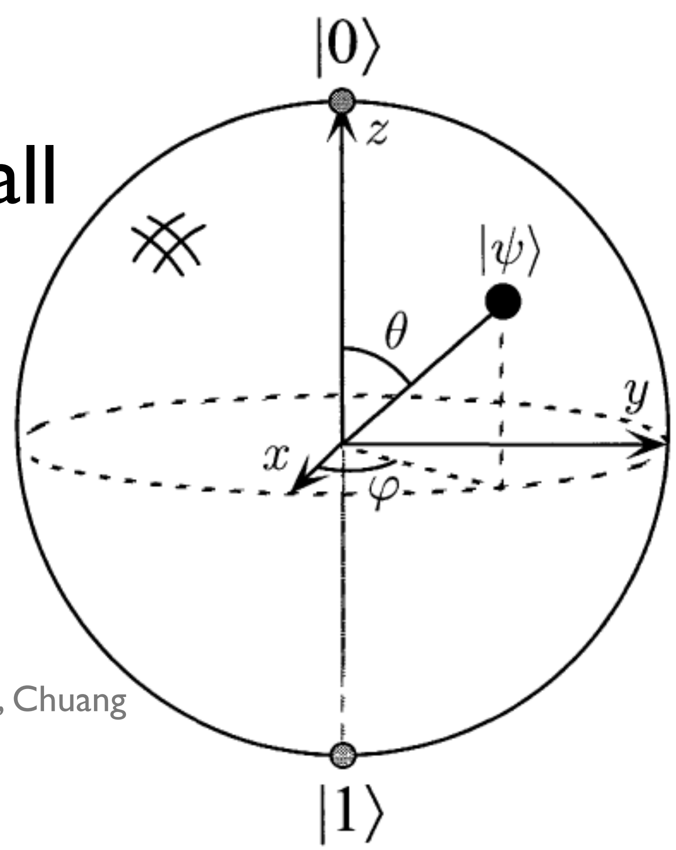
$$S_2 = \left\{ \left(\begin{array}{cc} \frac{1}{2} + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & \frac{1}{2} - r_3 \end{array} \right) \;\middle|\; |\vec{r}| \leq 1 \right\}$$

Bloch ball



Source: Nielsen, Chuang

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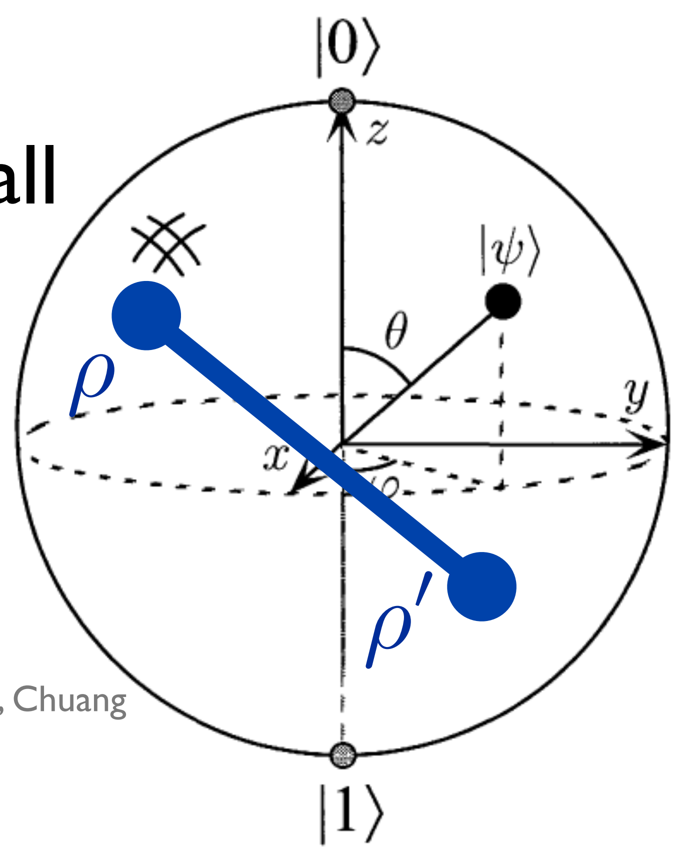

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- This is a particularly nice representation:

  $p\rho + (1-p)\rho' \;\; \mapsto \;\; p\vec{r} + (1-p)\vec{r}'$

  statistical mixtures ⟶ convex combinations

- S_2 is Euclidean and 3-dimensional. But so is physical space! Just a coincidence?

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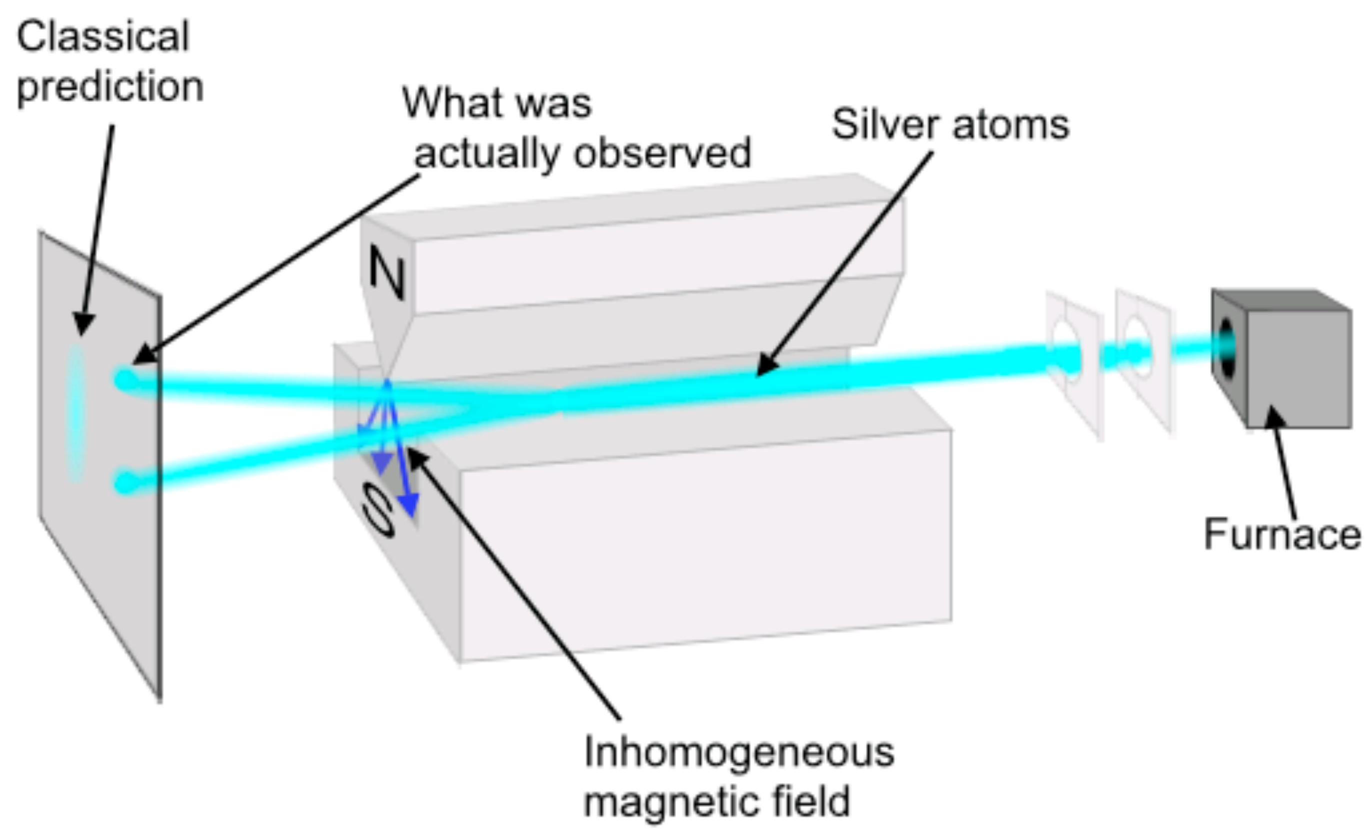
Physical consequence of ballness: 1:1 correspondence between noiseless measurements on 2-level systems and “directions“ (of magnetic field)



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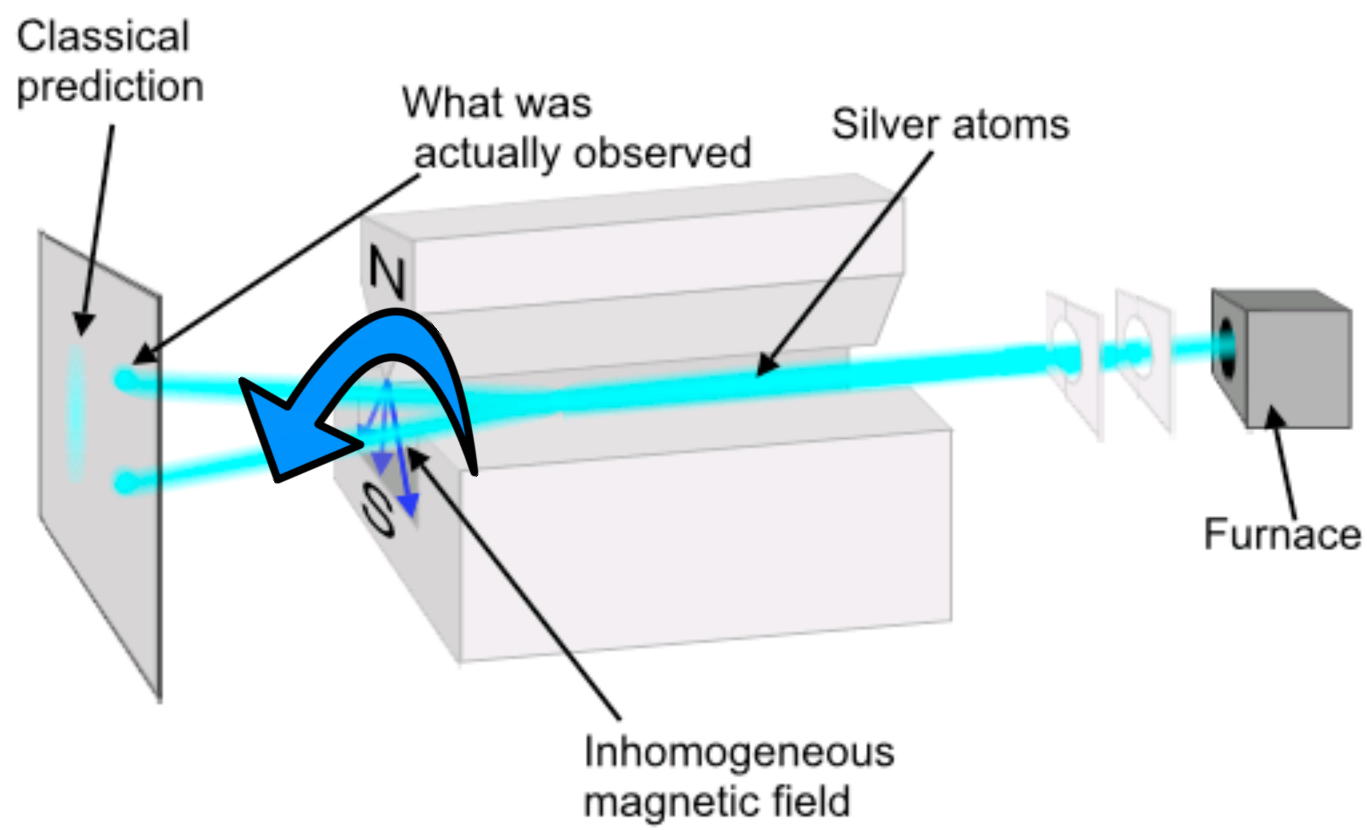

By “rotating the magnet“, we can implement all noiseless measurements.

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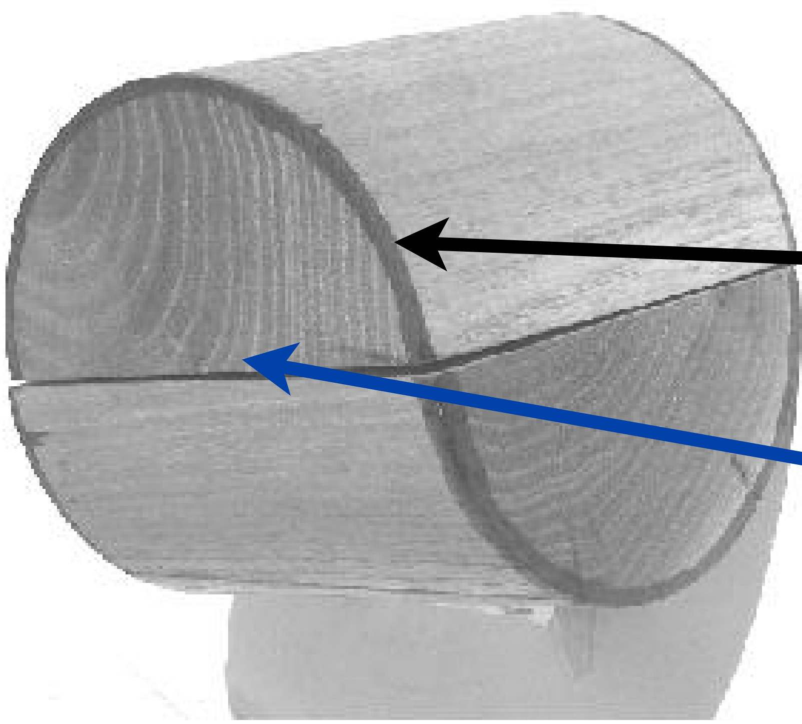

convex set S_n

pure states (extremal points)

mixed states in topological boundary:

$$\rho_{\mathrm{mix}} := \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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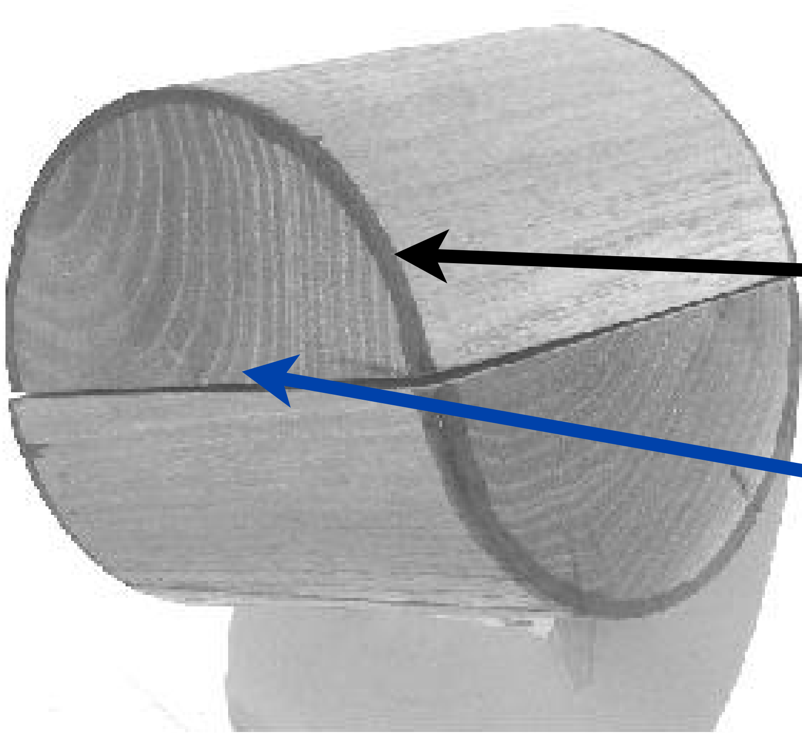

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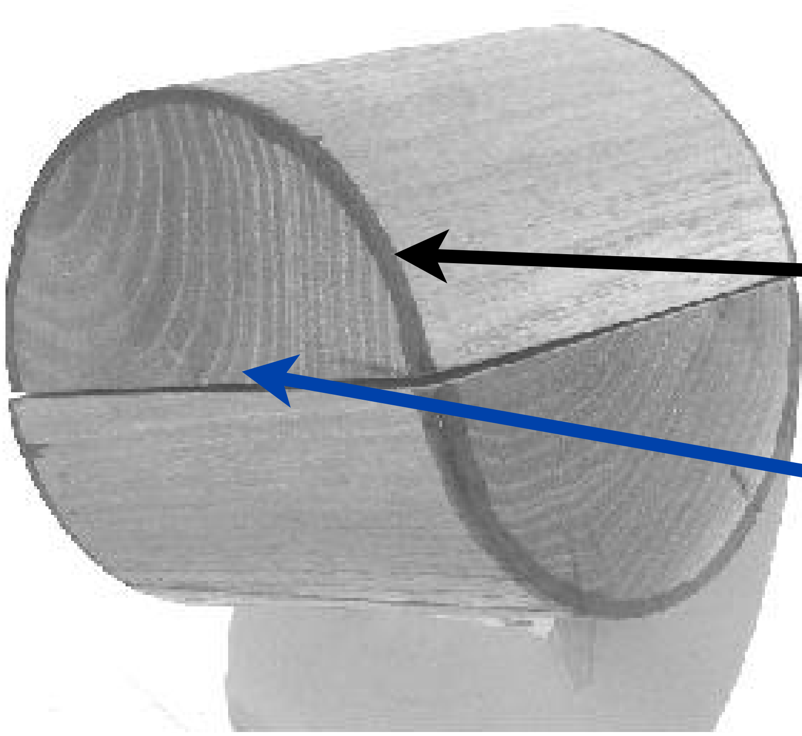

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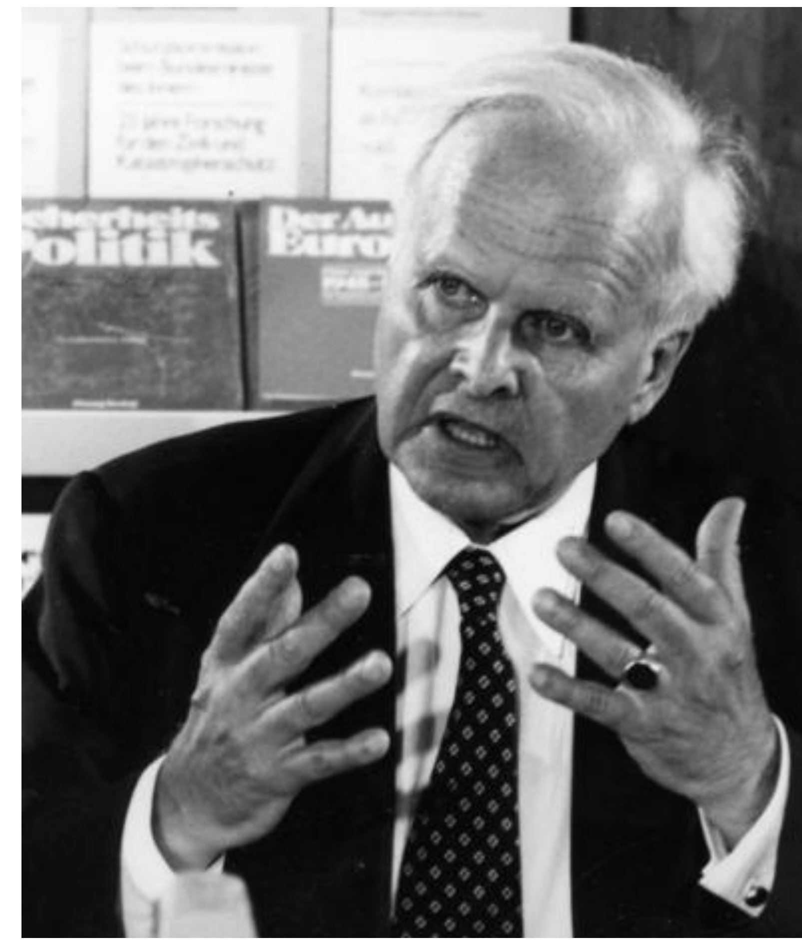
Carl-Friedrich von Weizsäcker: theory of “ur alternatives“ (1955+)



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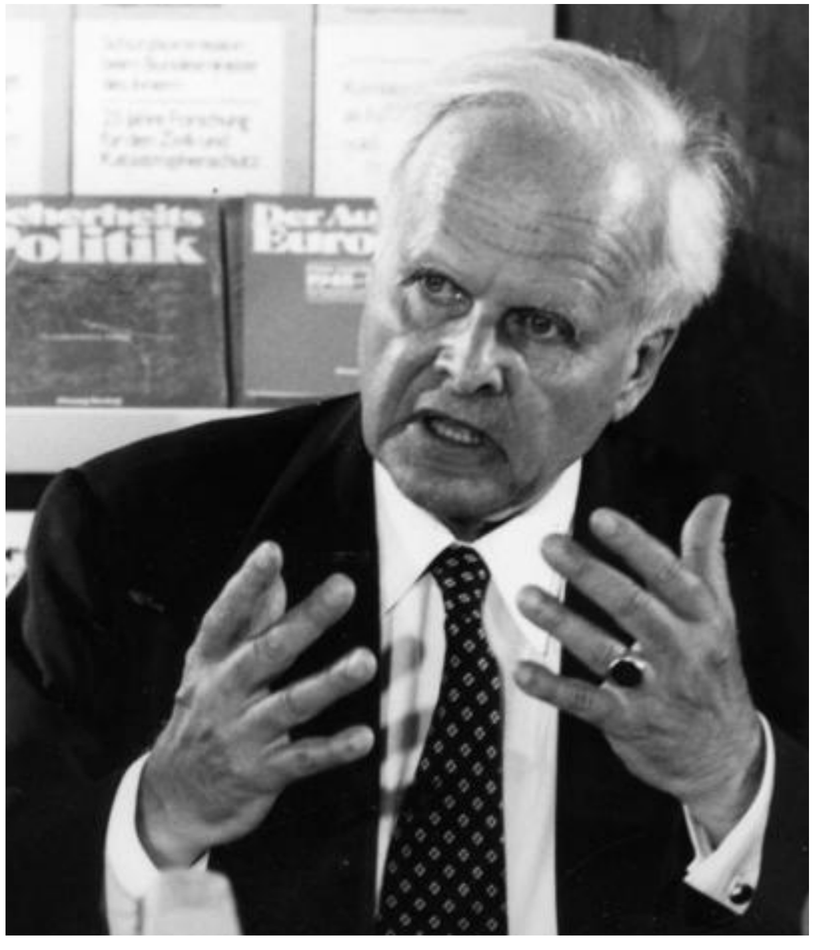

- “ur“ = (pure) qubit = quantum 2-level system
- everything is composed of (delocalized) urs
- symmetry group of ur

$$U(2) = SU(2) \otimes U(1) \sim S^3 \times S^1.$$

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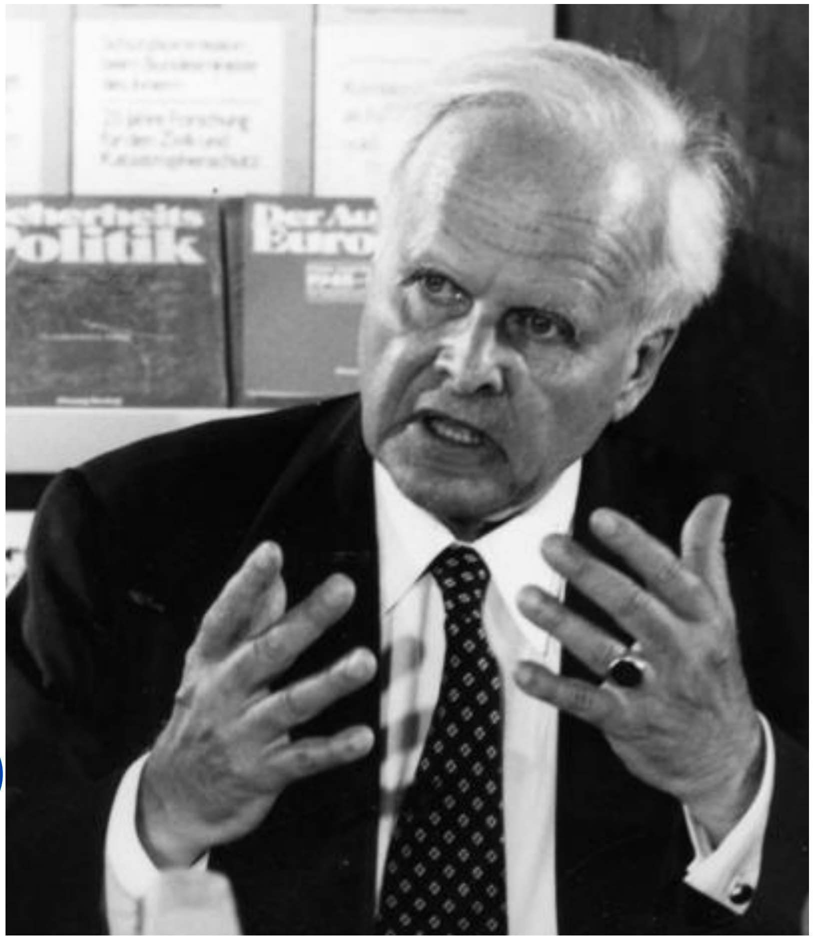

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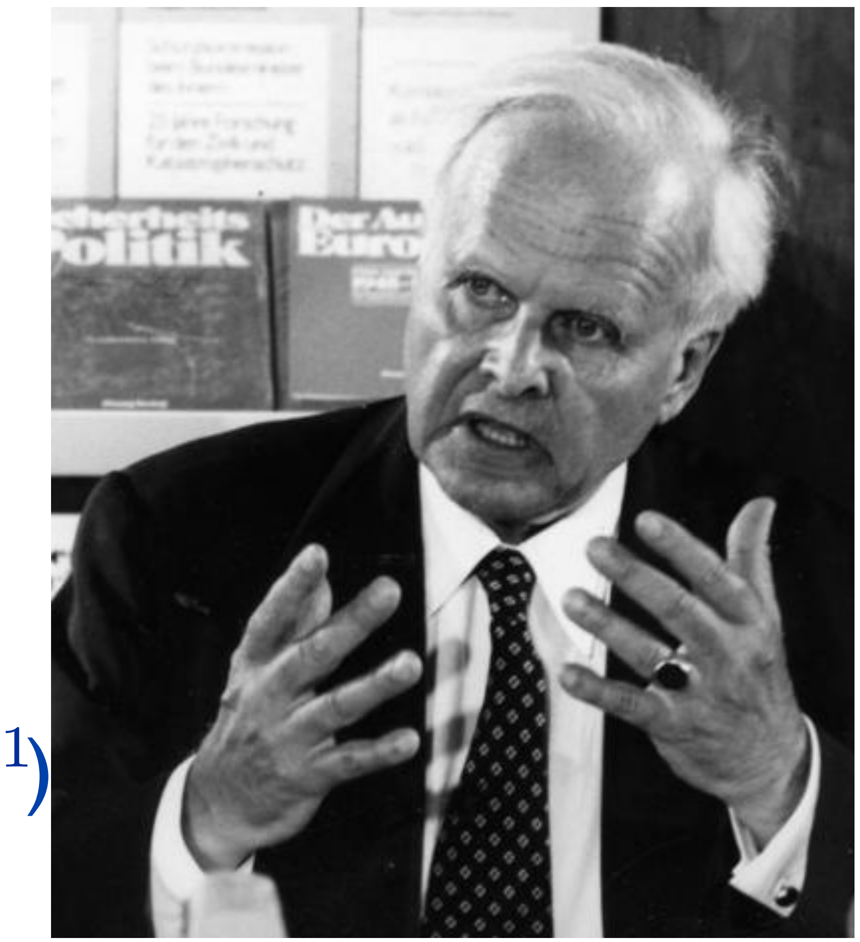

Very vague. What does that mean?

How is decomposition into *delocalized* urs chosen?

Why not ternary ur-alternatives w/ SU(3)?

Why is the result global cosmic space-time?

...

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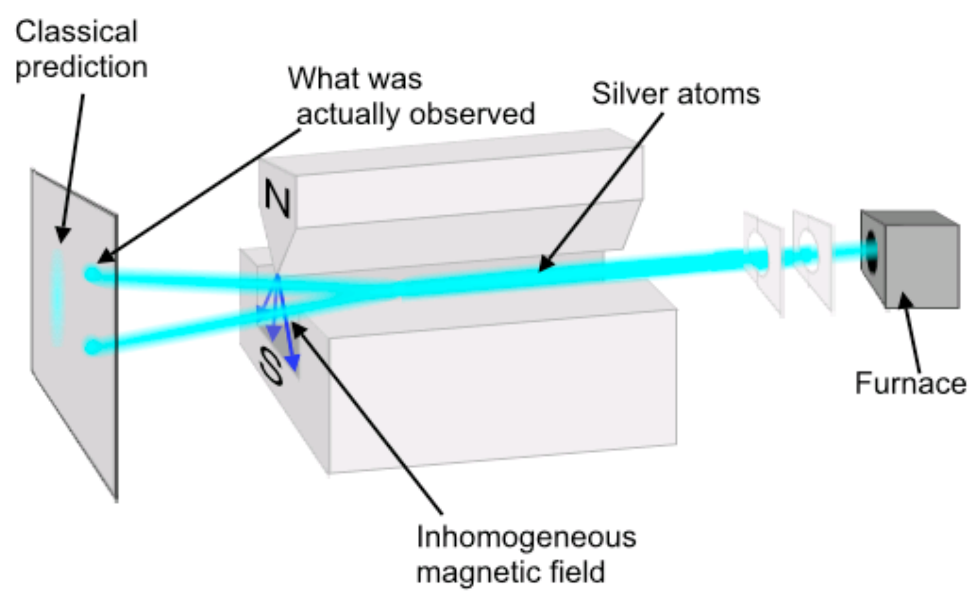

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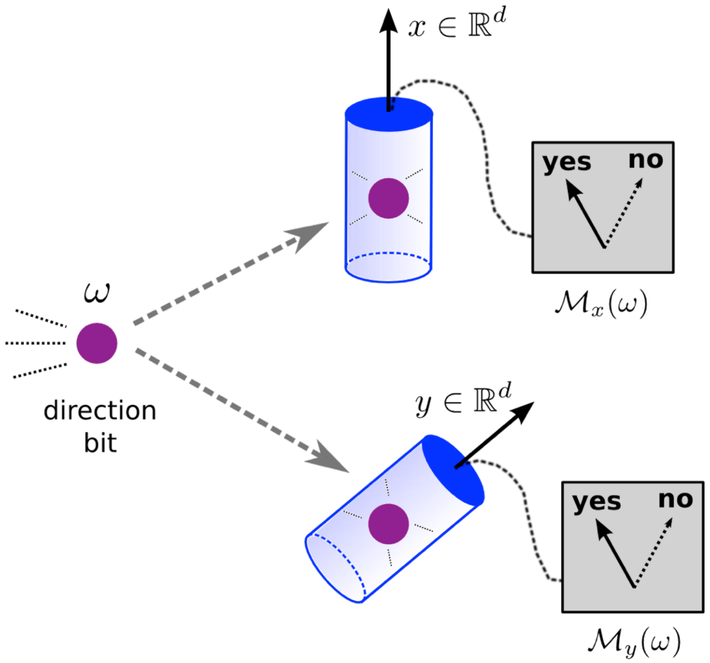

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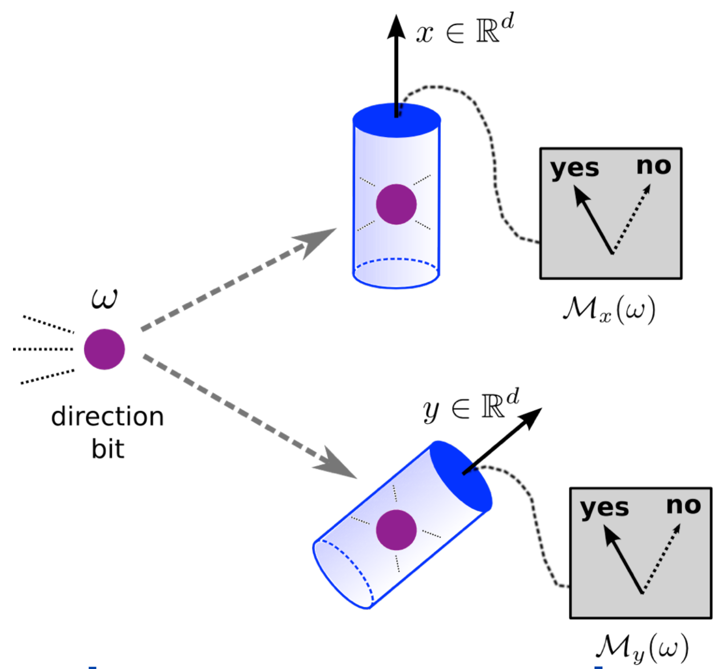
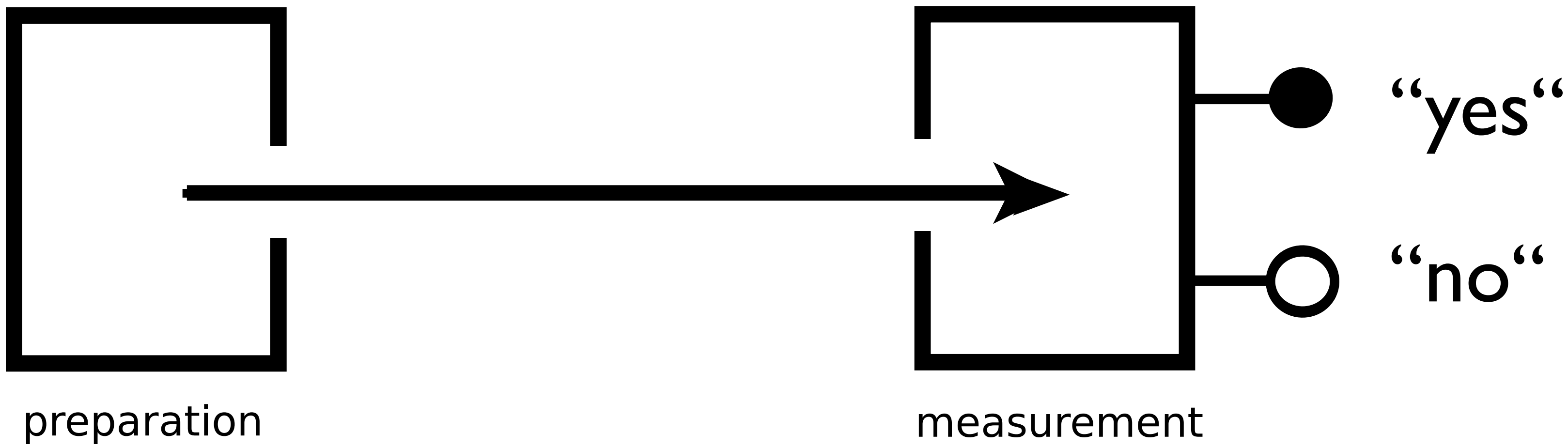


- Give three information-theoretic postulates on how probabilities and rotations are related.
- Prove that we must have *d*=3 and quantum theory necessarily.

2. The framework

Assumption: there are some events that happen probabilistically.



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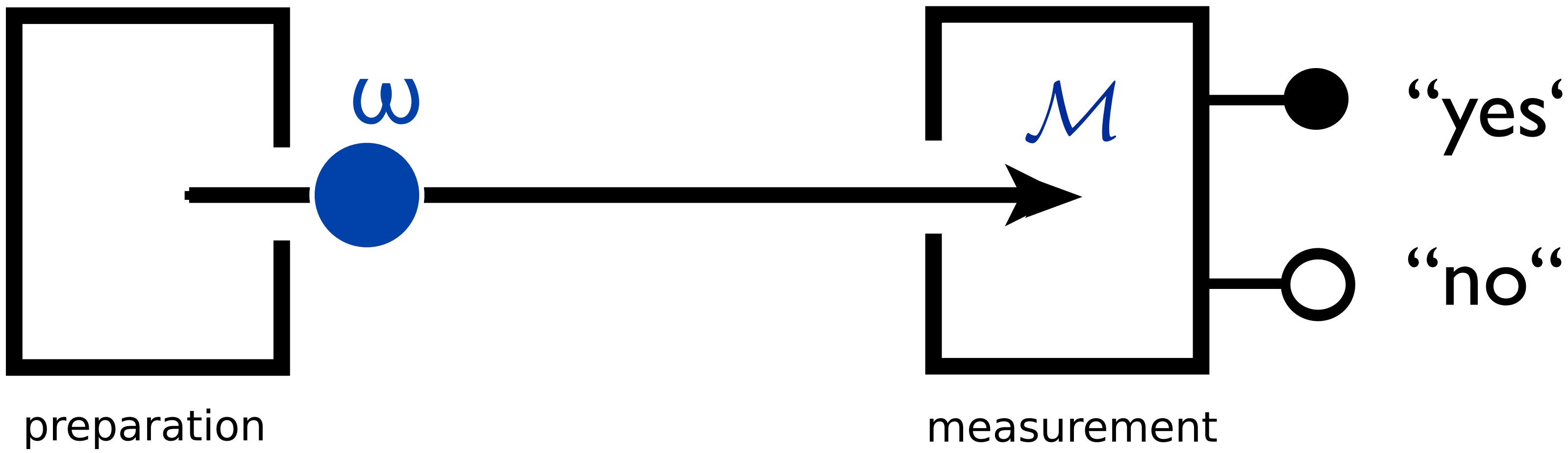

- Physical systems can be in some state ω. From this, all outcome probabilities of all subsequent events can be computed:

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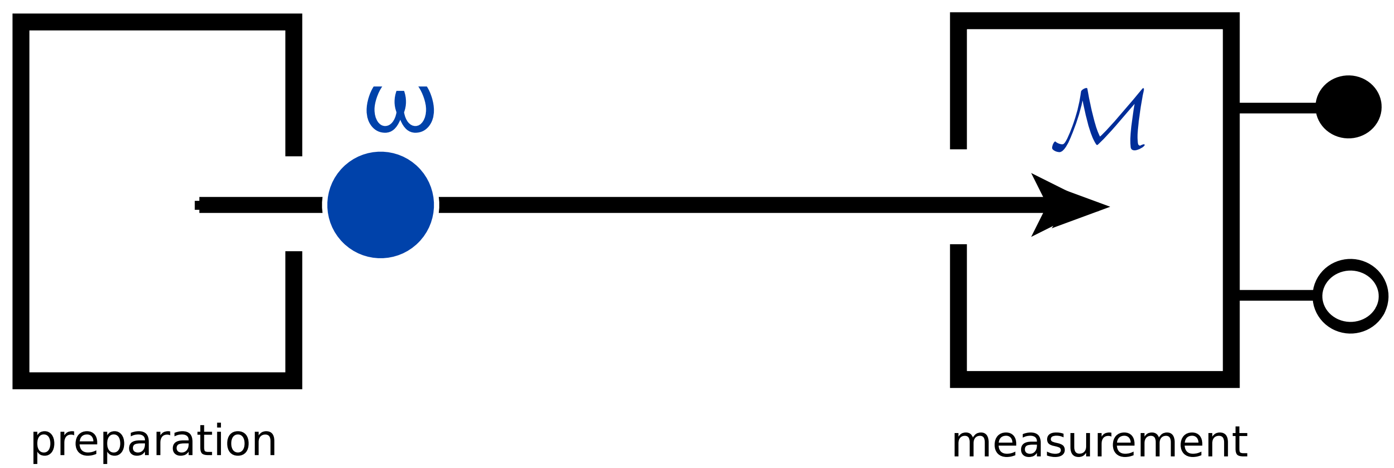

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- Statistical mixtures are described by convex combinations: prepare ω with prob. *p* and state φ with prob. *(1-p)*, result:

$$p\,\omega + (1-p)\varphi$$

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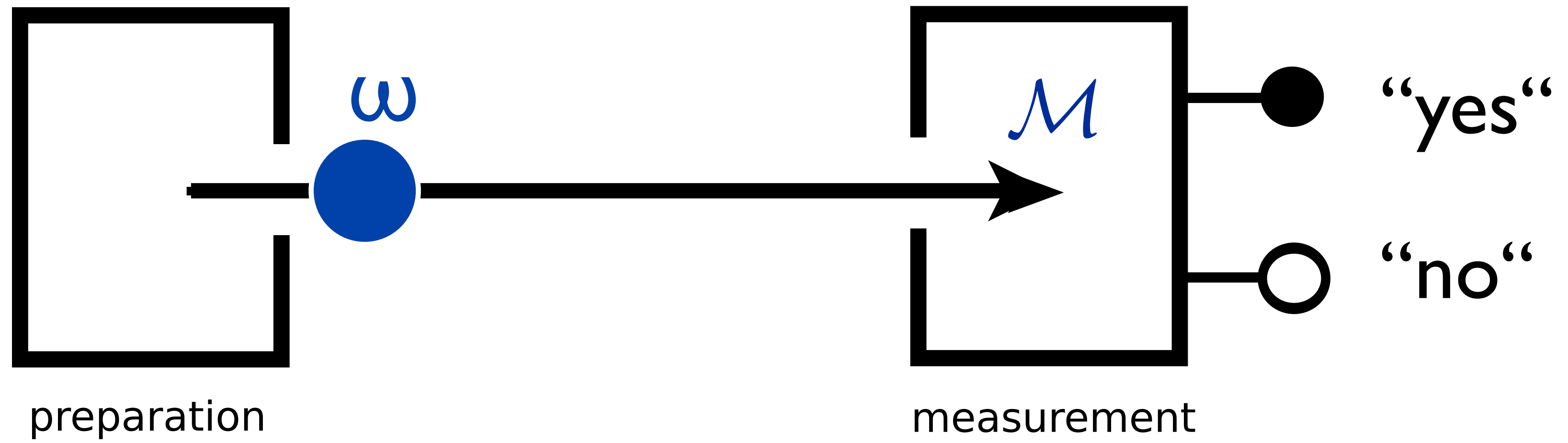

- Consequence: measurements ("effects") $\mathcal{M}$ are affine-linear:

$$\mathcal{M}(p\omega + (1-p)\varphi) = p\mathcal{M}(\omega) + (1-p)\mathcal{M}(\varphi).$$

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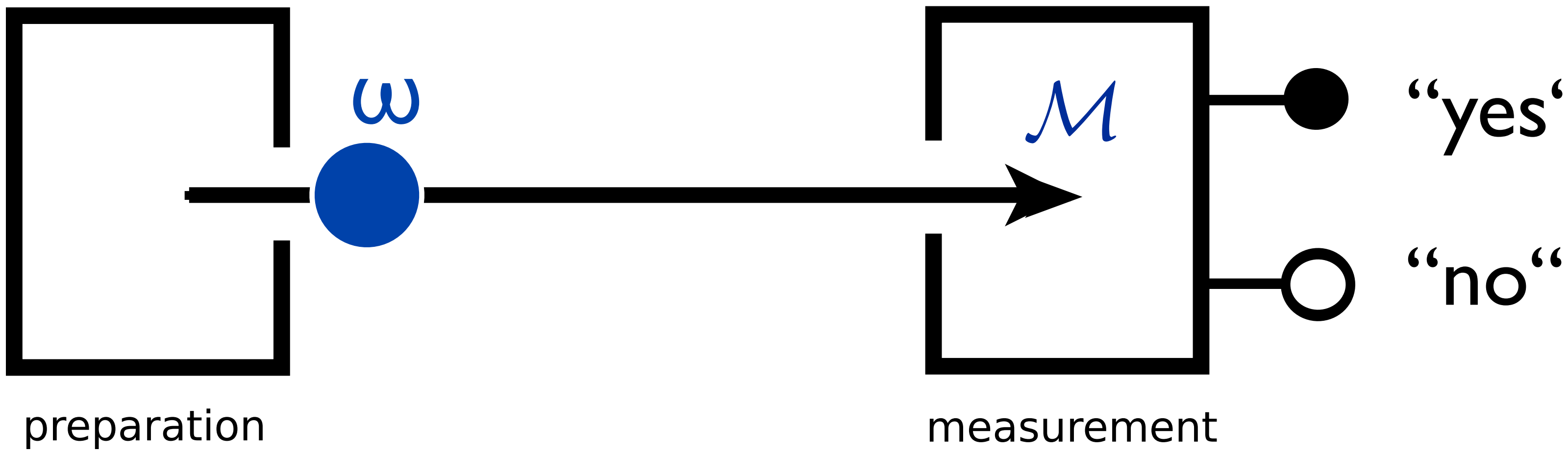

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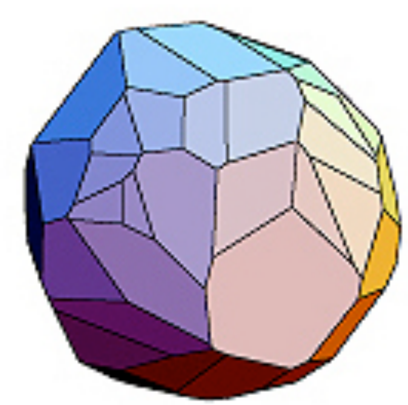
  Convex, compact, finite-dimensional.
  Otherwise arbitrary.



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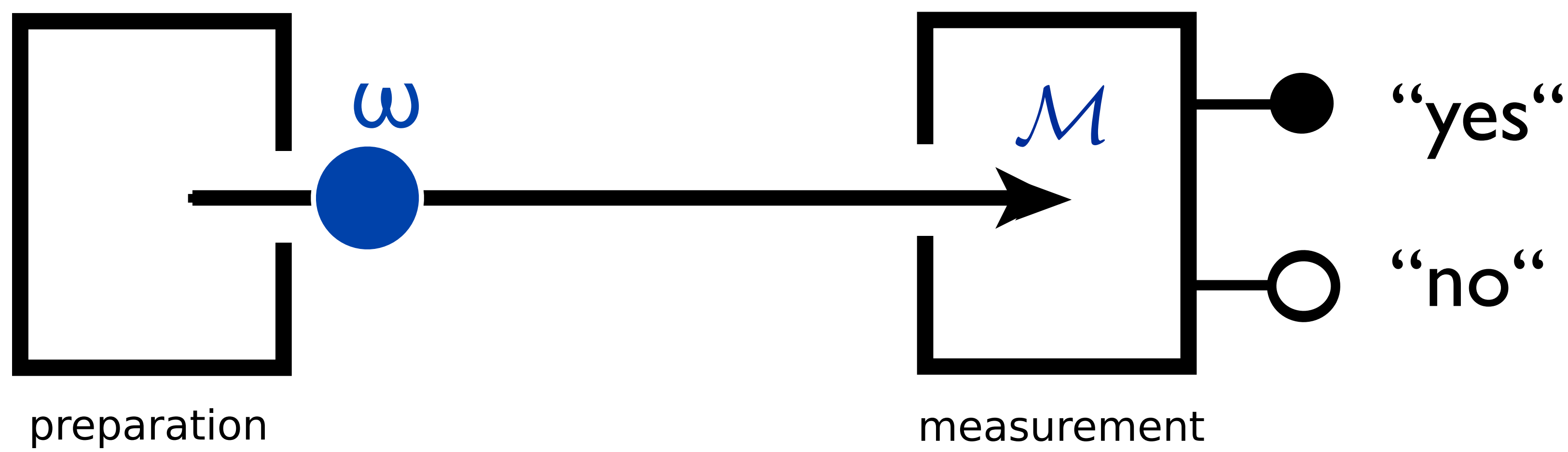

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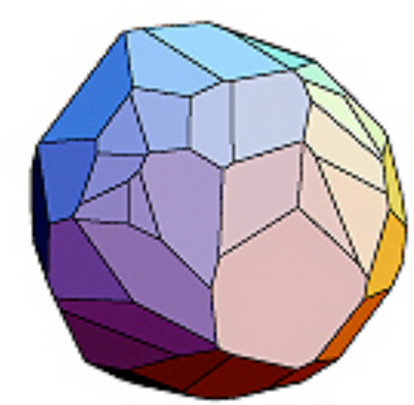
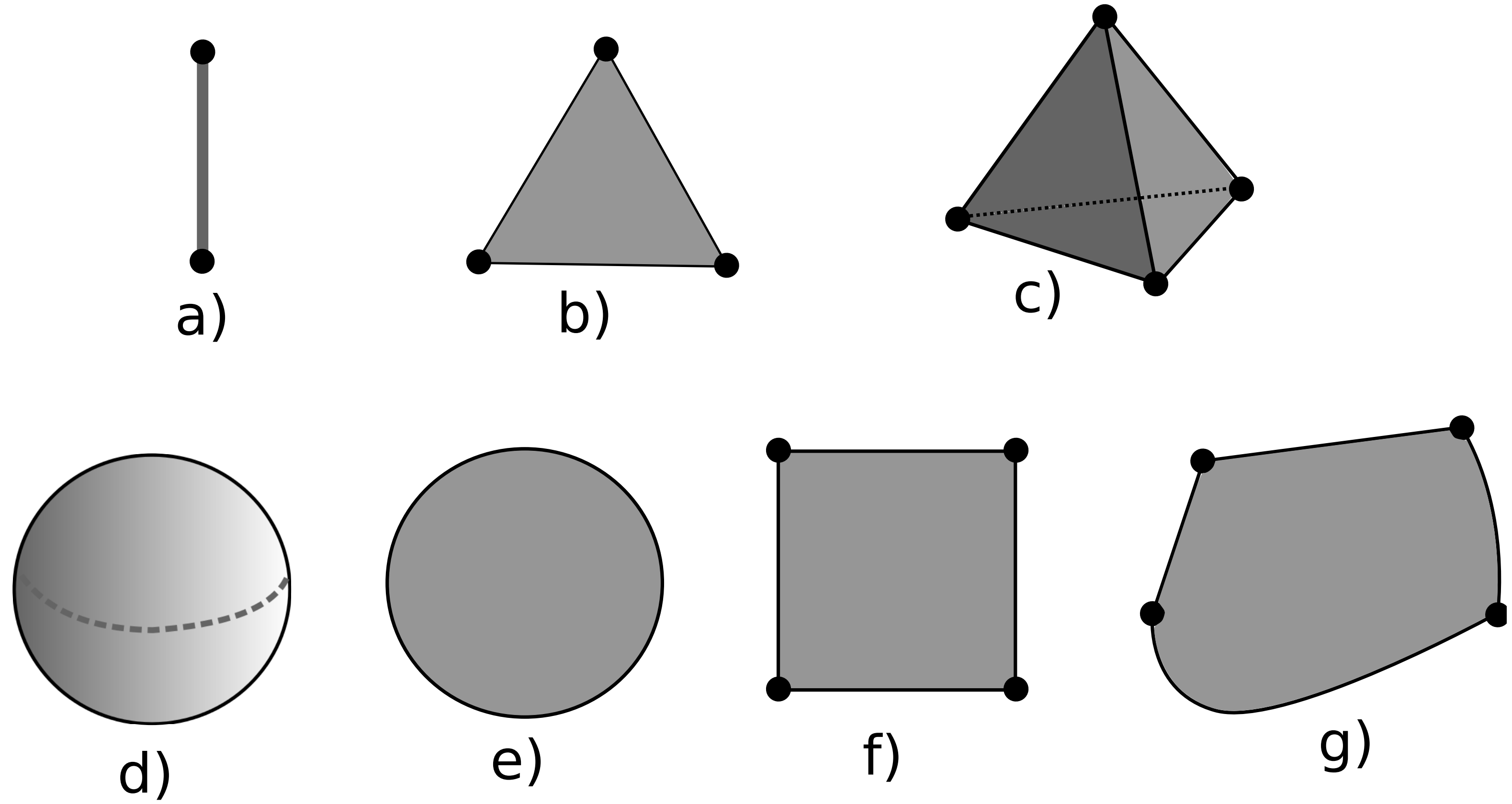
  Extremal points are pure states, others mixed.



Some examples:



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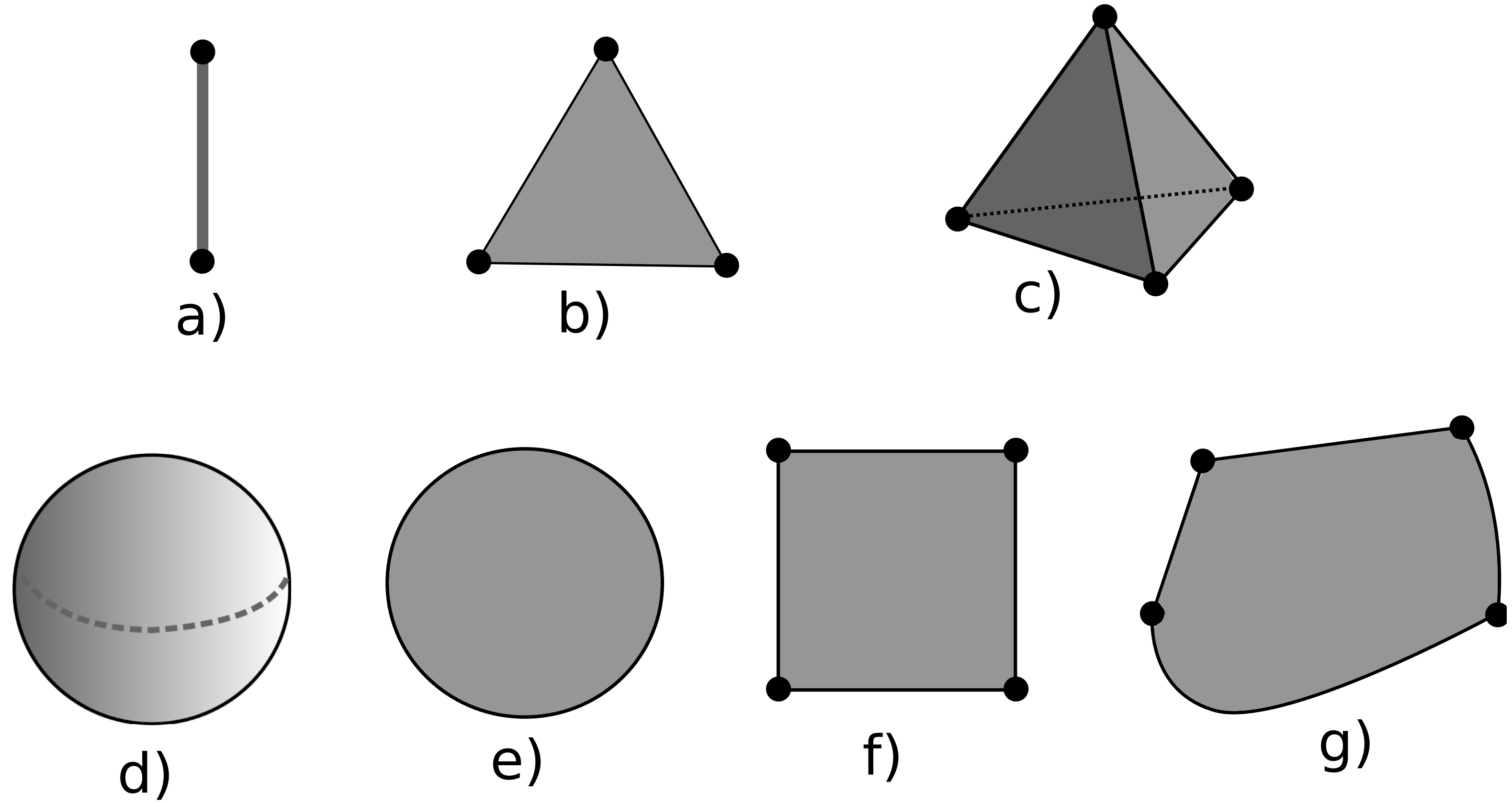

- Classical n-level system:

$$\Omega = \{\omega = (p_1, \ldots, p_n) \mid p_i \geq 0, \ \textstyle\sum_i p_i = 1\}.$$

n pure states: $\omega_1 = (1, 0, \ldots, 0), \ldots, \omega_n = (0, \ldots, 0, 1).$

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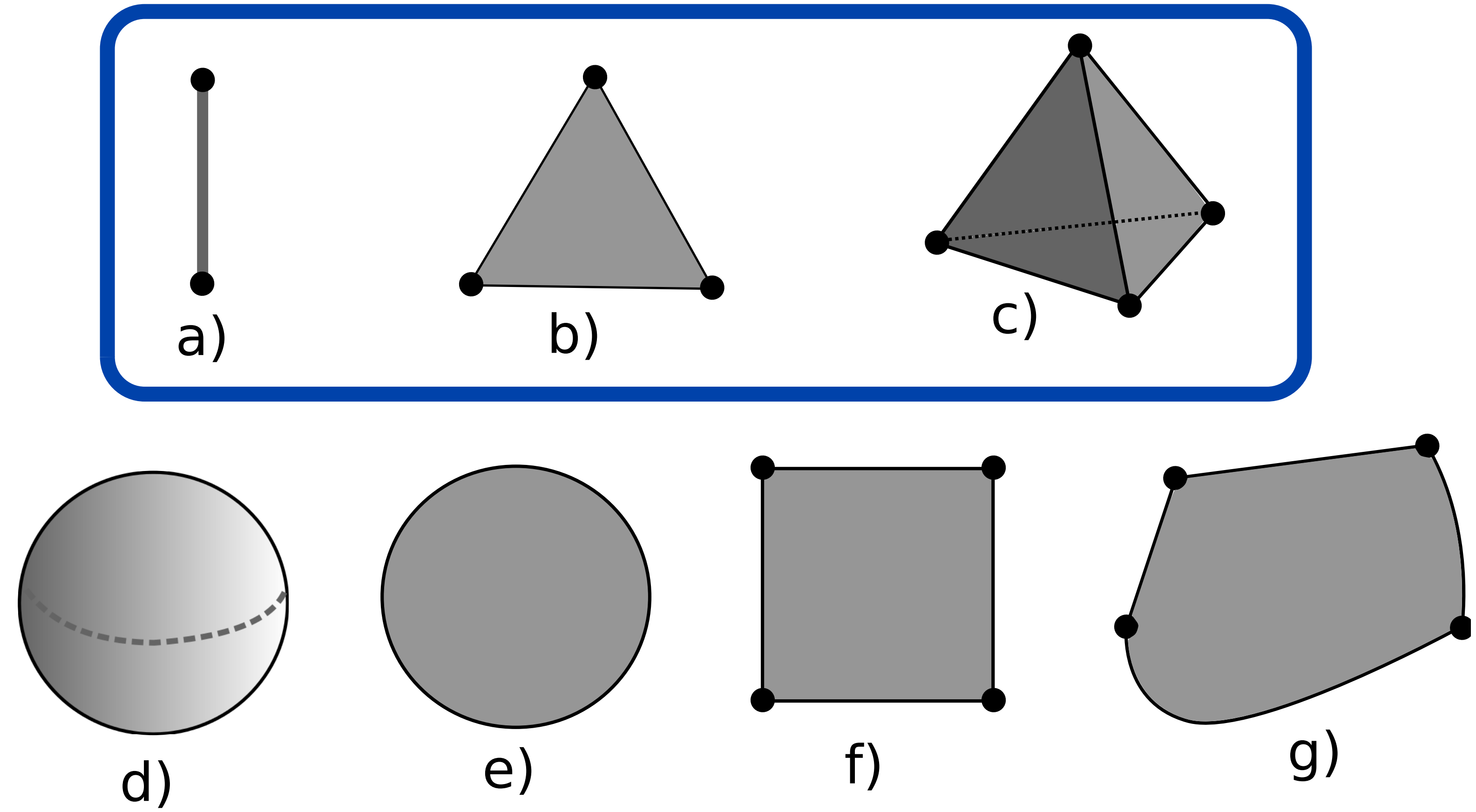

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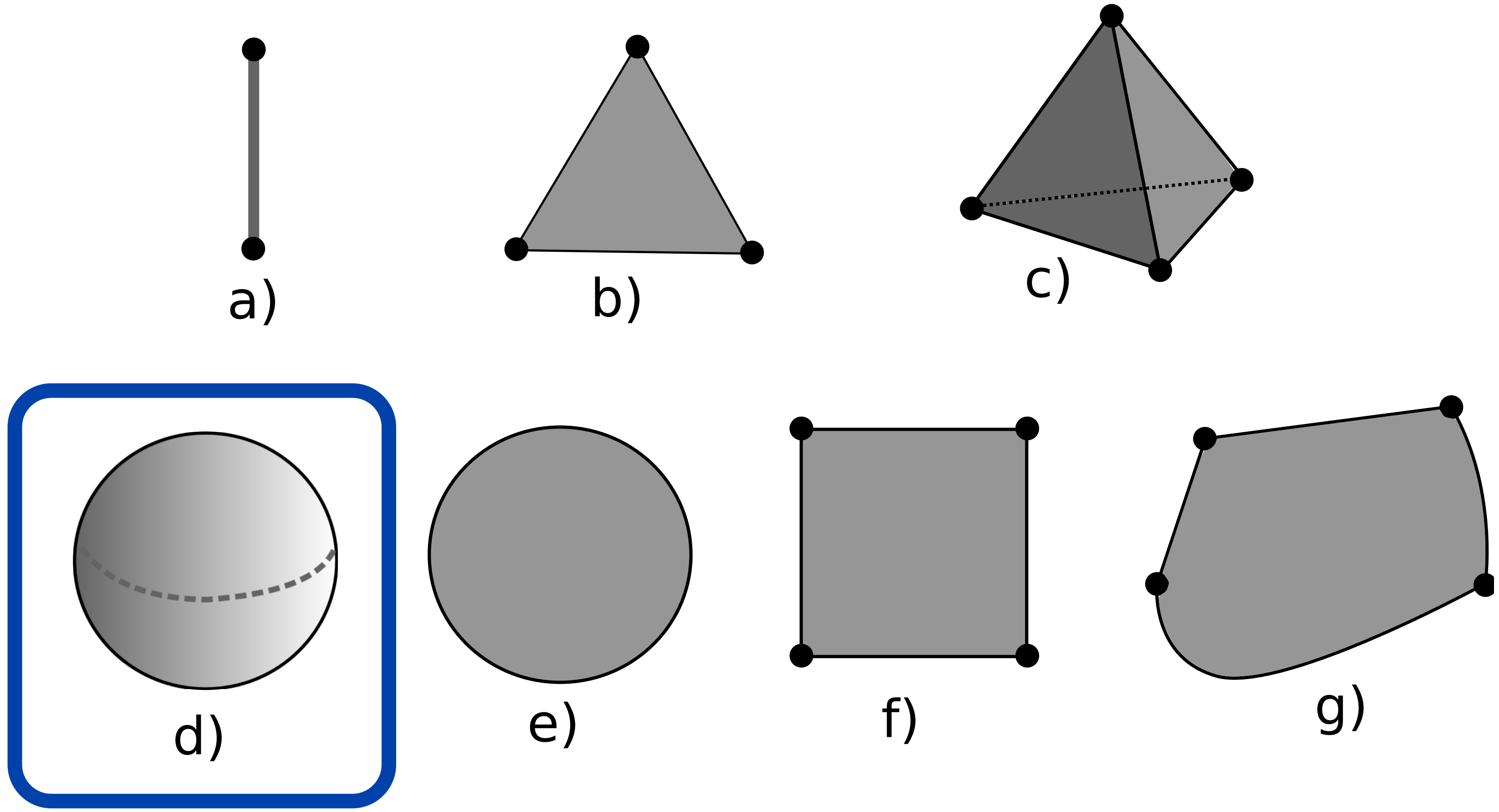

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- d): quantum 2-level system (qubit)

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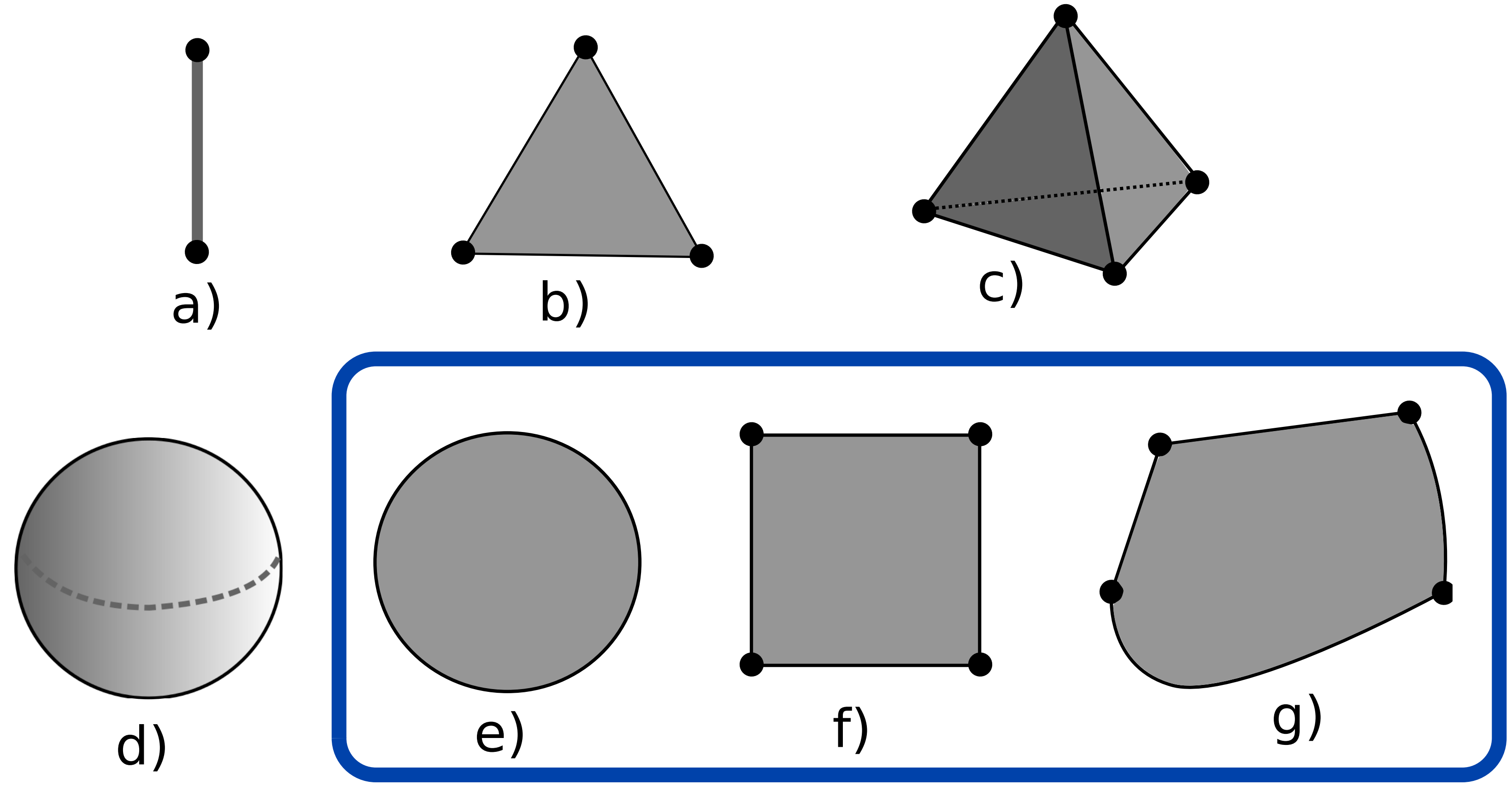
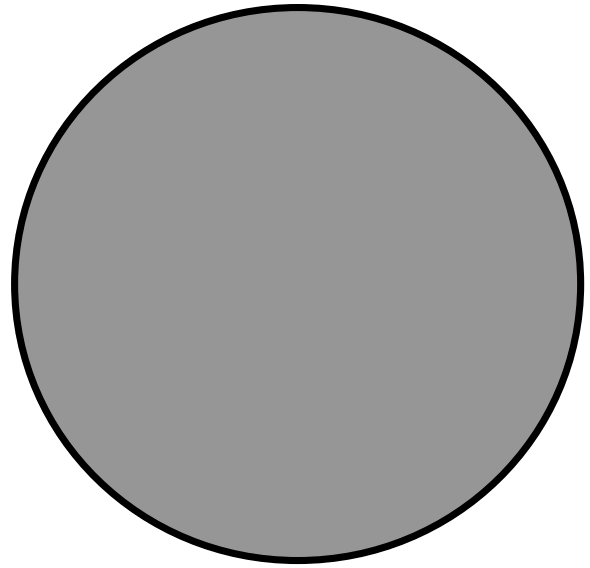


- e), f), g): neither classical nor quantum.



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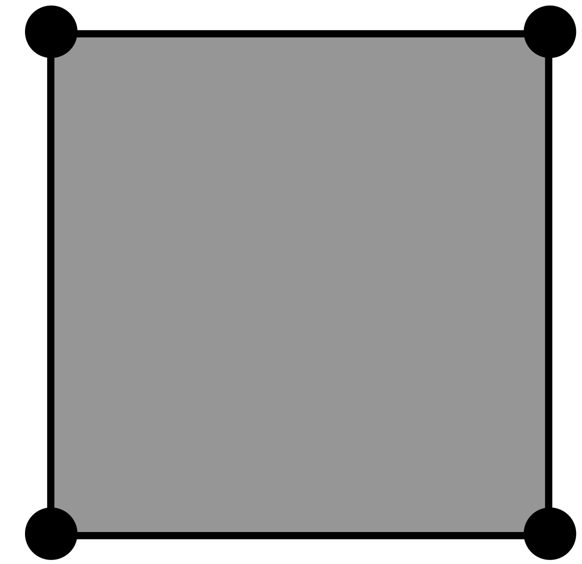

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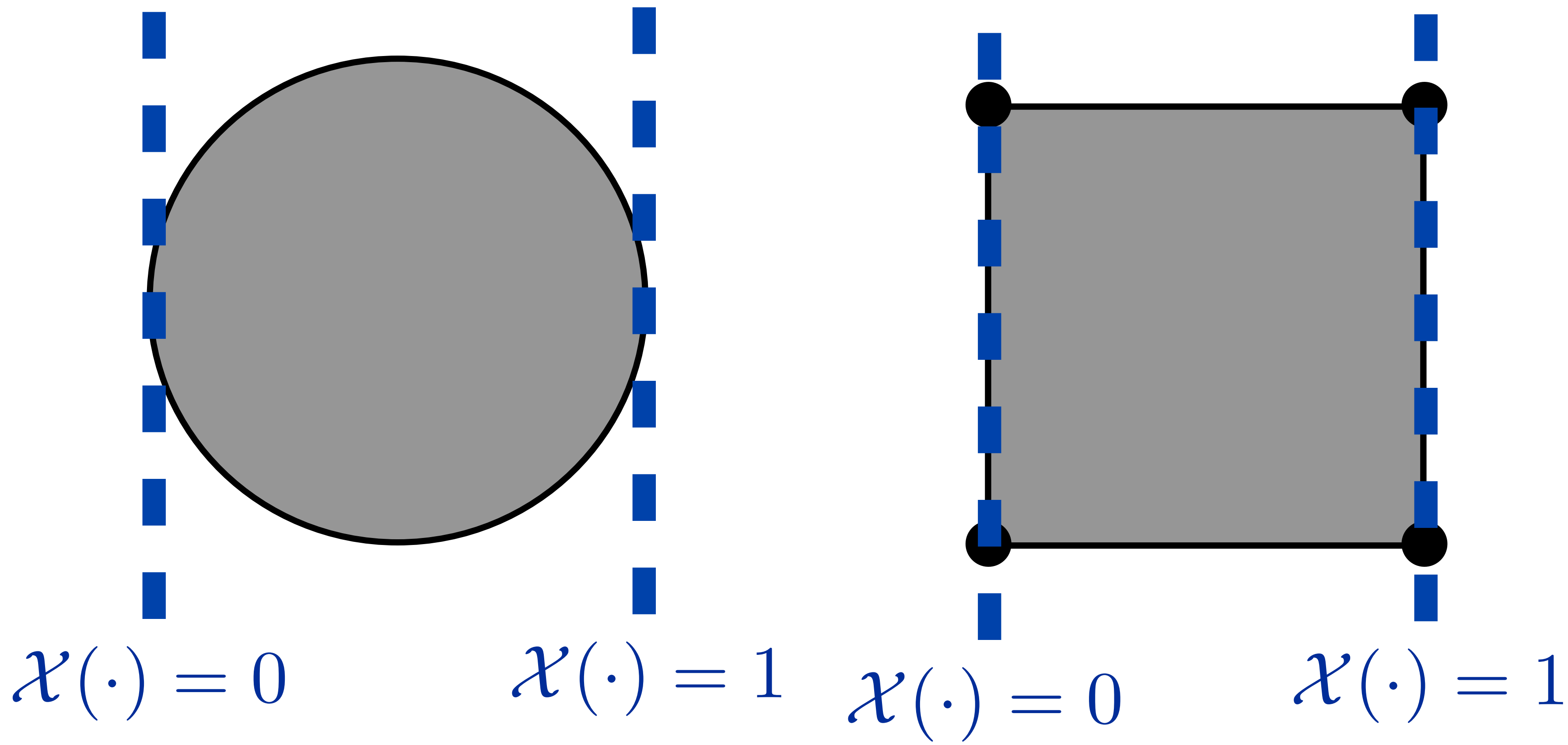

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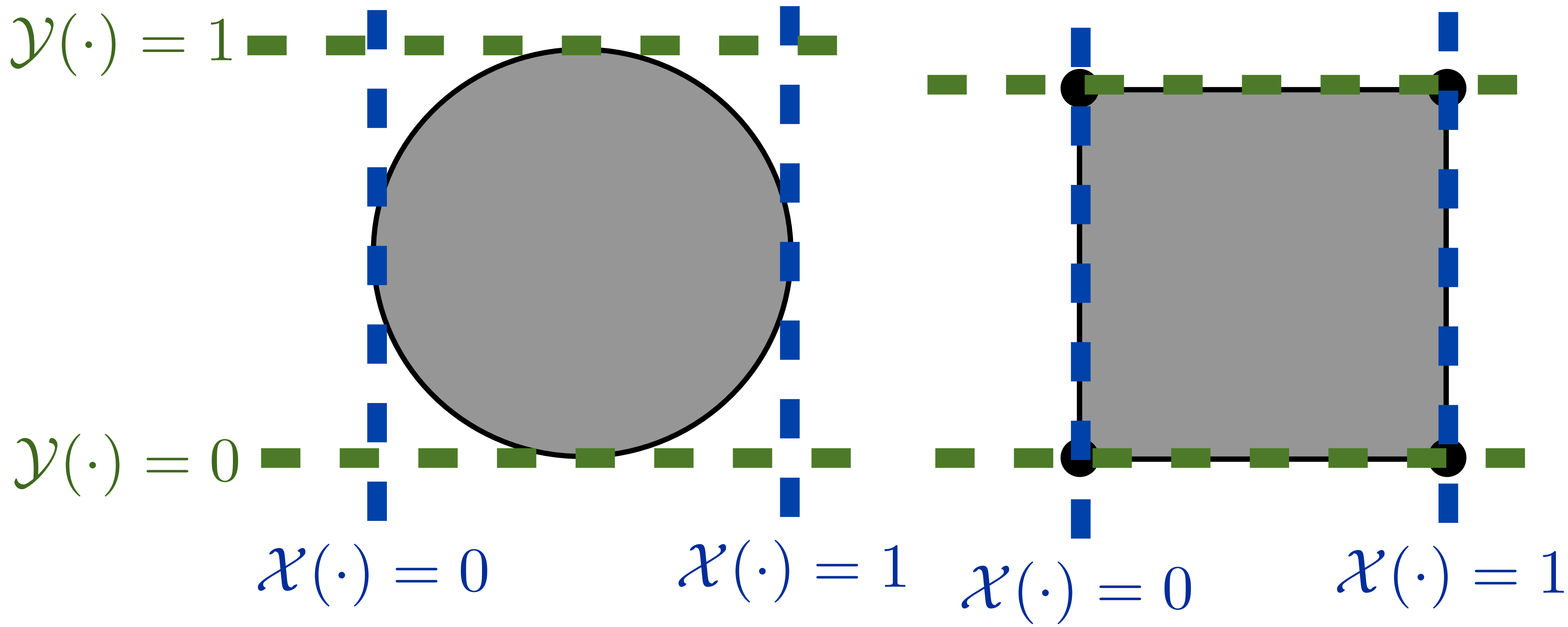
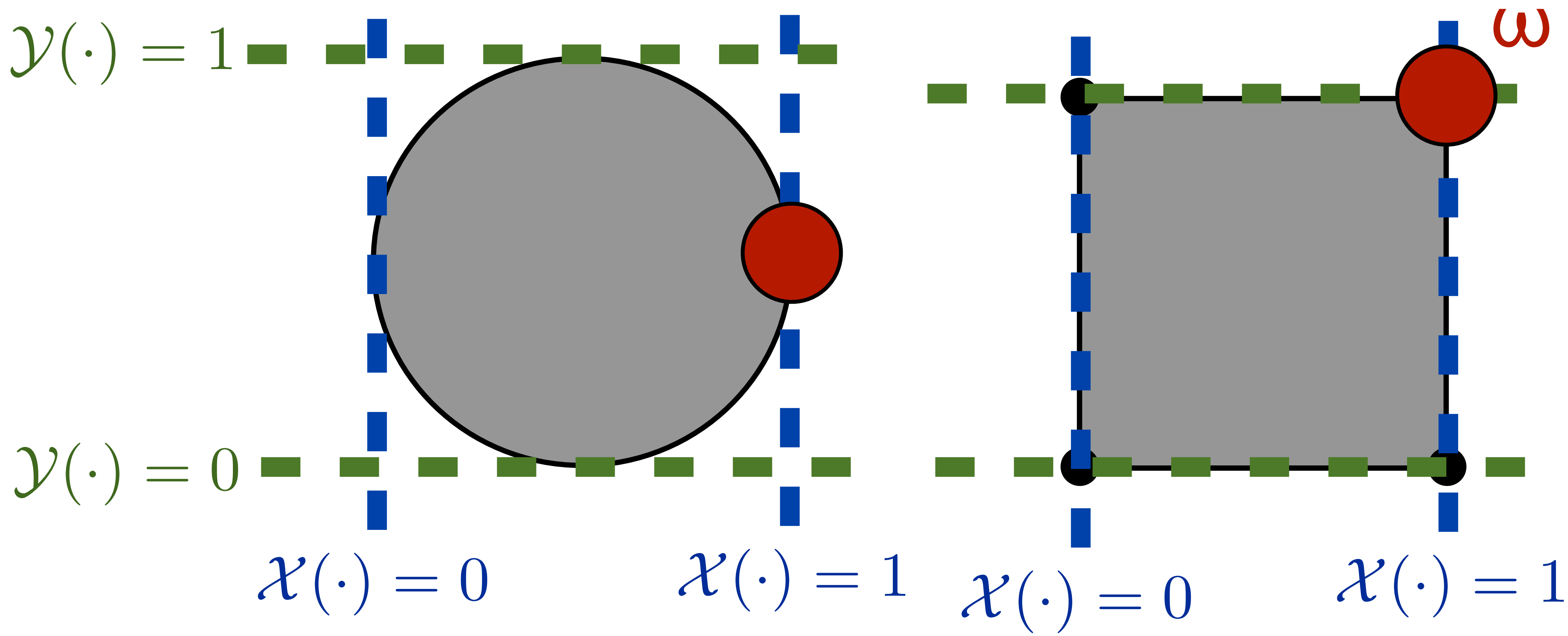


- e), f), g): neither classical nor quantum.
- e), f): let X=measurement "is spin up in x-direction?"

  analogously for Y.



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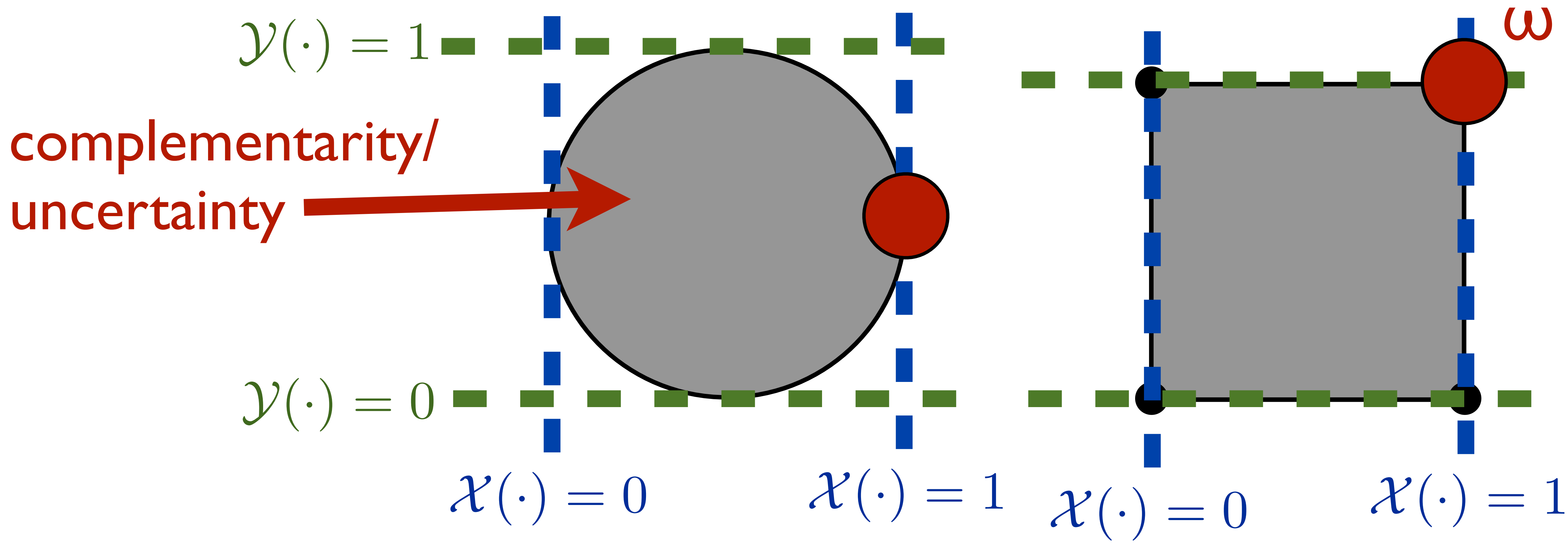

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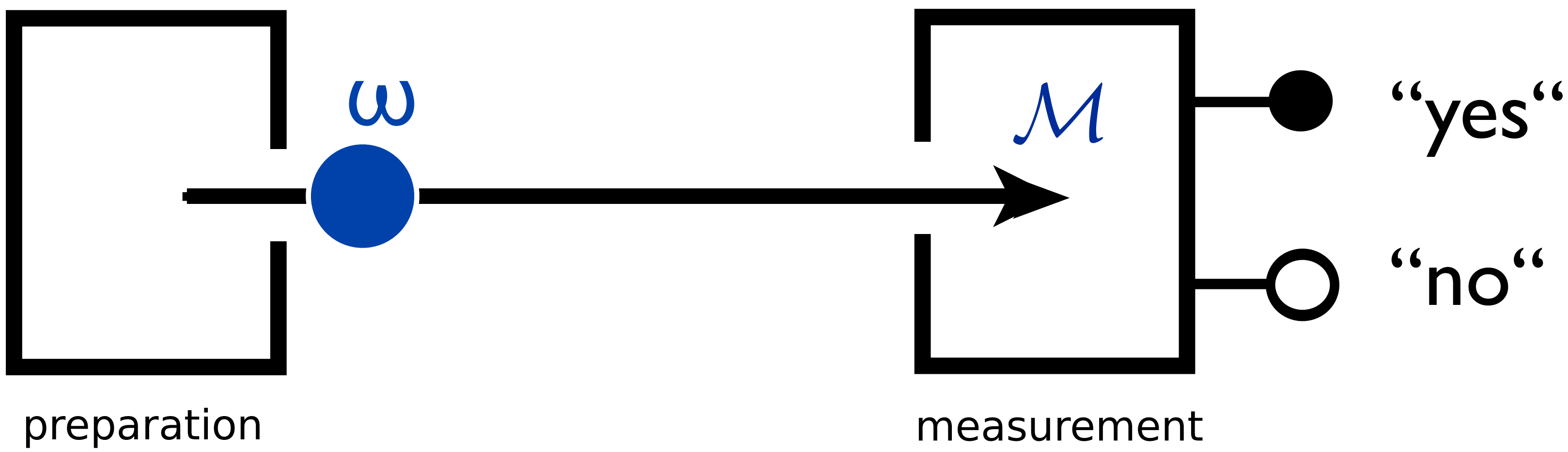

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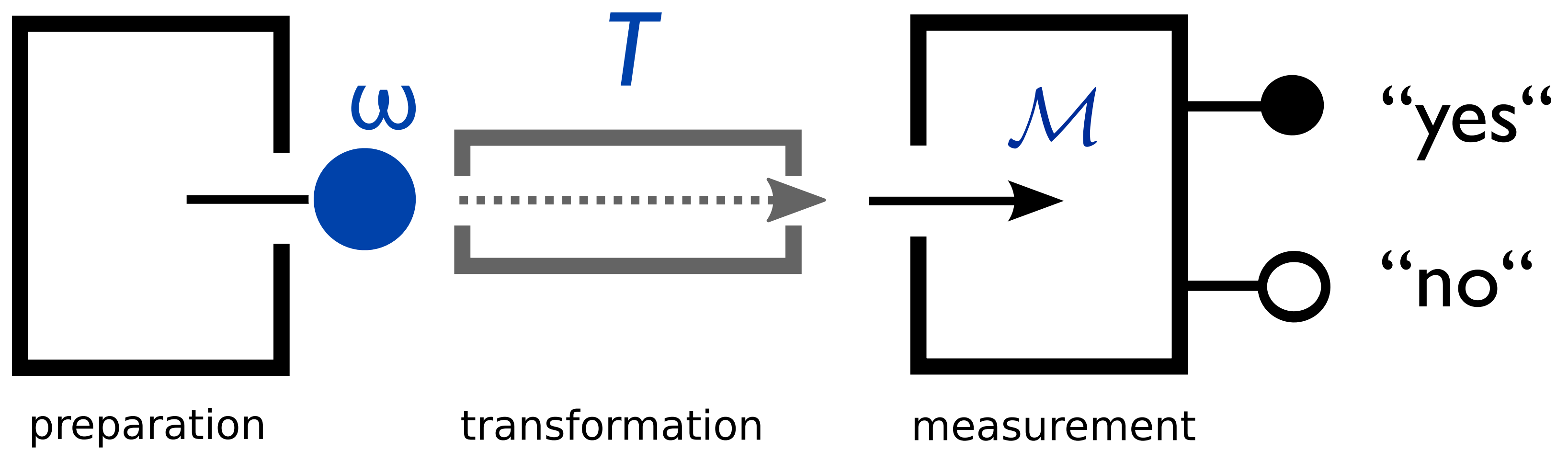

Transformations T map states to states and are linear.

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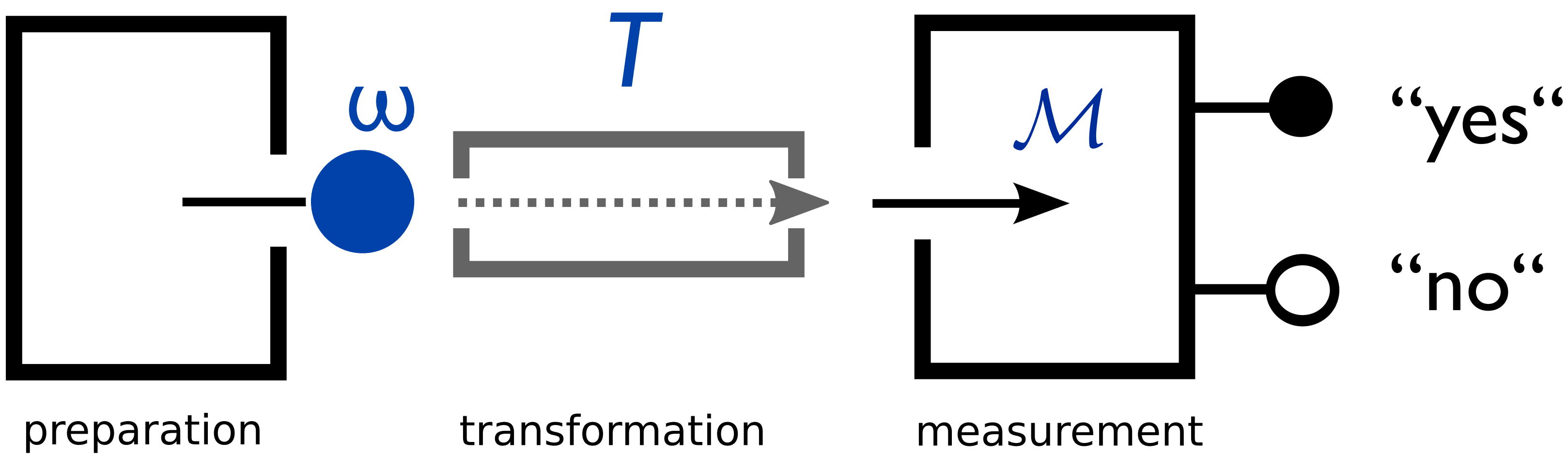

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- Here, only interested in reversible transformations T (i.e. invertible).
- They form a compact (maybe finite) group $\mathcal{G}$.
- In quantum theory, these are the unitaries:

$$\rho \mapsto U \rho U^\dagger .$$

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Assumption: physics takes place in *d* spatial dimensions (+ time).
All we consider happens locally + at rest ⟶ Euclidean space.

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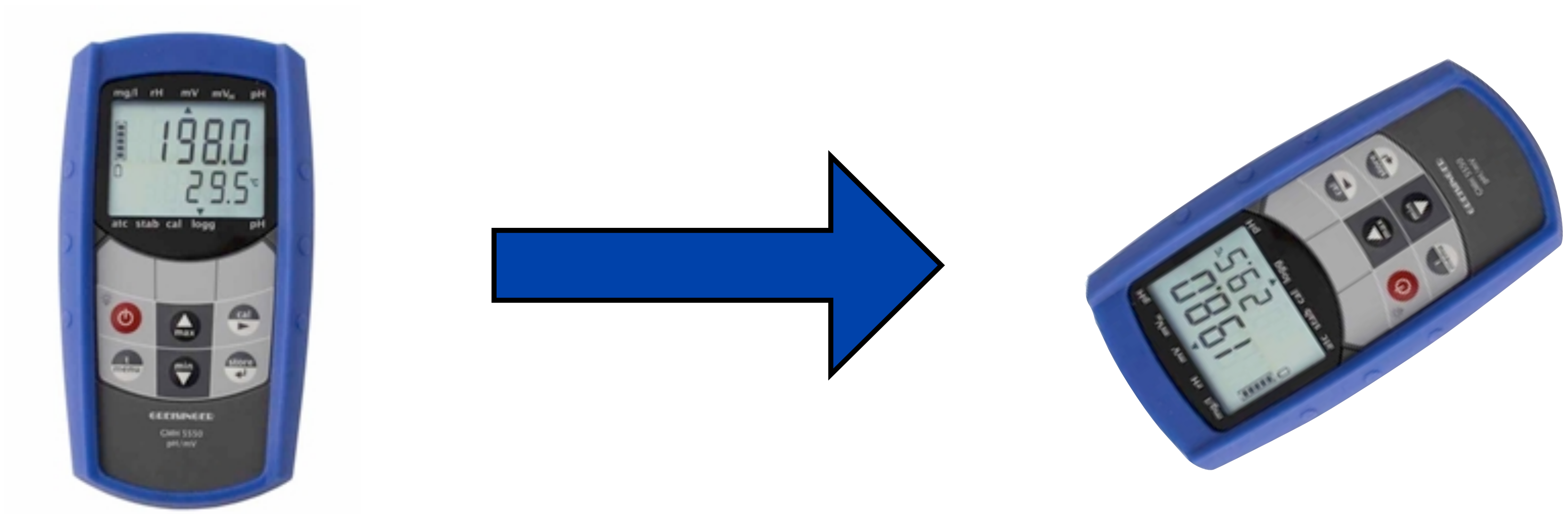

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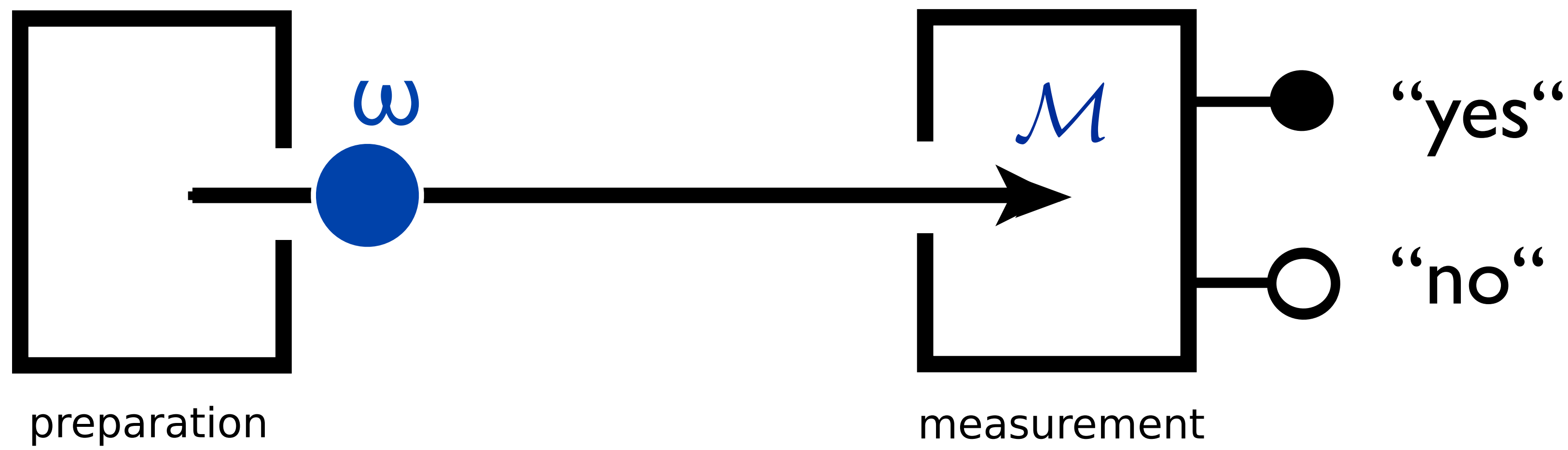

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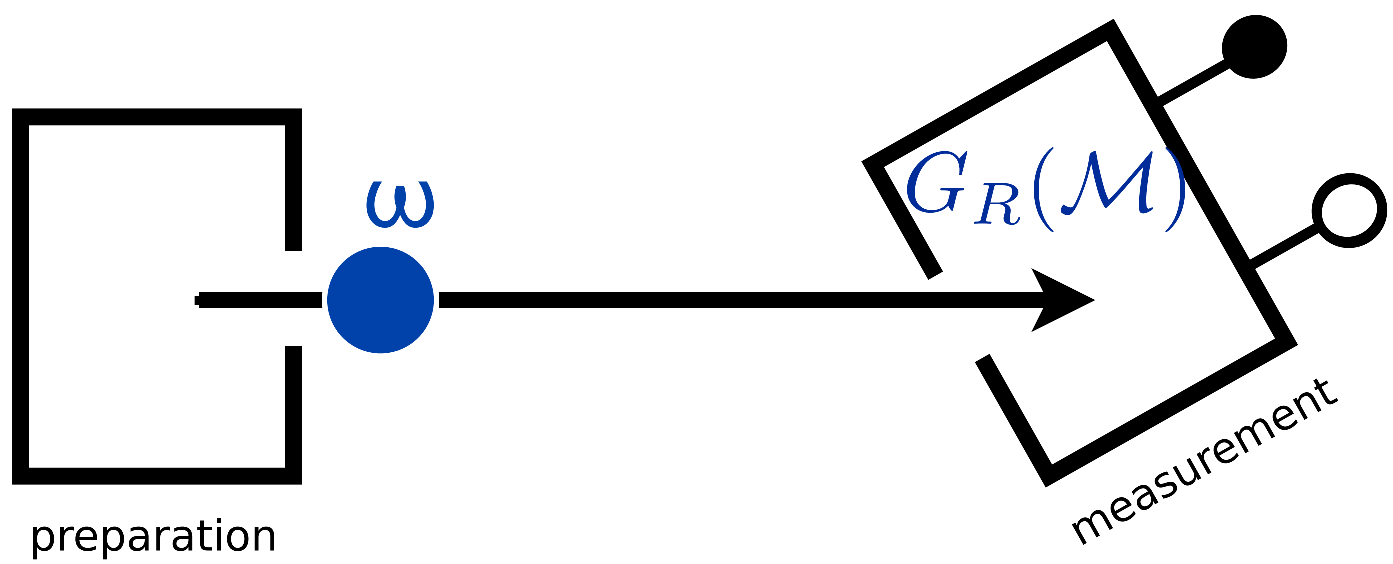

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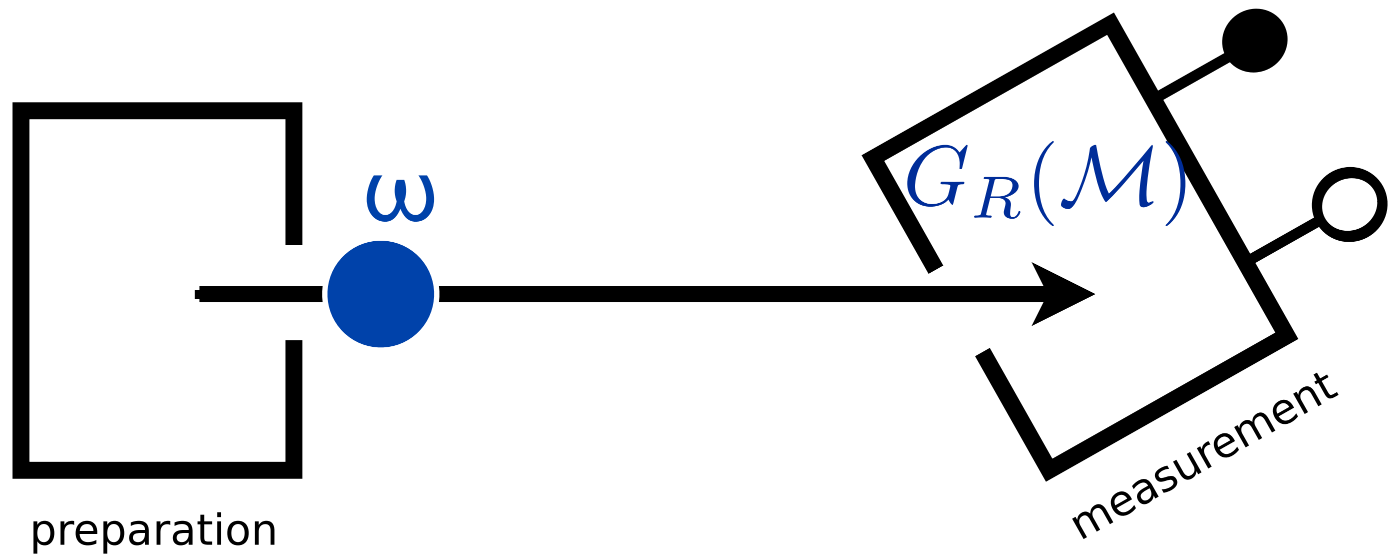

- Macroscopic objects can be subjected to SO(*d*) rotations.
- Rotation of measurement device $\mathcal{M}$: linear group representation $G_R \quad (R \in SO(d))$ such that $\mathcal{M} \mapsto G_R(\mathcal{M})$.

$$G_R(\mathcal{M})(\omega) = \mathcal{M}(G_R^*(\omega)).$$

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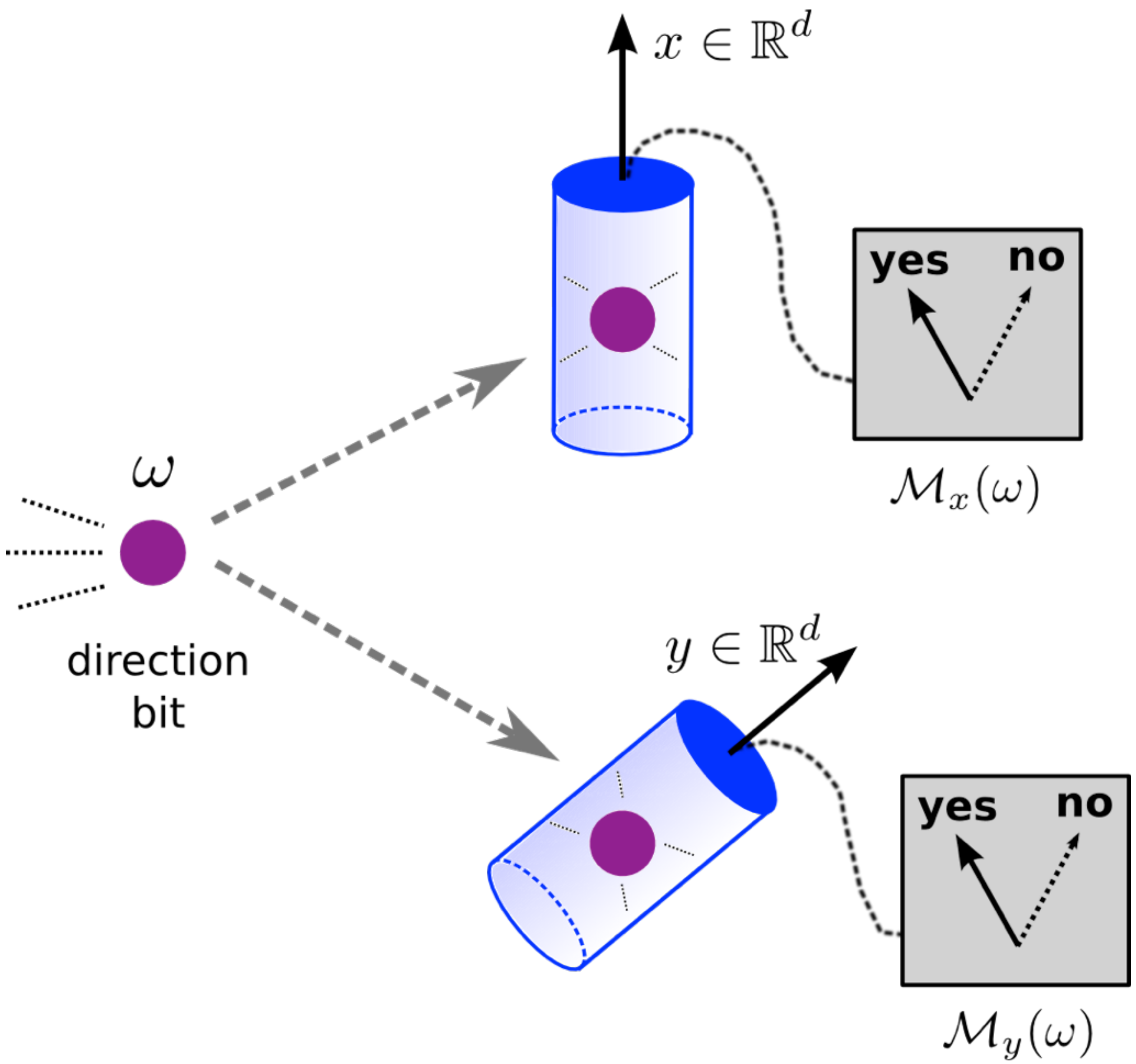

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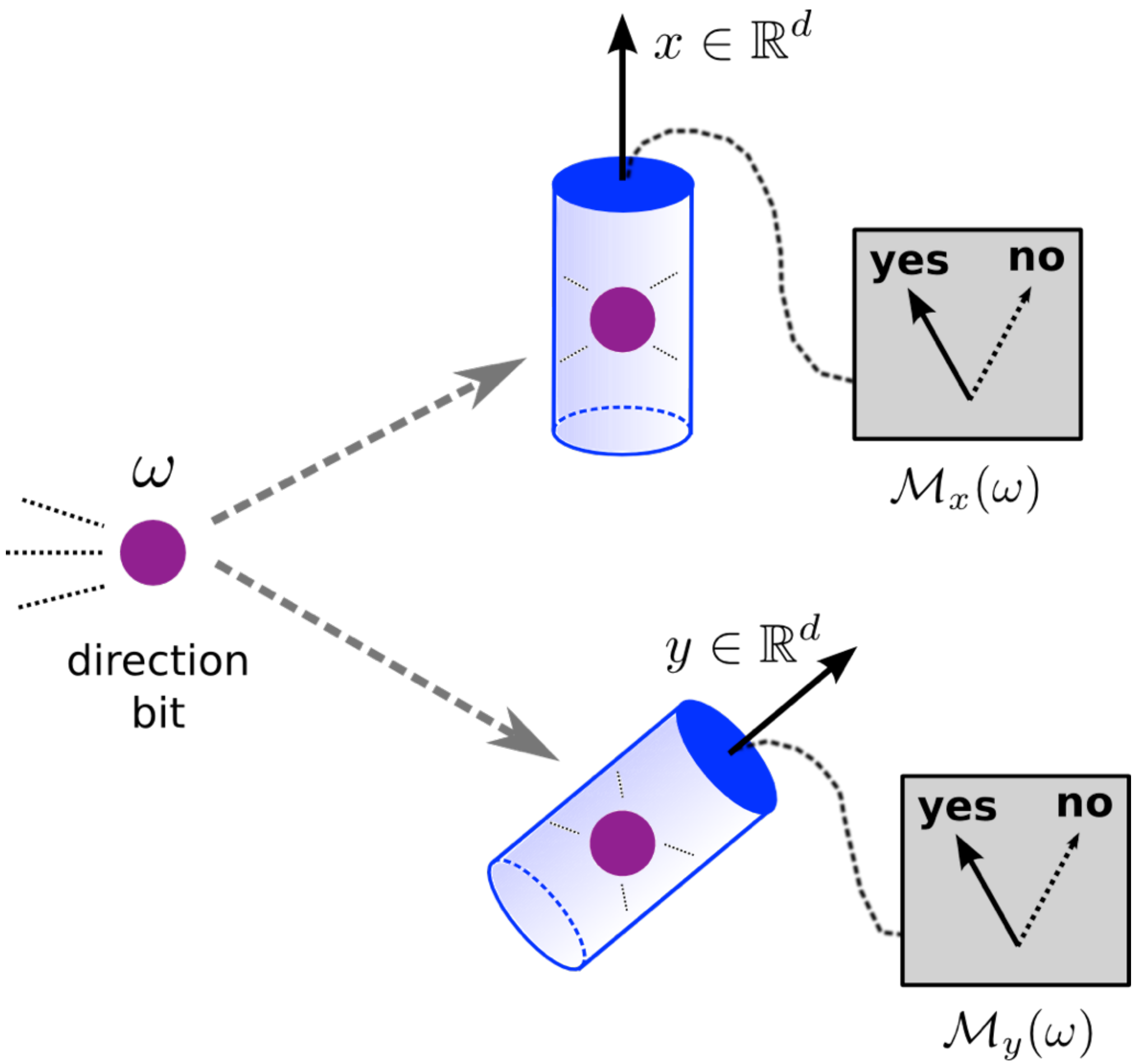

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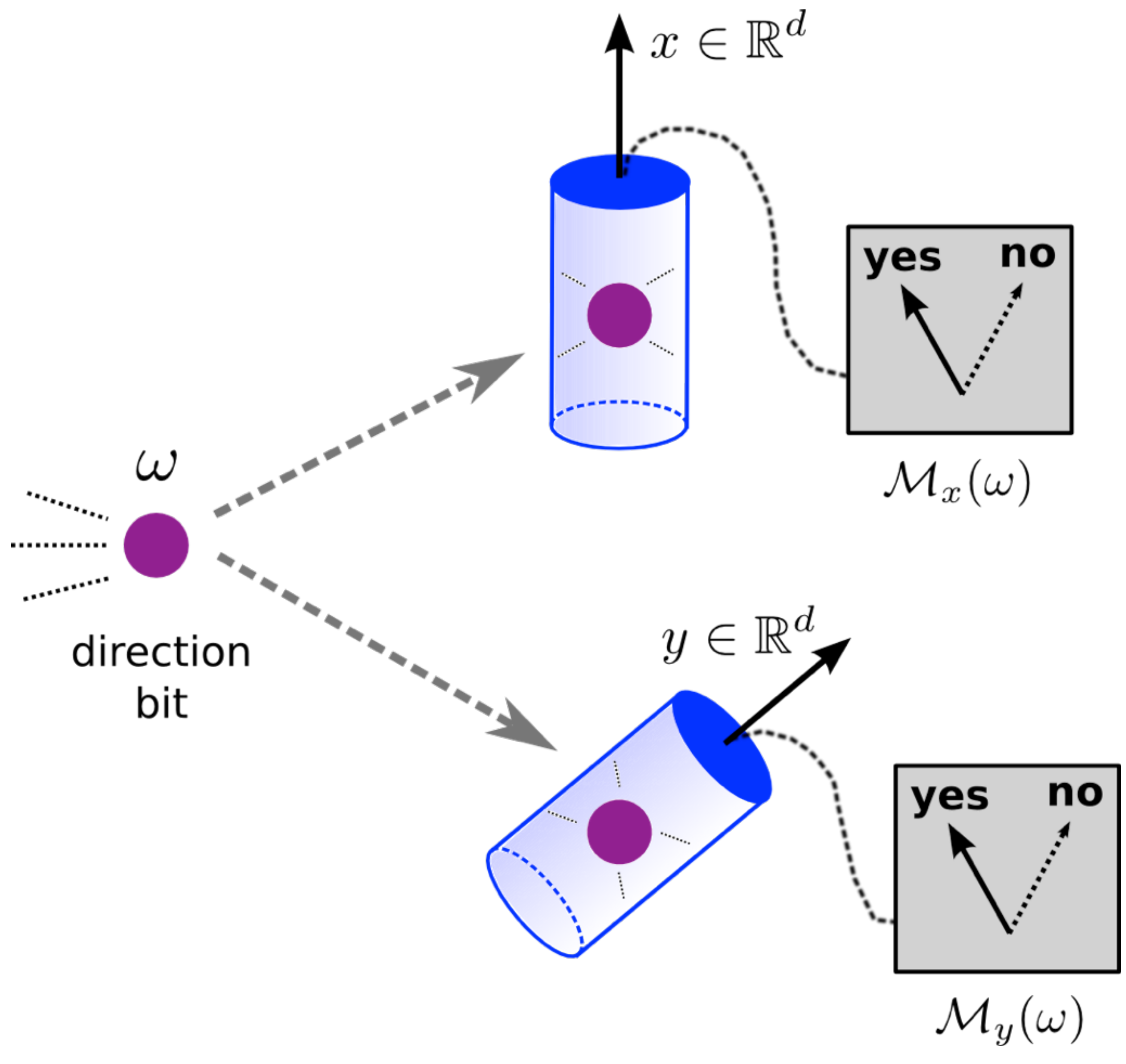

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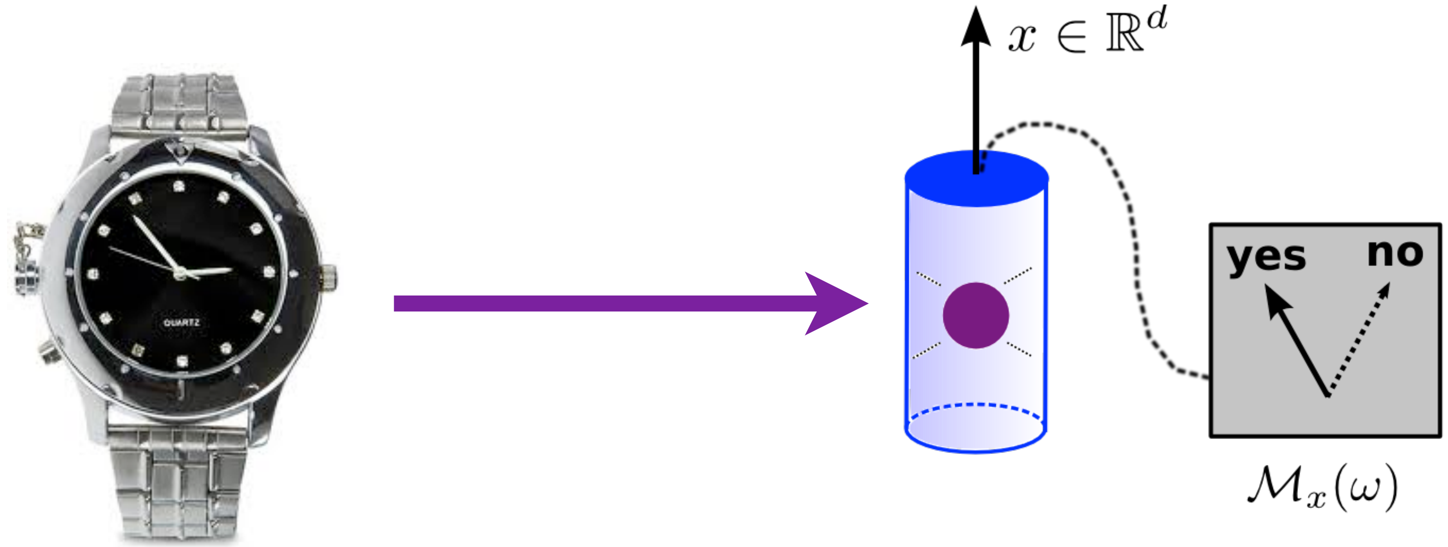

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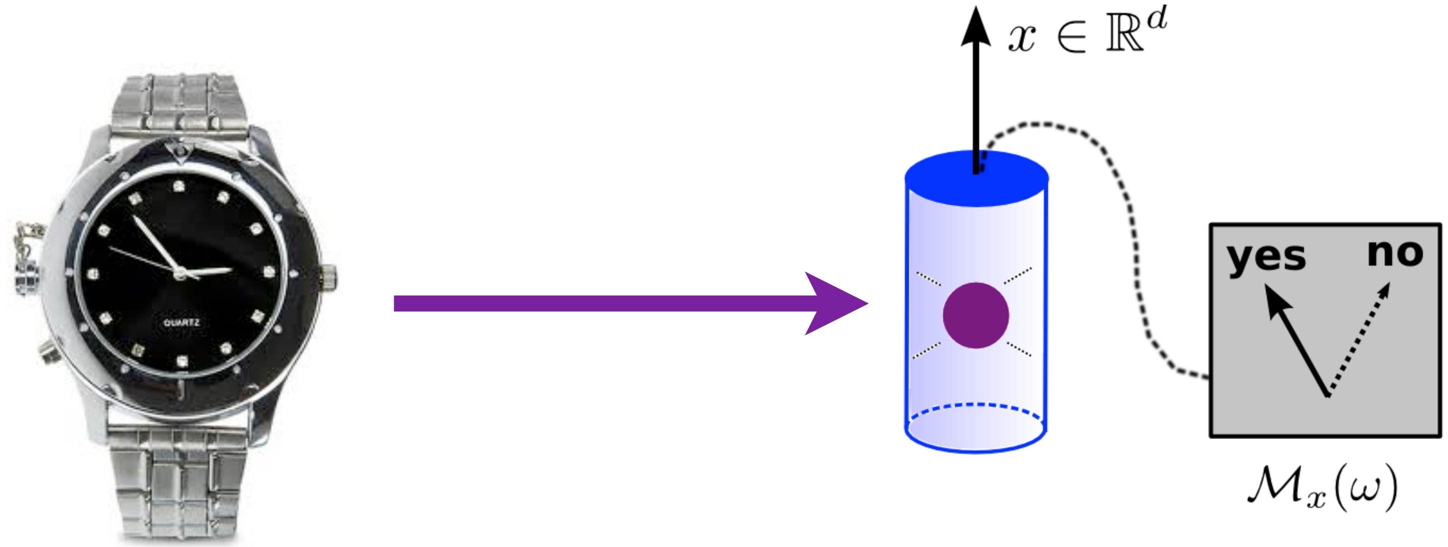

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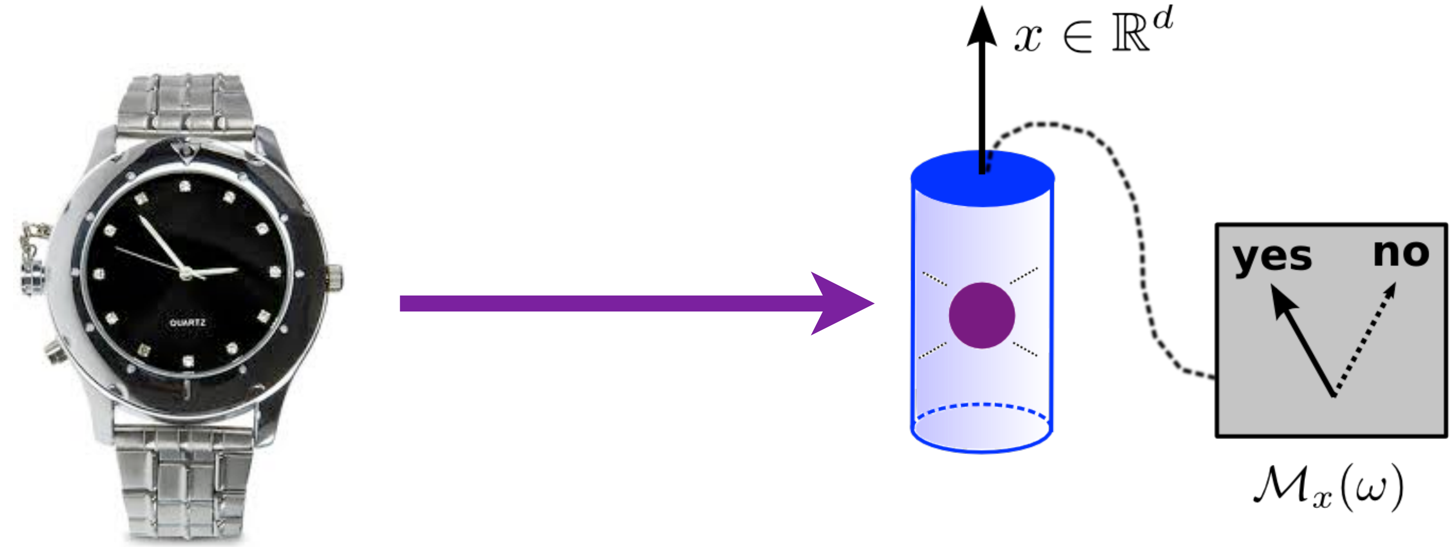

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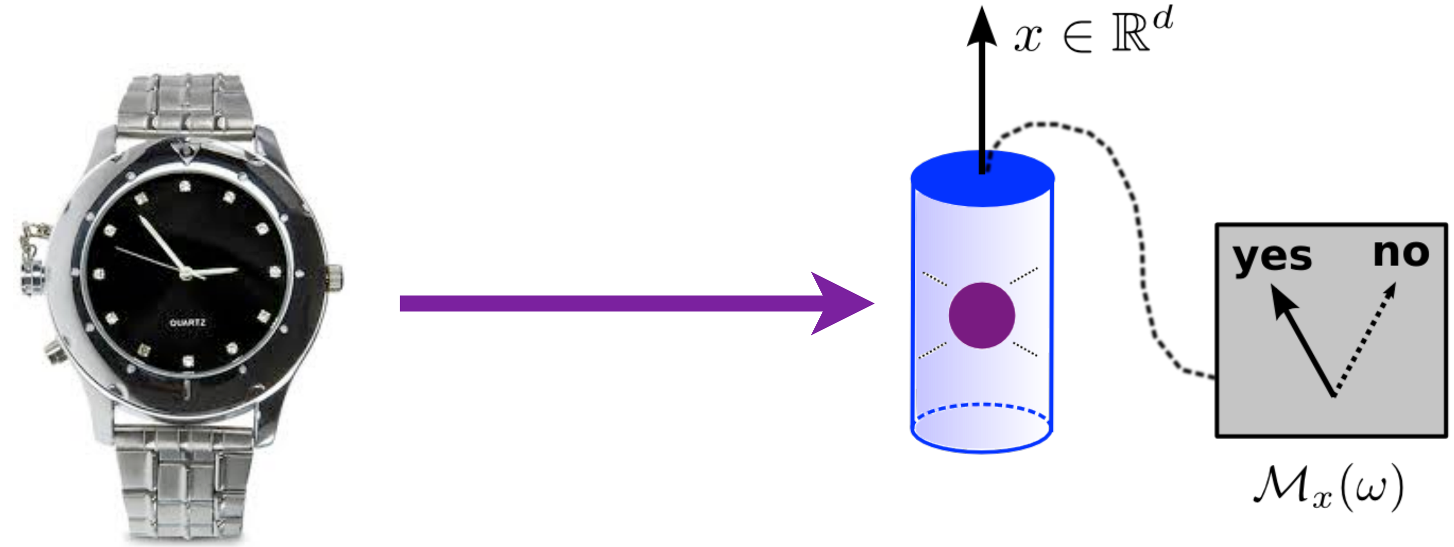

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- Interpretation: system carries information on direction *x* (and intensity) *and nothing else*.

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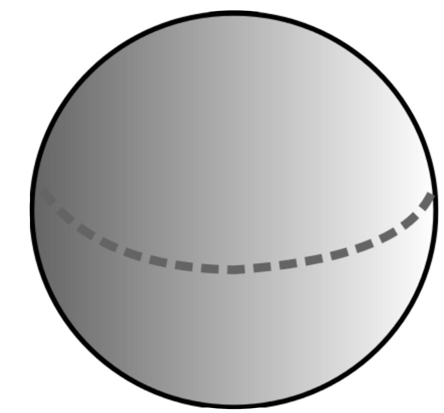

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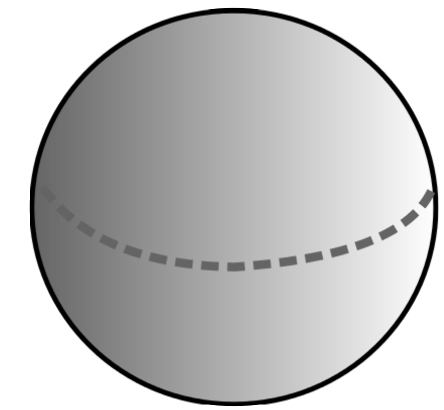

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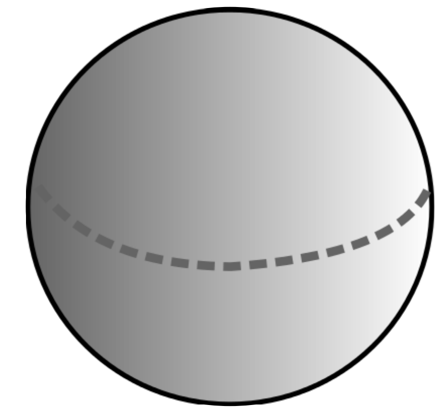

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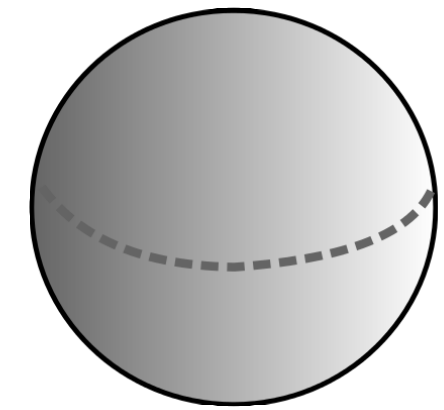

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- Group rep. theory: inner product such that $|\vec{\omega}_y| = 1 \text{ for all } y.$
- Postulate B $\Rightarrow$ every state can be written $\omega = \lambda \omega_x + (1 - \lambda)\mu.$

$\Rightarrow$ *D*-dim. ball. Dimension counting $\Rightarrow$ *D*=d.

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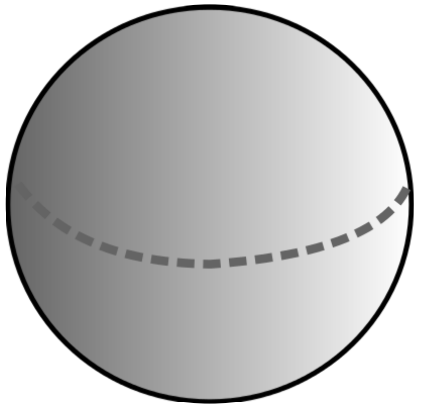

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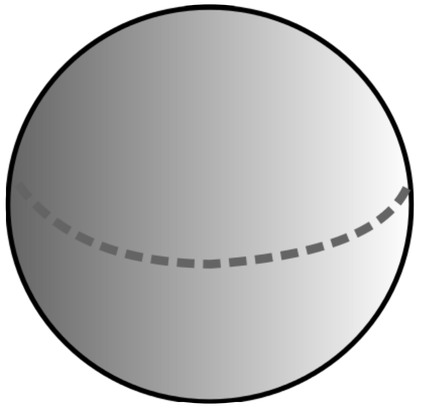

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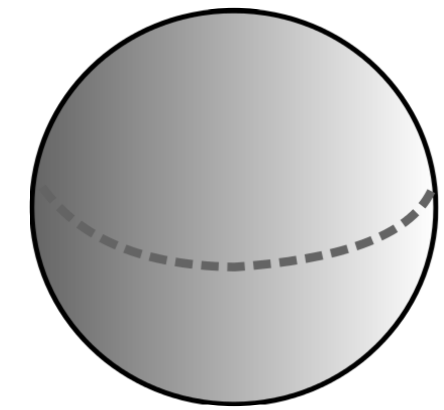

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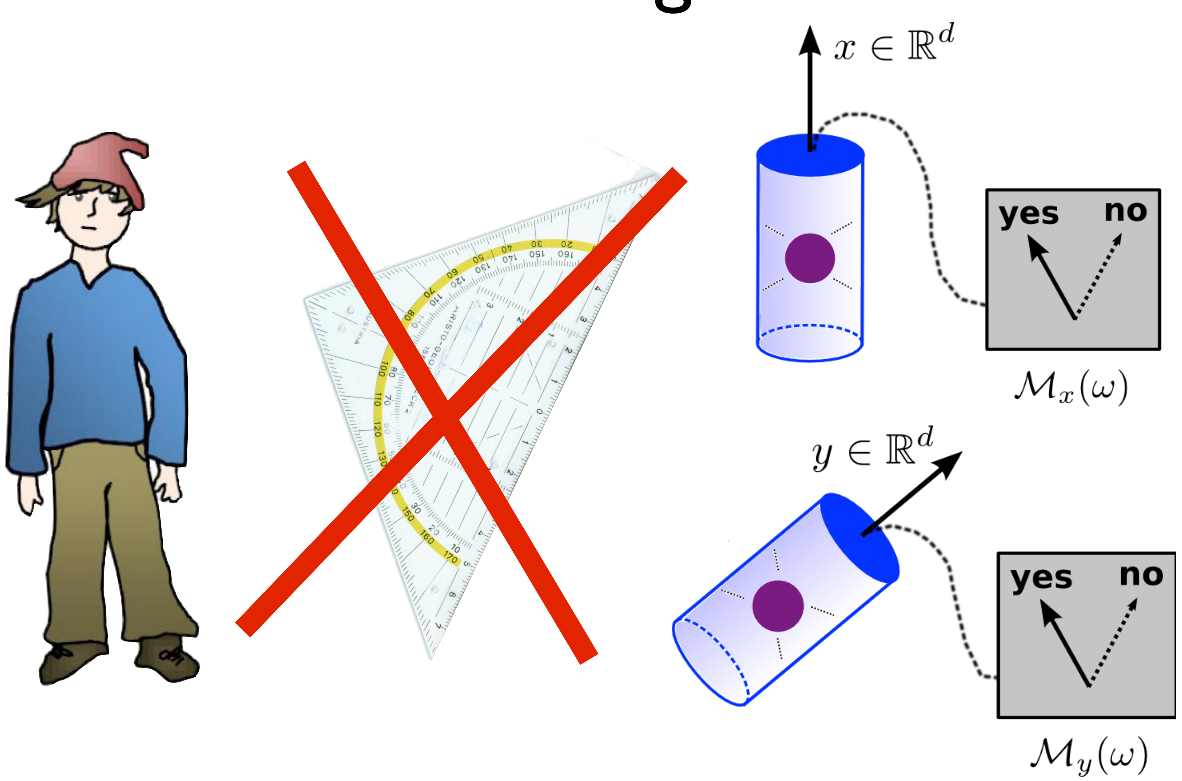

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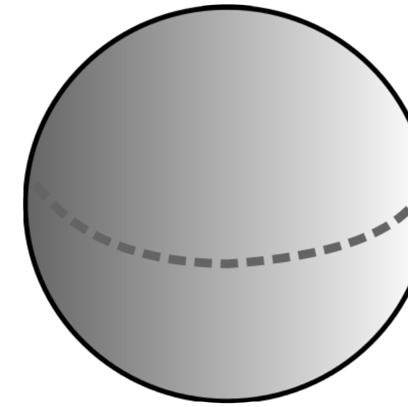

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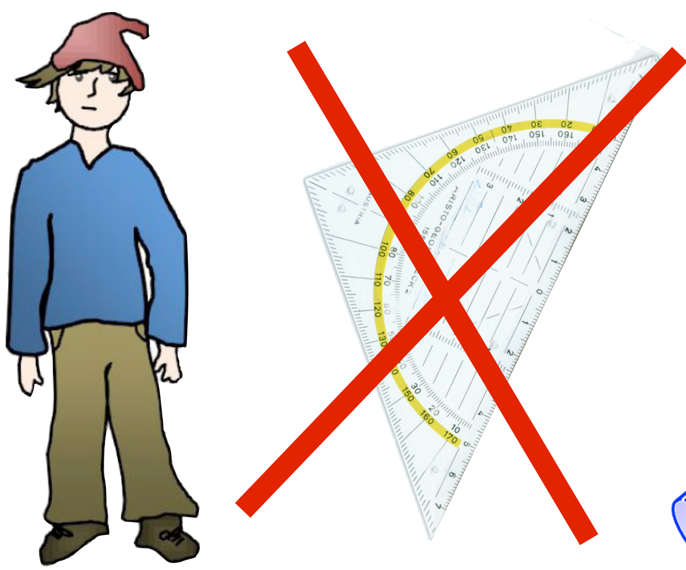

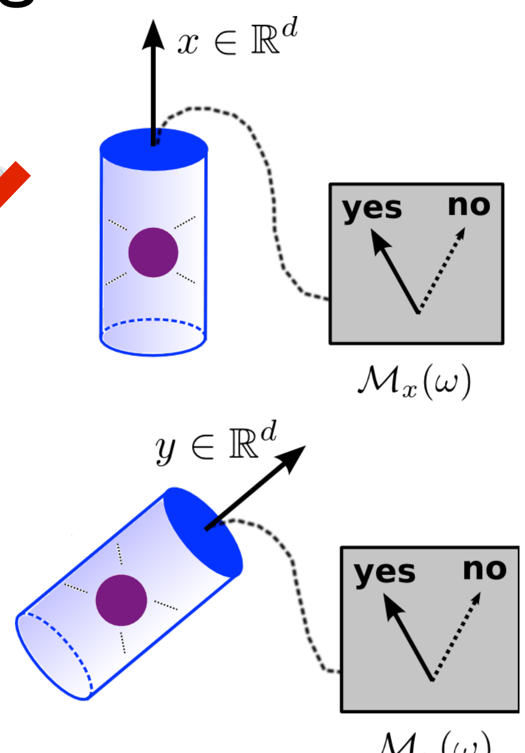

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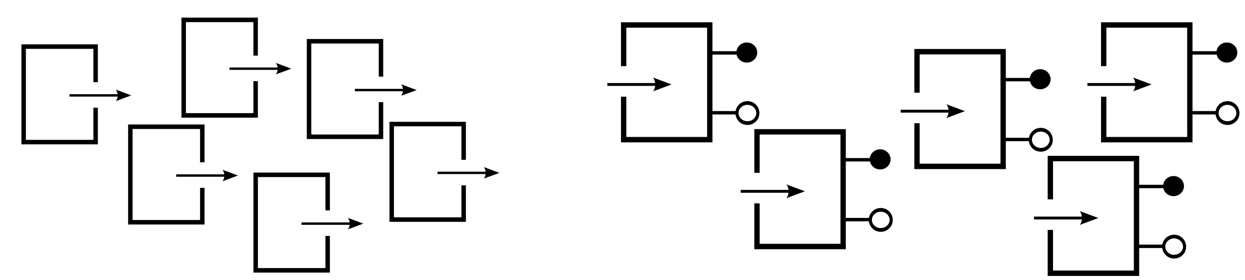

3. Postulates A+B

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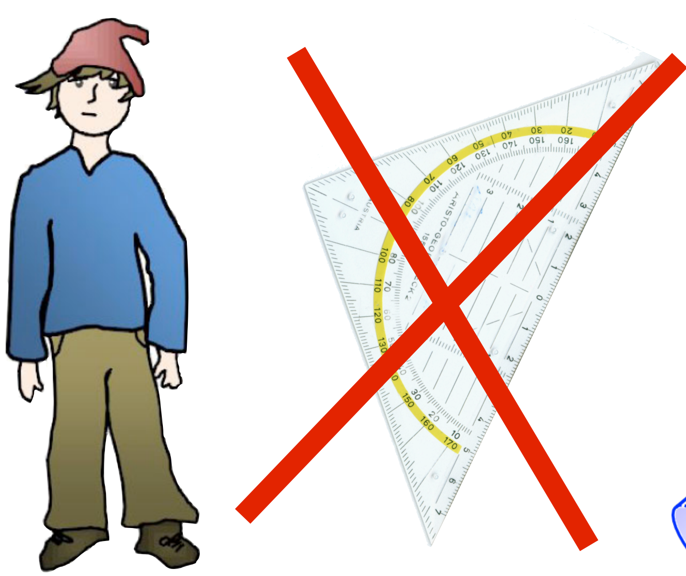

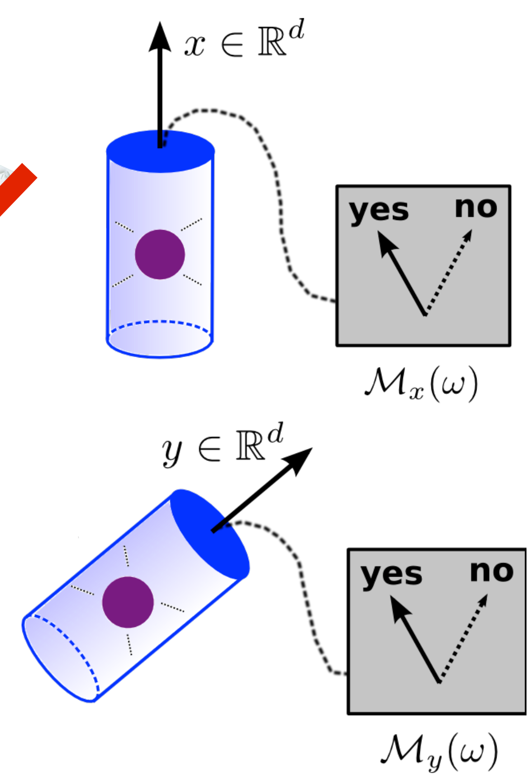

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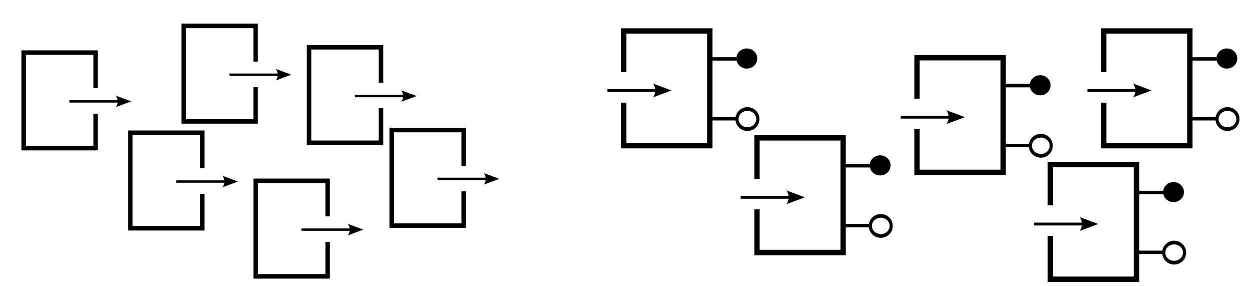

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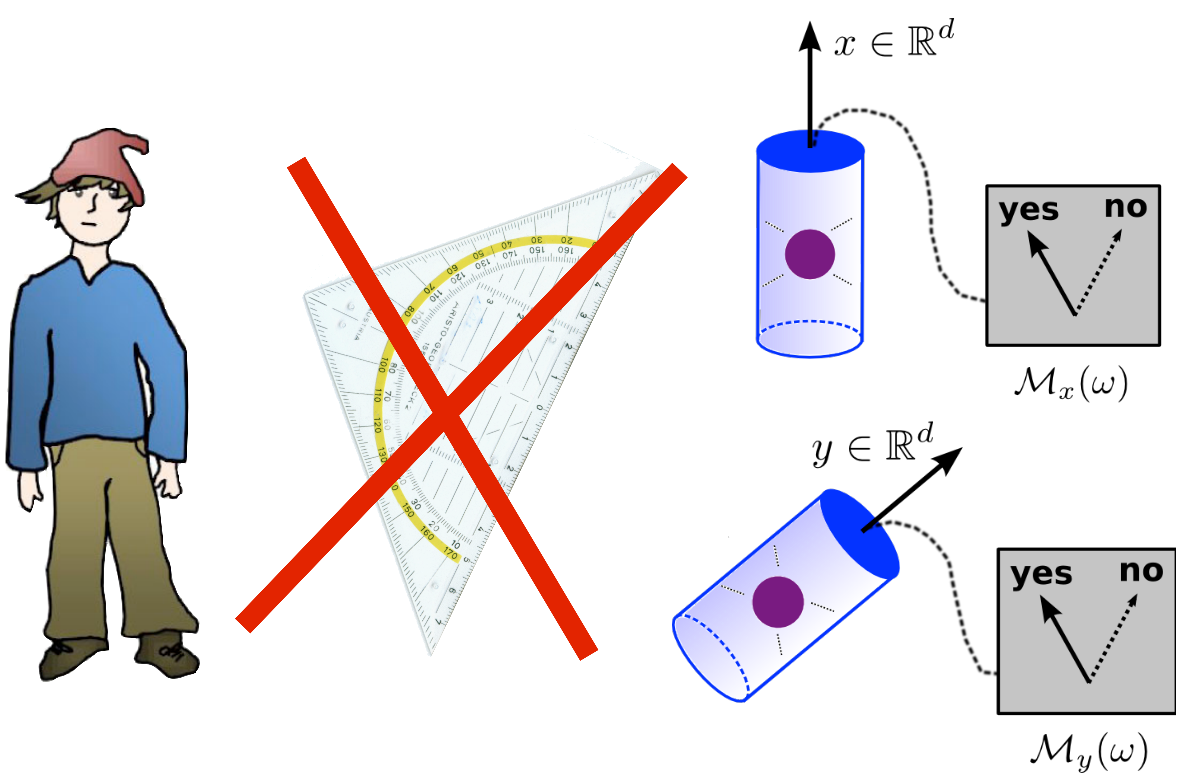

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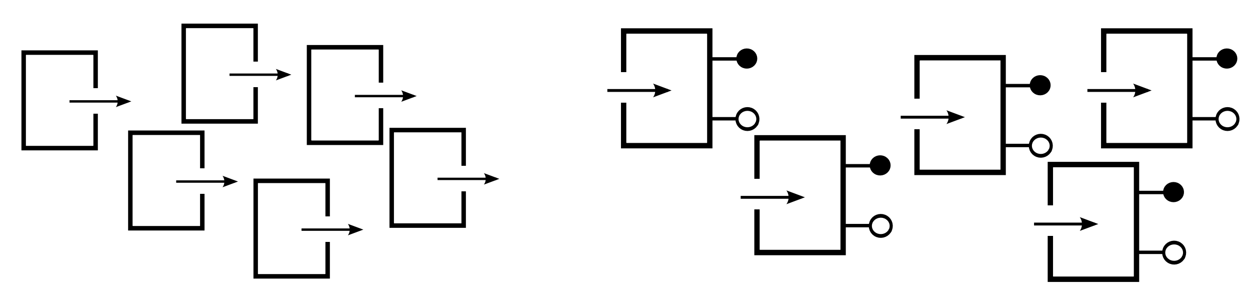

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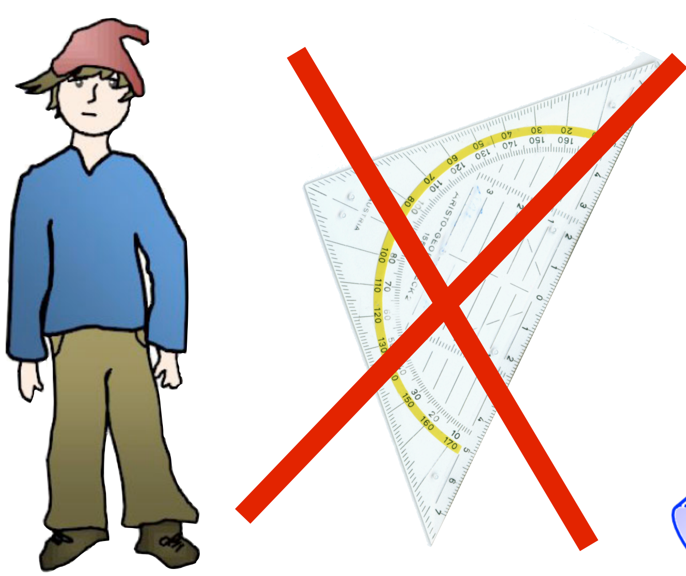

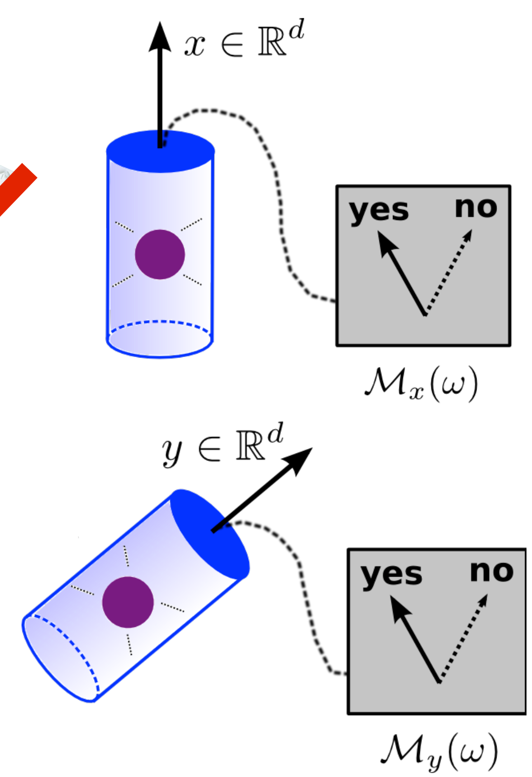

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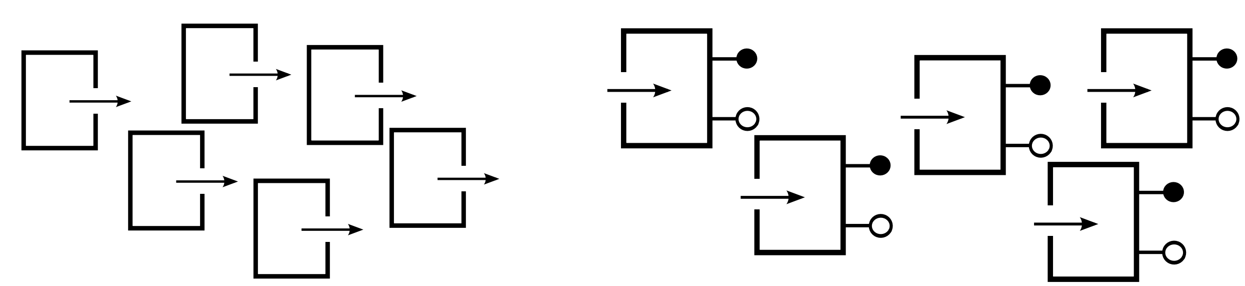

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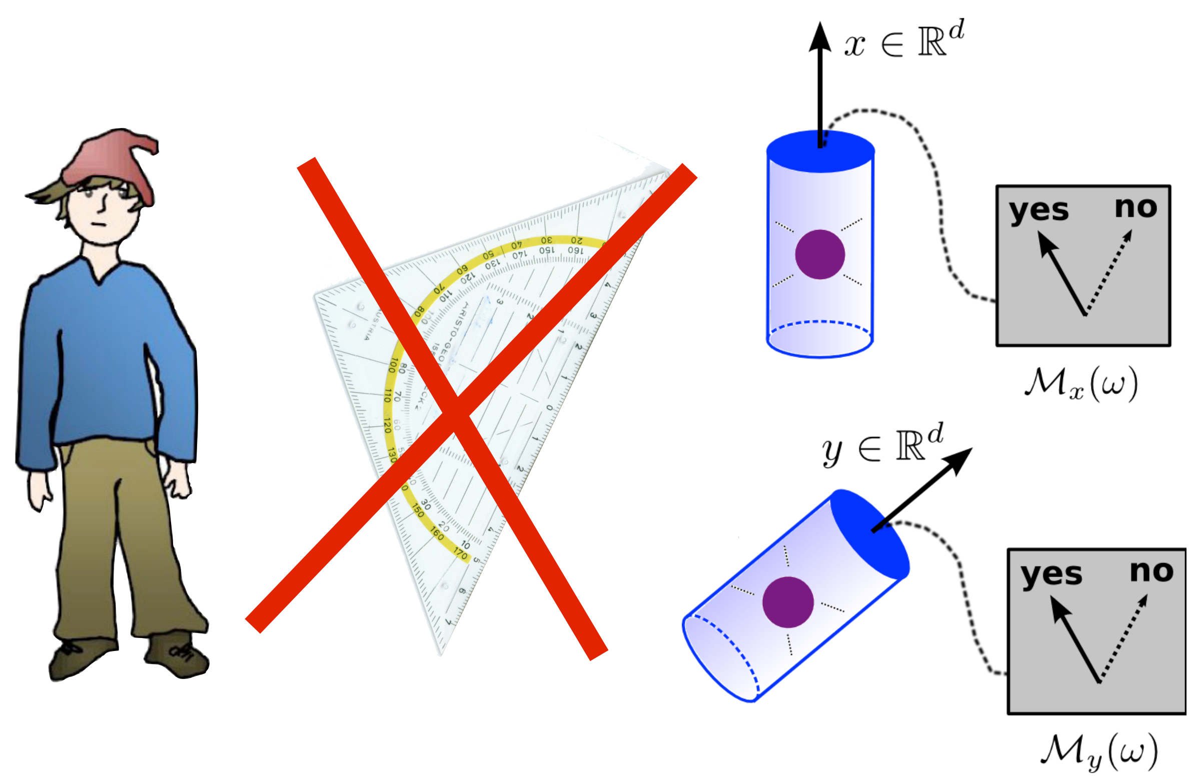
- Columns of *S* give rep. of $\vec{\omega}_1, \ldots, \vec{\omega}_d$ in some ONB.
- From $\mathcal{M}_x(\omega_i)$ and $\mathcal{M}_y(\omega_i)$ obtain rep. of $\vec{\omega}_x$ and $\vec{\omega}_y$ in ONB. Then $\angle(x,y) = \angle(\vec{\omega}_x, \vec{\omega}_y)$.



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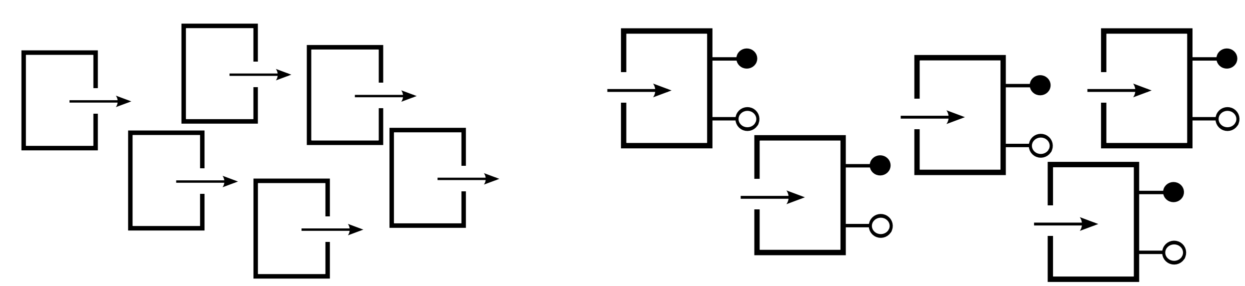

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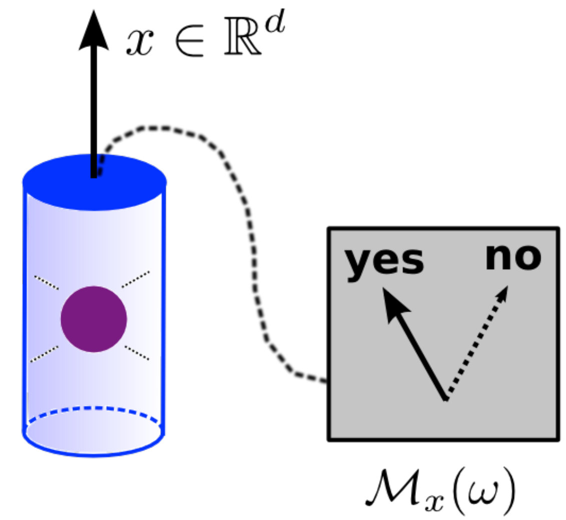

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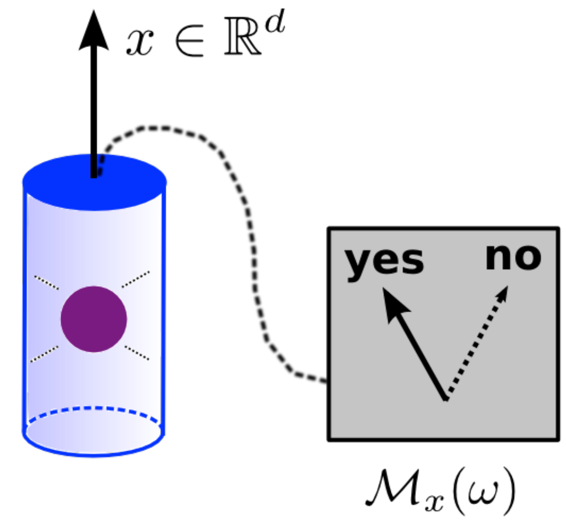

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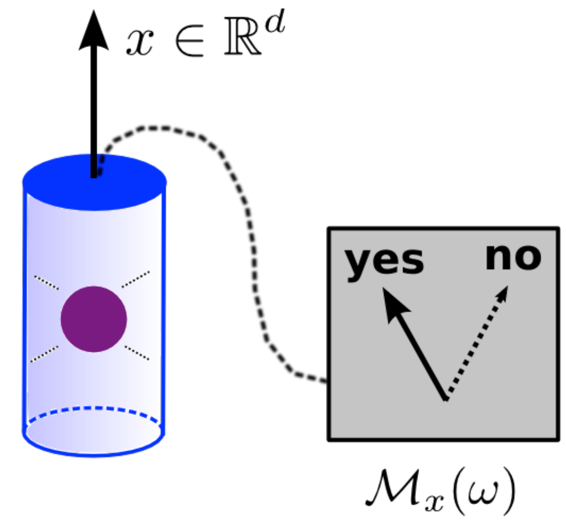

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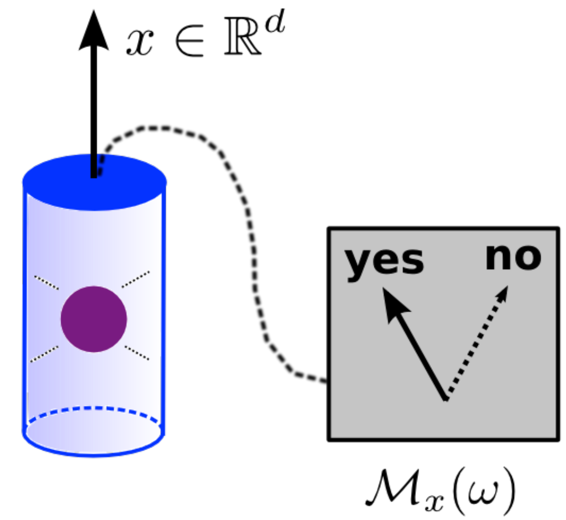

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Proof: State space would again be a unit ball. Pure states: $\{\omega_X\}_{X \in SO(d)}$
But SO(d) is not simply connected, and the sphere is.

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Our final postulate says that two direction bits can interact via some continuous reversible time evolution:

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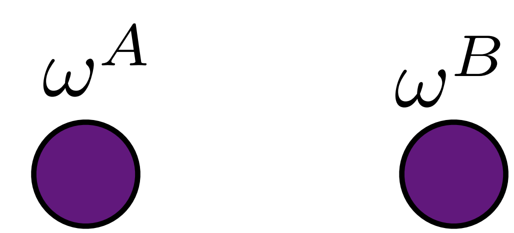

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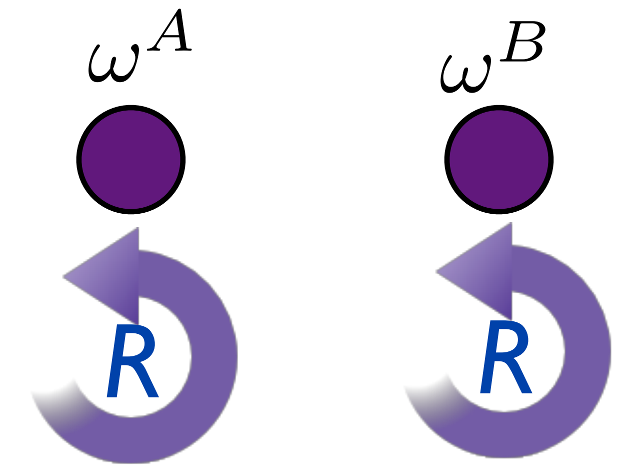
$$\mathcal{M}^A \mathcal{M}^B(\omega^A \omega^B) = \mathcal{M}^A(\omega^A) \cdot \mathcal{M}^B(\omega^B).$$


ω^A
ω^B

Given $R \in SO(d)$, we want a unique way to specify the global rotation on the composite system.



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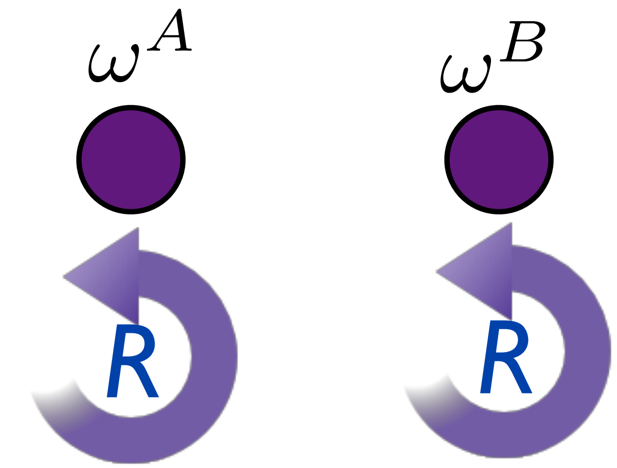

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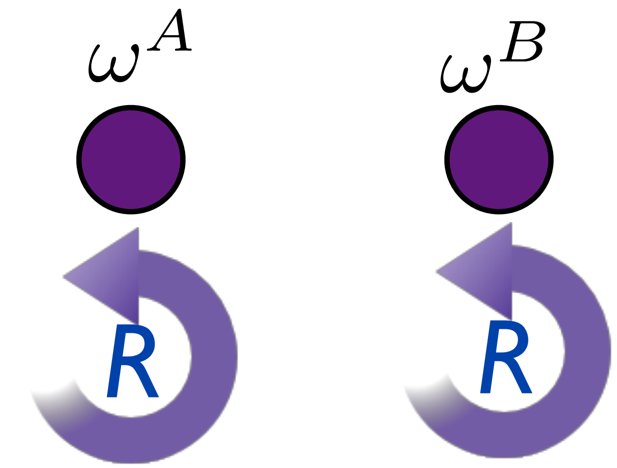

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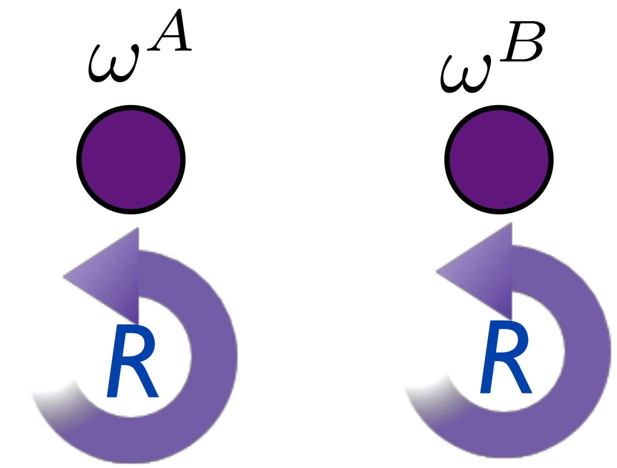

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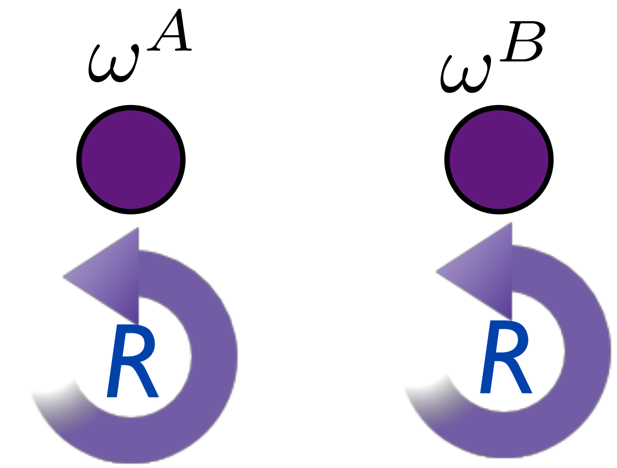

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- Equivalent to "tomographic locality": global states are uniquely determined by probabilities of local measurements and their correlations.
- Allows to represent product states via tensor product:

$\omega^A \omega^B = \omega^A \otimes \omega^B.$ $\omega^{AB} \mapsto G_R \otimes G_R(\omega^{AB}).$

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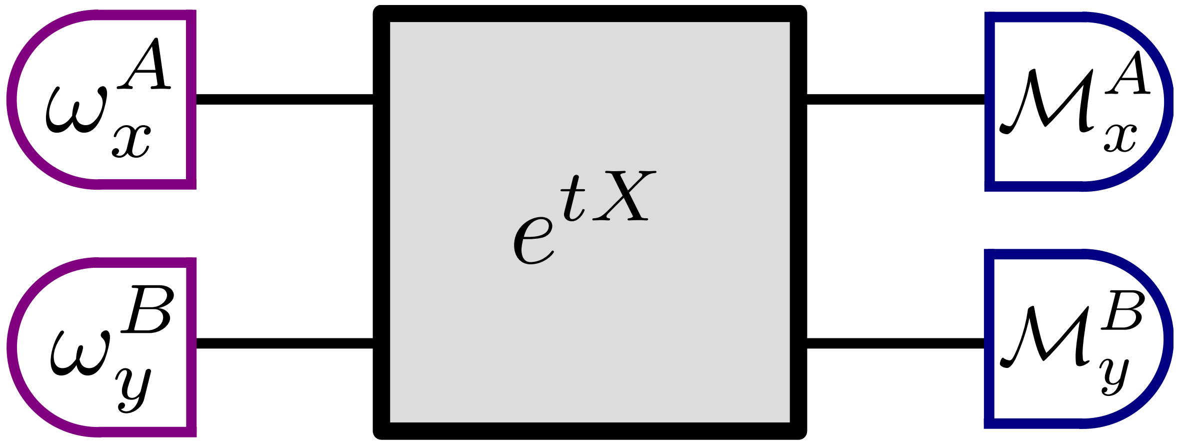

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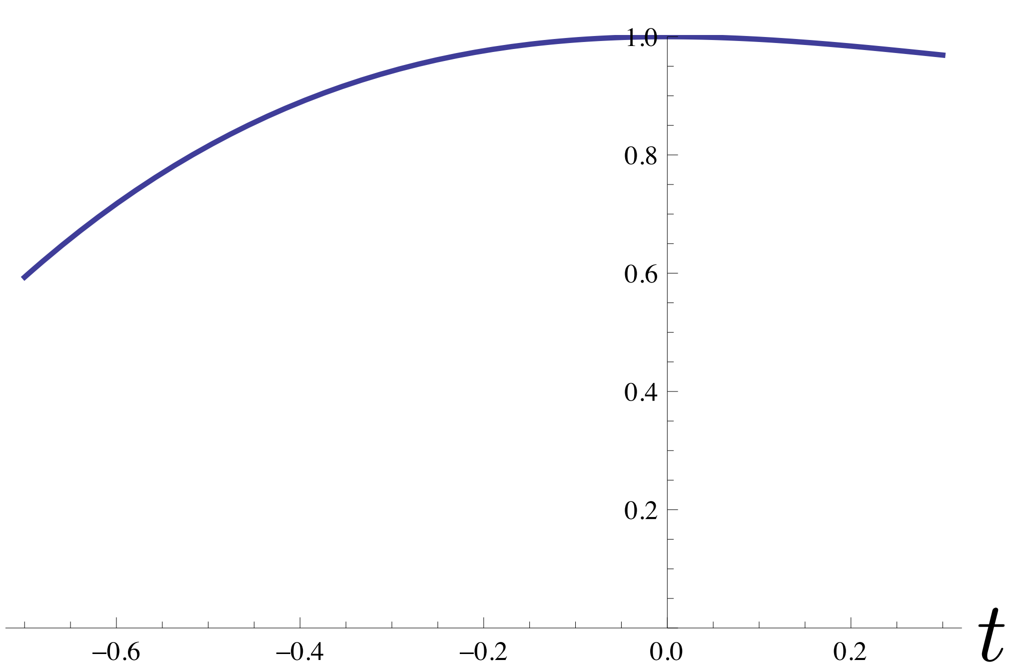

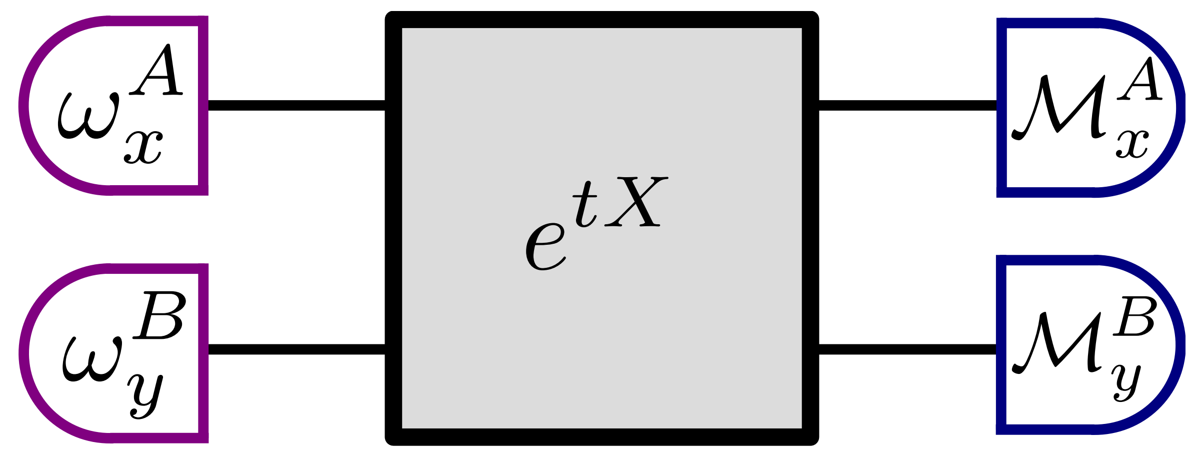

- But this equals 1 for $t=0$, thus

$$\mathcal{M}_x \otimes \mathcal{M}_y\, X\, \omega_x \otimes \omega_y = 0,$$

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- We get several constraints on $X \in \mathfrak{g}^{AB}$:

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- For $d \geq 3$, evaluating constraints involves integrals like

$$X \mapsto \int_{SO(d-1)} G^A \otimes \mathbf{1}^B X (G^A)^{-1} \otimes \mathbf{1}^B \, dG^A.$$

This behaves very differently if *SO(d-1)* is Abelian, i.e. iff *d*=3. □

Theorem: From Postulates A, B and C, it follows that the state space of two direction bits is 2-qubit quantum state space (i.e. the set of 4x4 density matrices), and time evolution is given by a one-parameter group of unitaries,

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- We have *d*=*3*. Embed the *3*-ball in the unit trace matrices of $\mathbb{C}^{2\times 2}_{s.a.}$

$$(r_1, r_2, r_3) \mapsto \begin{pmatrix} \frac{1}{2} + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & \frac{1}{2} - r_3 \end{pmatrix}.$$

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- Now some $X \neq X^A + X^B$ satisfy constraints. But they all generate maps of the form $e^{tX}(\rho) = U\rho U^\dagger$ with $U \in SU(4)$.

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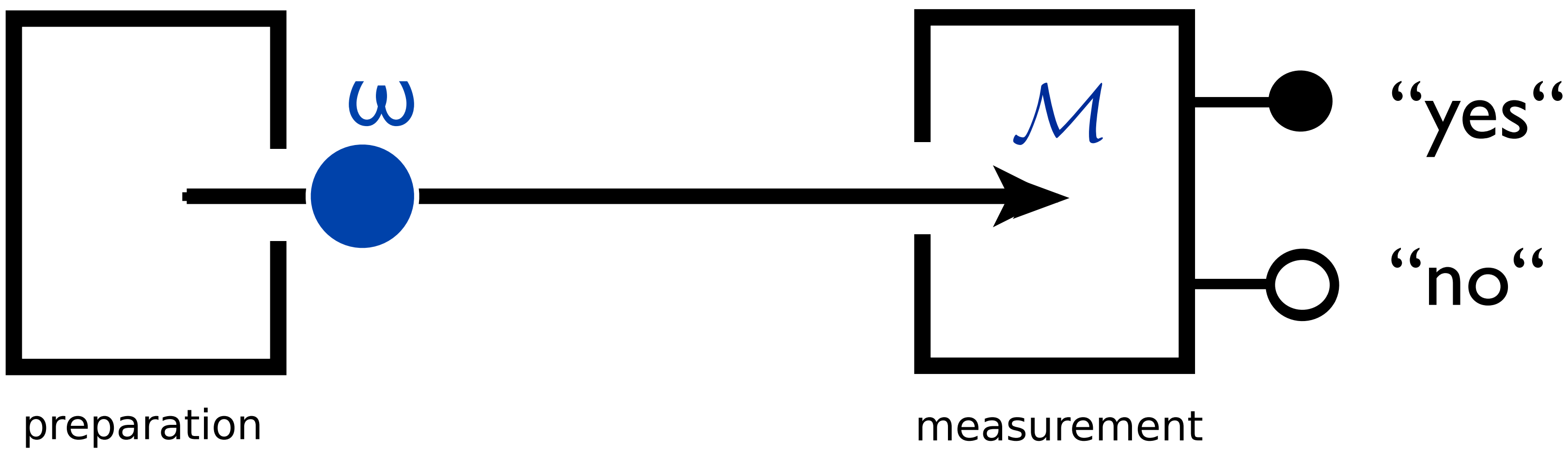
- But these generate all 4-level quantum states.
- If there were additional states, these would generate negative probabilities. $\square$

5. Conclusions

Attempt to clarify the relationship between spatial geometry and the qubit (based on old ideas & new techniques):



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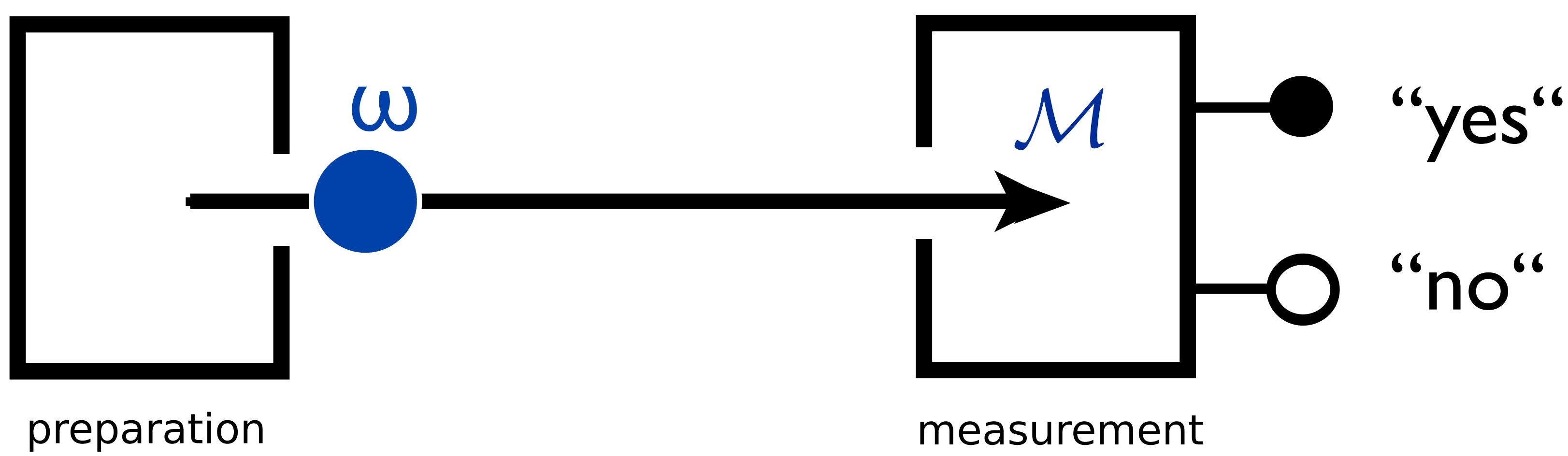

- Start with *d* spatial dimensions, not assuming quantum theory.
- Three "information-theoretic" postulates on the relation between spatial geometry (rotations) and probability
- Proof that these determine *d=3* and quantum theory on 2 bits.

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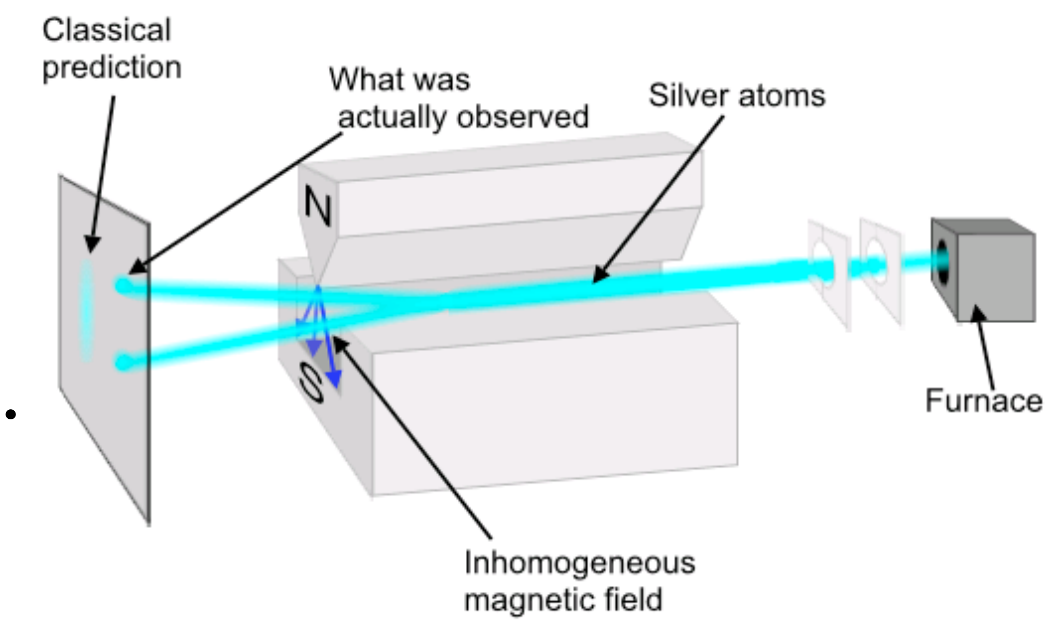

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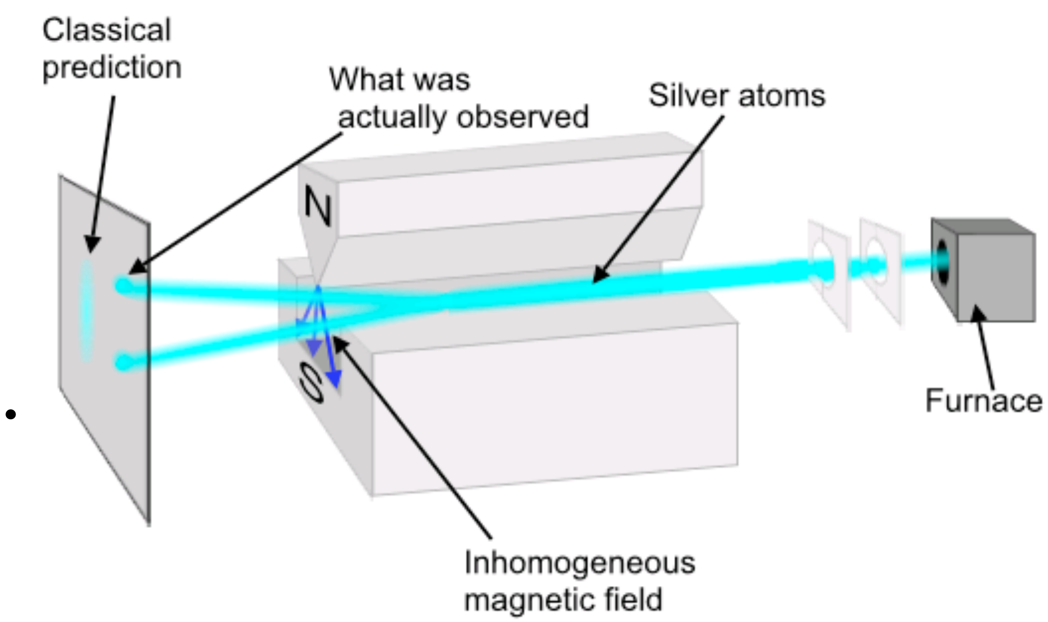

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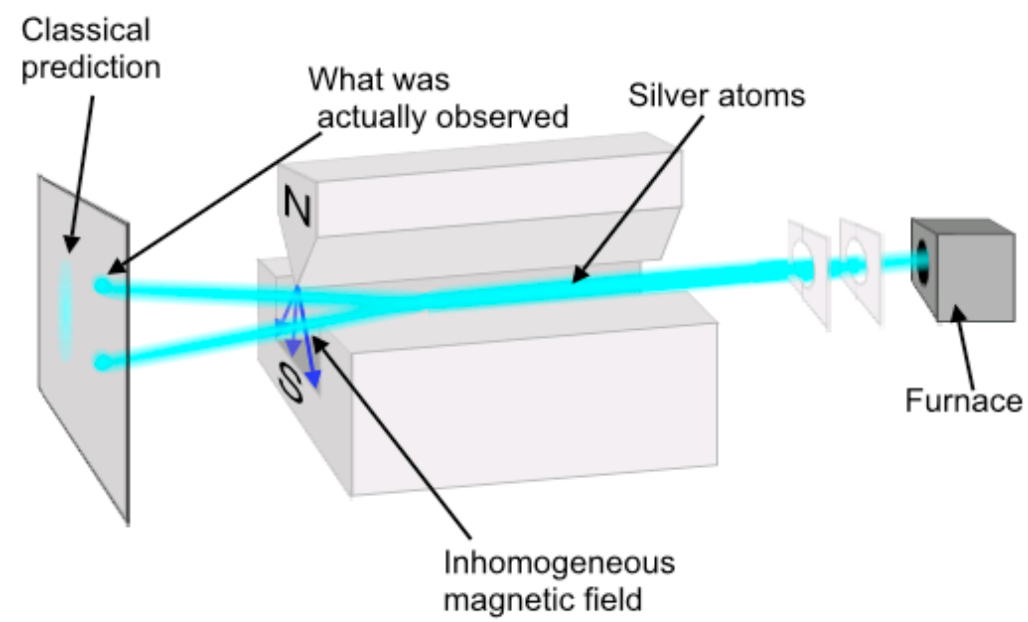

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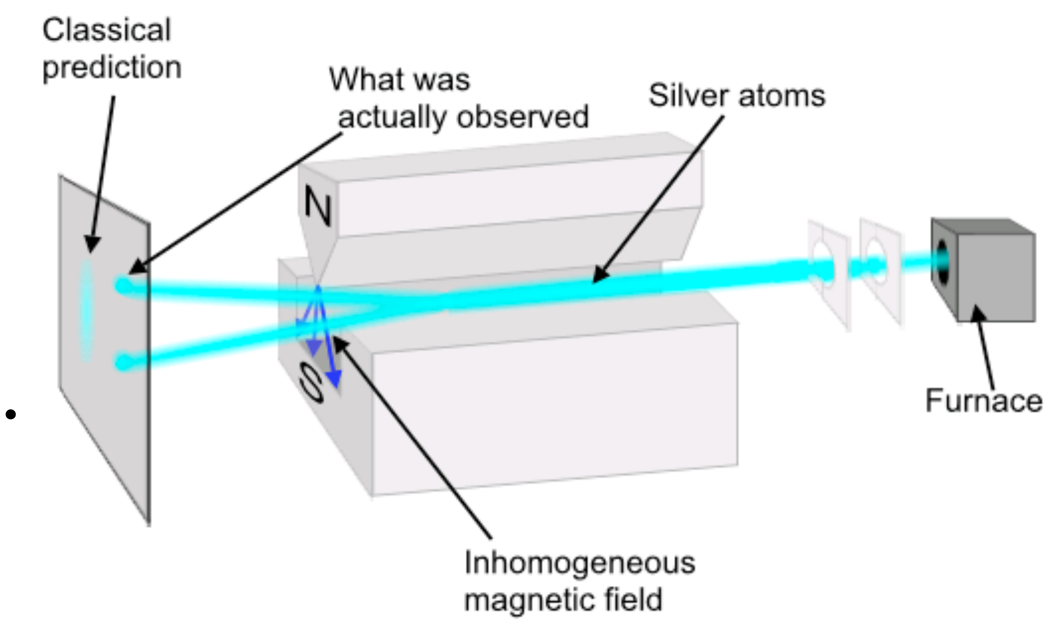

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- Possible (relativistic) generalizations of the result?

- Speculation: do space(-time) and quantum theory have a *common* information-theoretic origin?

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Thank you to Lucien Hardy, Lee Smolin; my co-authors; Danny Terno, FJ Schmitt, Hilary Carteret, Mauro d'Ariano, Raymond Lal, Tobias Fritz, ...

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- introduction to convex probabilistic theories:
  J. Barrett, arXiv:quant-ph/0508211
- ruling out $d \neq 3$:
  Ll. Masanes, MM, D. Pérez-García, R. Augusiak, arXiv:1111.4060
- $d=3$ implies quantum theory:
  G. de la Torra, Ll. Masanes, A. J. Short, MM, arXiv:1110.5482
- results of this talk:
  MM, Ll. Masanes, arXiv:hopefully.soon