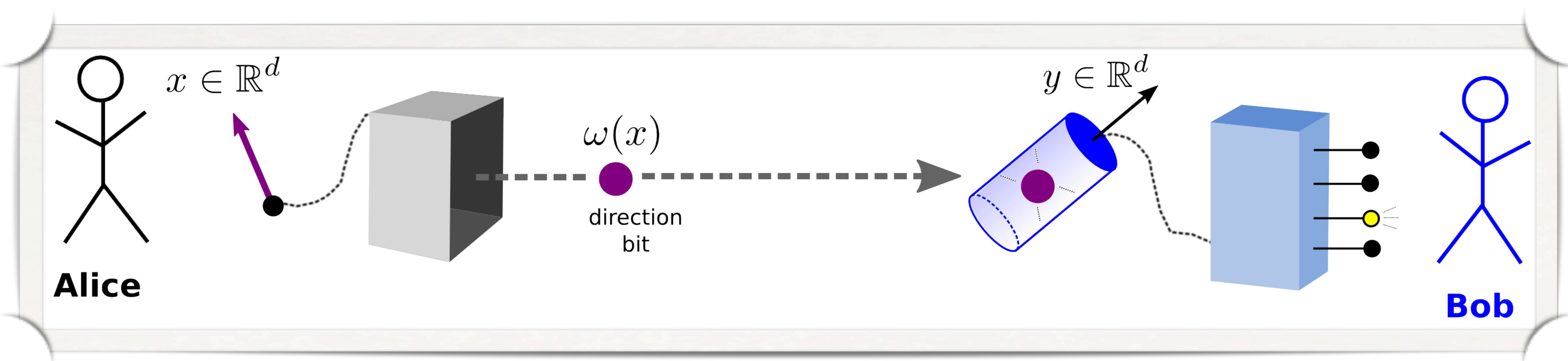
Three-dimensionality of space and the quantum bit: an information-theoretic approach

Markus P. Müller
Perimeter Institute for Theoretical Physics, Waterloo (Canada)



Joint work with Lluís Masanes (ICFO Barcelona)

PERIMETER INSTITUTE
FOR THEORETICAL PHYSICS

Overview

geometry vs. probability

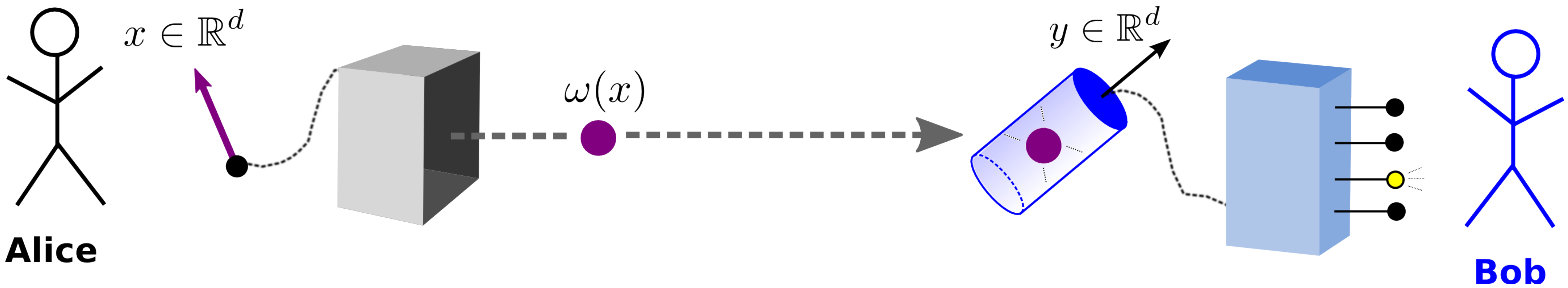
Overview

Geometry and probability are fundamentally intertwined.

Overview

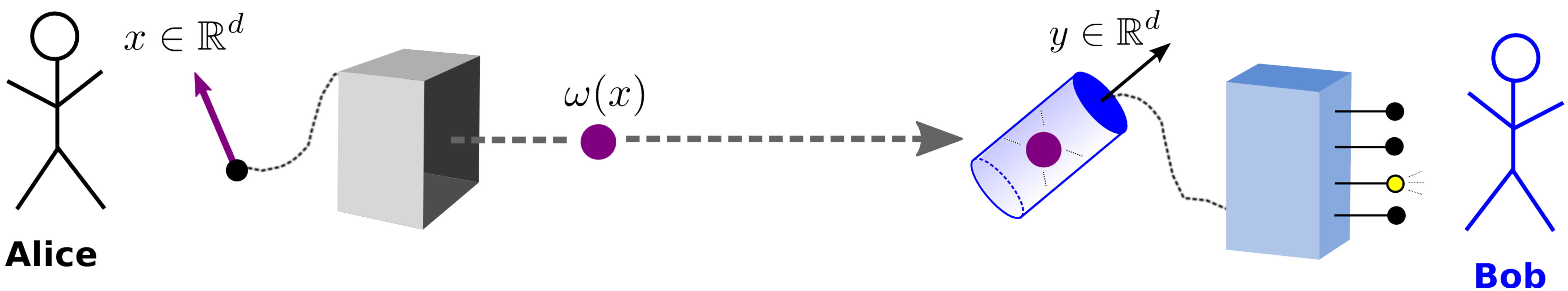
Overview

Our result: task for Alice and Bob in *d* spatial dimensions:



Overview

Our result: task for Alice and Bob in *d* spatial dimensions:

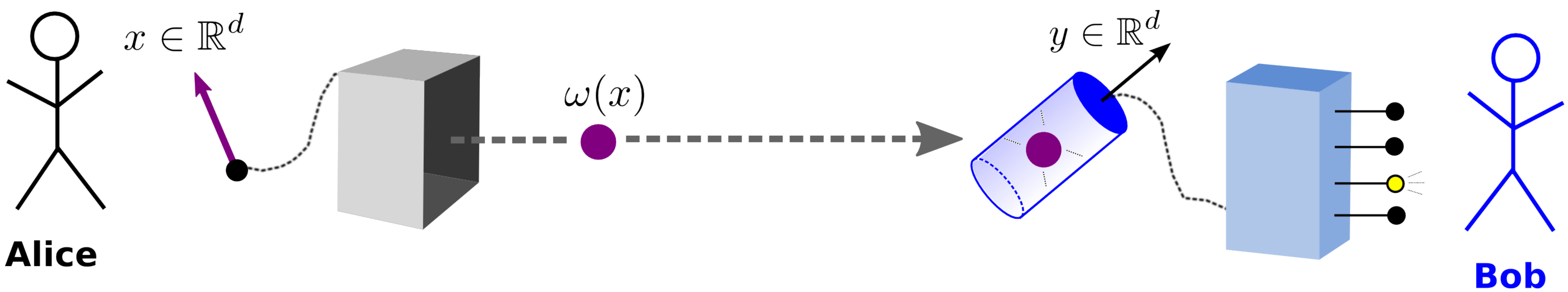


IF

- Alice and Bob can accomplish this task
- with "minimal overhead" and
- with interacting information carriers
- that allow for global coordinate transformations

Overview

Our result: task for Alice and Bob in *d* spatial dimensions:



IF

- Alice and Bob can accomplish this task
- with "minimal overhead" and
- with interacting information carriers
- that allow for global coordinate transformations

THEN

- automatically *d=3* and
- quantum theory holds for information carriers

(entanglement, unitary time evolution, complementarity, ...)

Overview

1. Overview

2.

3. The task

4.

5.

Overview

1. Overview

2. General-probabilistic state spaces

3. The task

4.

5.

M. Müller*, Ll. Masanes

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Overview

1. Overview

2. General-probabilistic state spaces

3. The task

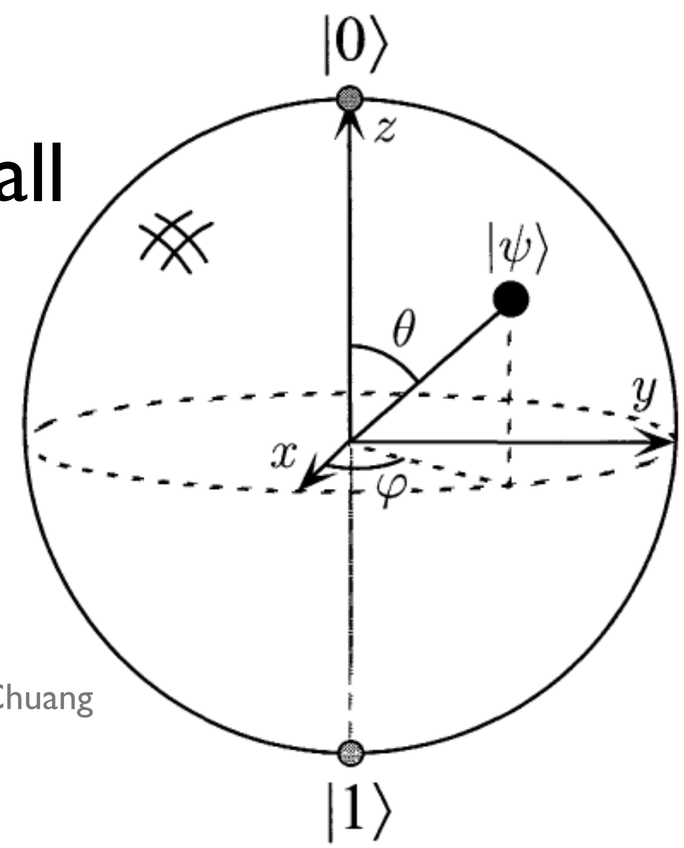
4. Deriving *d=3* and quantum theory

5. Some speculation

Overview

State space of quantum 2-level system is a 3D Euclidean ball:

Bloch ball



© Nielsen, Chuang

Same as space! Coincidence?

Overview

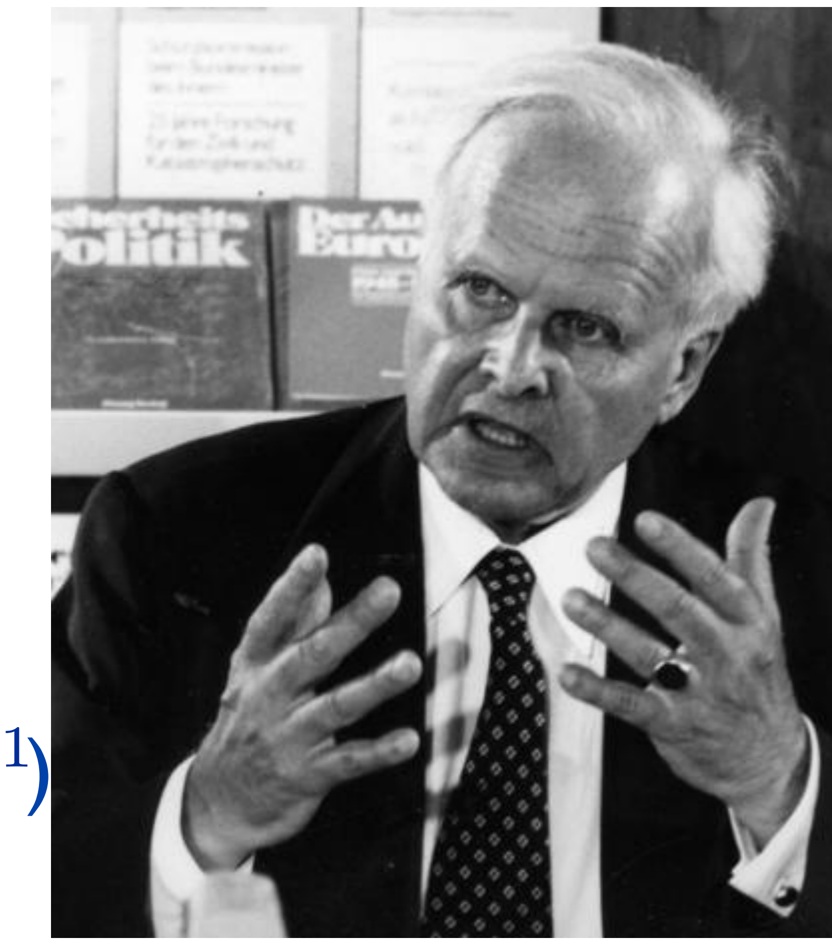
Carl-Friedrich von Weizsäcker: theory of "ur alternatives" (1955+)

- "ur" = (pure) qubit = quantum 2-level system
- everything is composed of (delocalized) urs
- symmetry group of ur

$$U(2) = SU(2) \otimes U(1) \sim S^3 \times S^1.$$

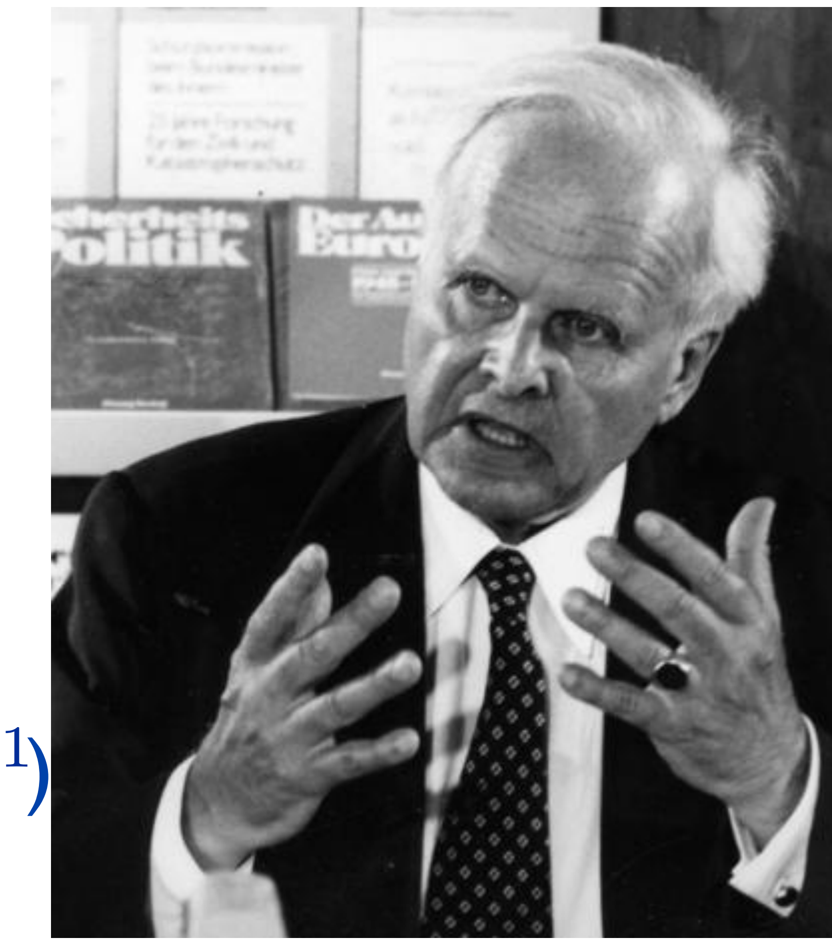
becomes global symmetry group of universe.

space (?!)   time (replaced by $\mathbb{R}^1$)



Overview

Carl-Friedrich von Weizsäcker: theory of “ur alternatives“ (1955+)



- “ur“ = (pure) qubit = quantum 2-level system
- everything is composed of (delocalized) urs
- symmetry group of ur

$$U(2) = SU(2) \otimes U(1) \sim S^3 \times S^1.$$

becomes global symmetry group of universe.

space (?!) time (replaced by $\mathbb{R}^1$)

Vague. What does this mean?

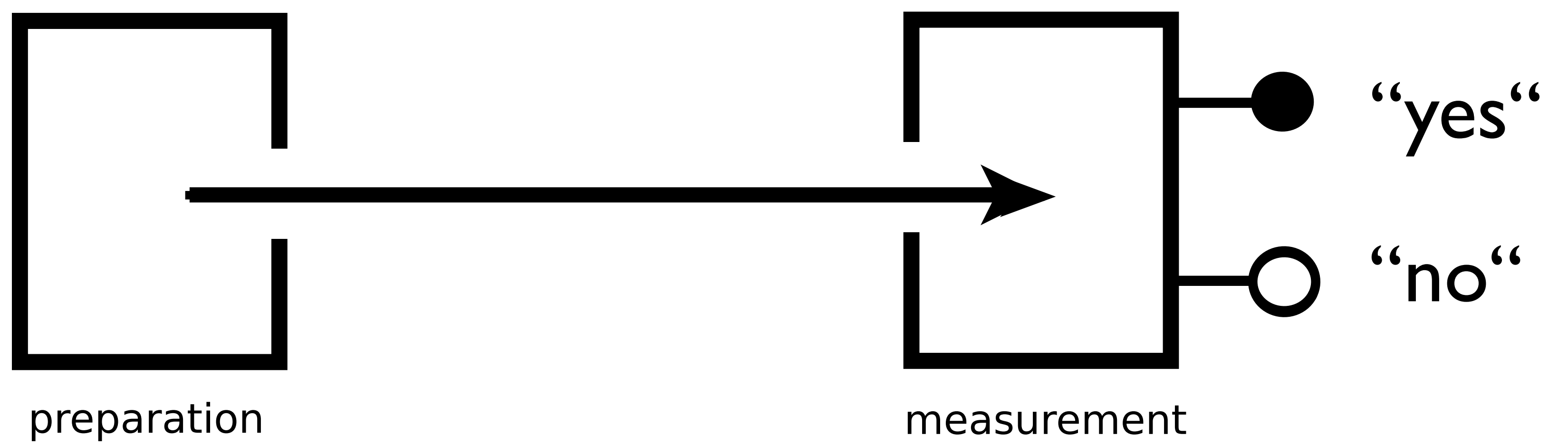
How is decomposition into *delocalized* urs chosen?

Why bits, not trits?

Mathematically not rigorous, conceptually unclear.

2. The framework

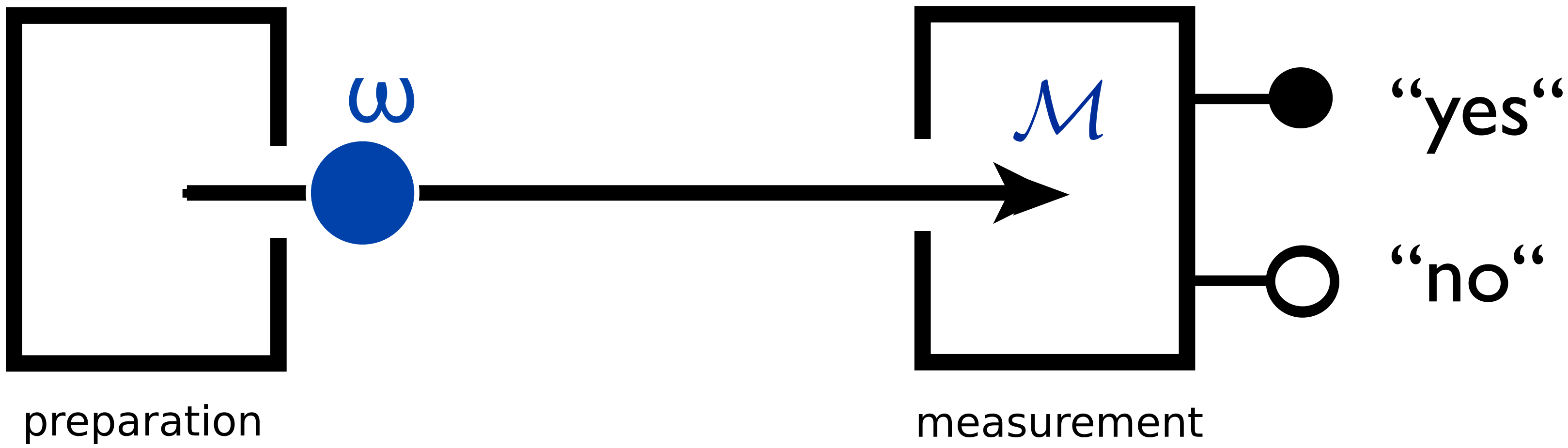
Assumption: there are some events that happen probabilistically.



M. Müller*, Ll. Masanes

2. The framework

Assumption: there are some events that happen probabilistically.

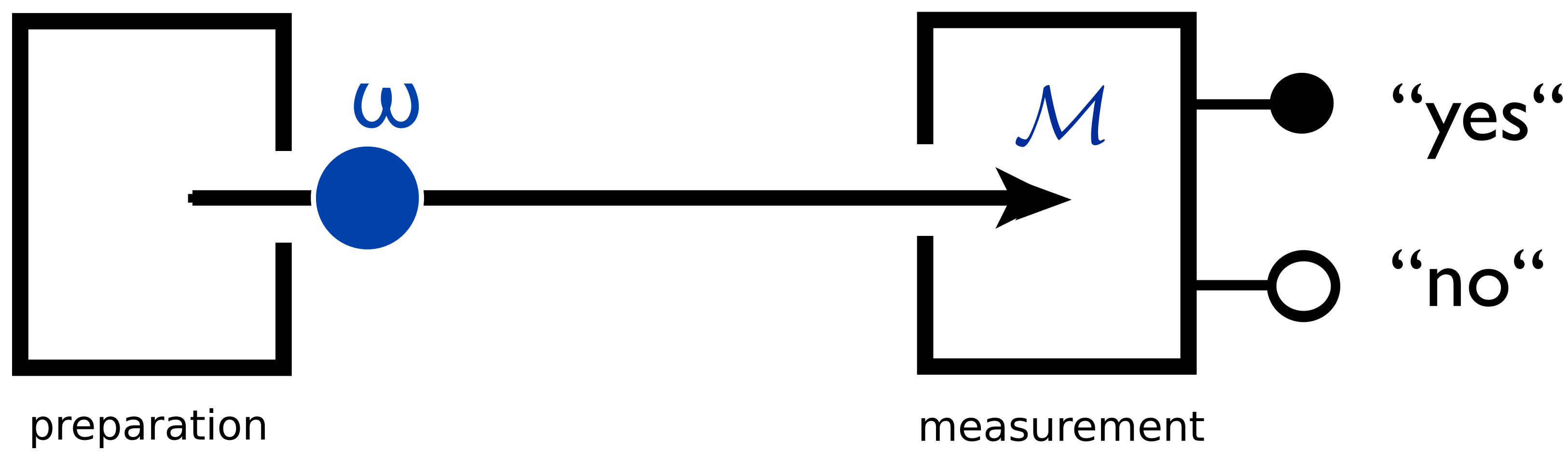


- Physical systems can be in some state ω. From this, all outcome probabilities of all subsequent events can be computed:

$$\text{Prob(outcome ”yes” } | \text{ meas. } \mathcal{M} \text{ on state } \omega) =: \mathcal{M}(\omega).$$

2. The framework

Assumption: there are some events that happen probabilistically.



- Physical systems can be in some state ω. From this, all outcome probabilities of all subsequent events can be computed:

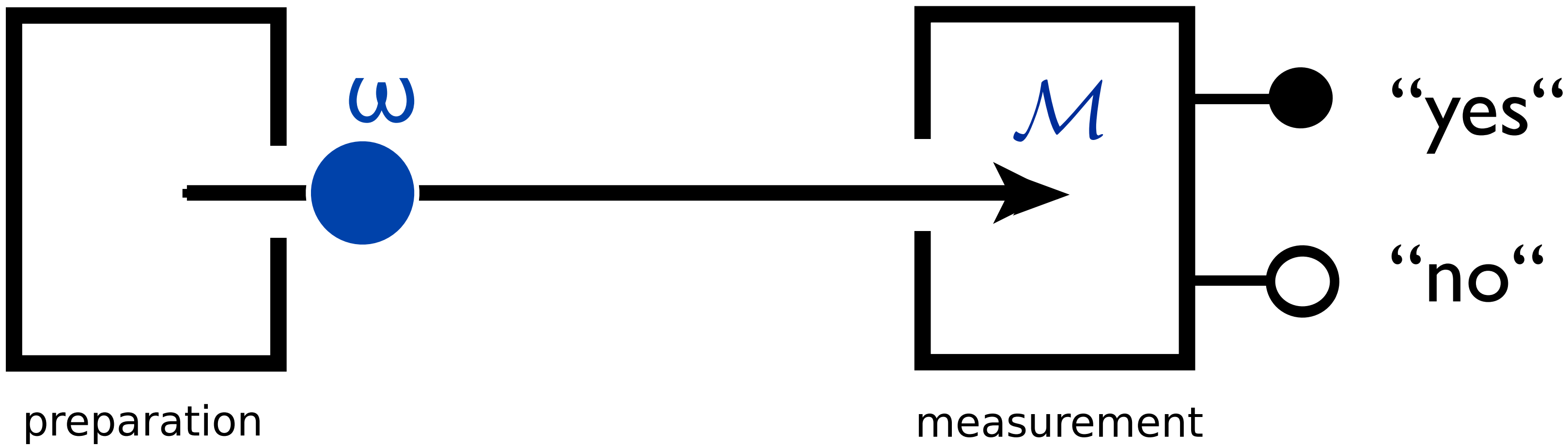
$$\text{Prob(outcome "yes" } | \text{ meas. } \mathcal{M} \text{ on state } \omega) =: \mathcal{M}(\omega).$$

- Statistical mixtures are described by convex combinations: prepare ω with prob. p and state φ with prob. $(1-p)$, result:

$$p\,\omega + (1-p)\varphi$$

2. The framework

Assumption: there are some events that happen probabilistically.

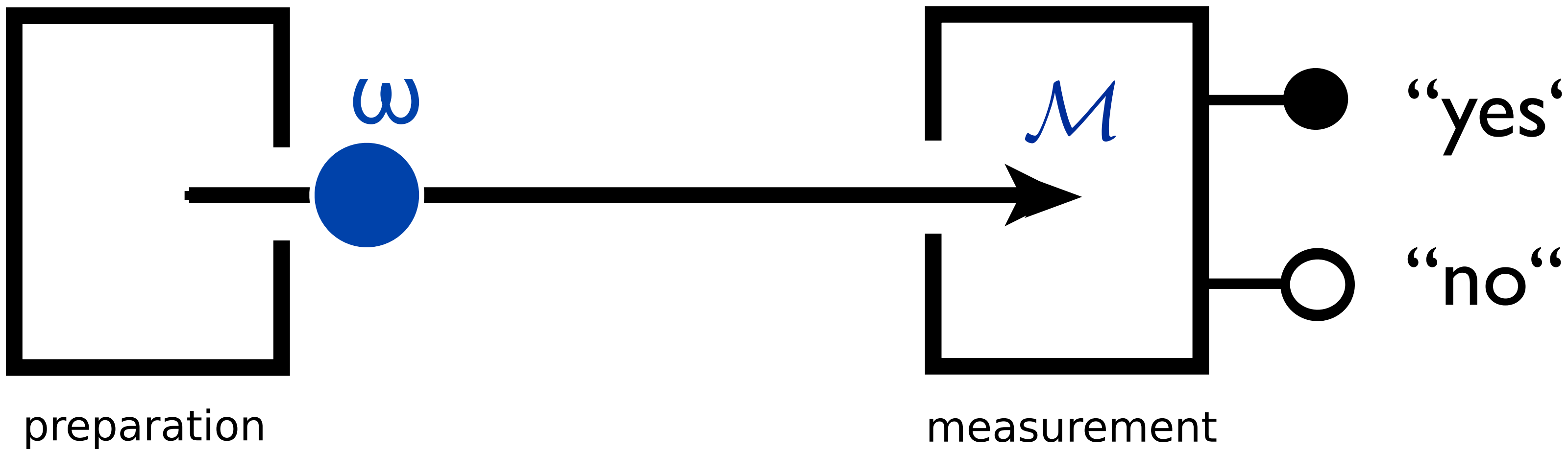


- Consequence: measurements ("effects") $\mathcal{M}$ are affine-linear:

$$\mathcal{M}(p\omega + (1-p)\varphi) = p\mathcal{M}(\omega) + (1-p)\mathcal{M}(\varphi).$$

2. The framework

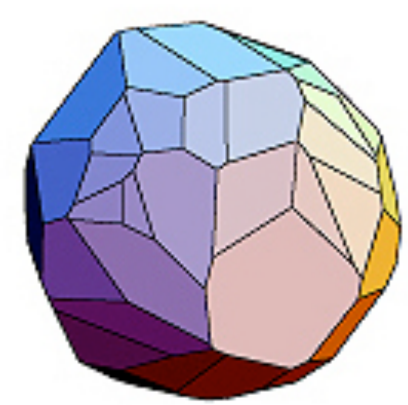
Assumption: there are some events that happen probabilistically.



- Consequence: measurements (“effects“) $\mathcal{M}$ are affine-linear:

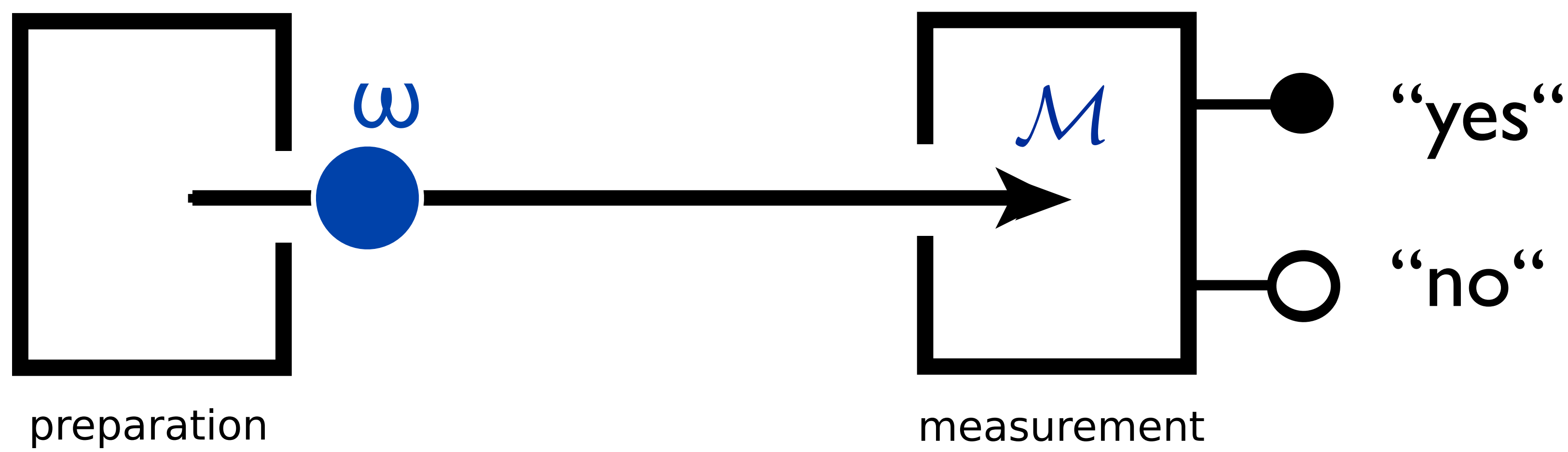
$$\mathcal{M}(p\omega + (1-p)\varphi) = p\mathcal{M}(\omega) + (1-p)\mathcal{M}(\varphi).$$

- State space Ω = set of all possible states ω.
  Convex, compact, finite-dimensional.
  Otherwise arbitrary.



2. The framework

Assumption: there are some events that happen probabilistically.

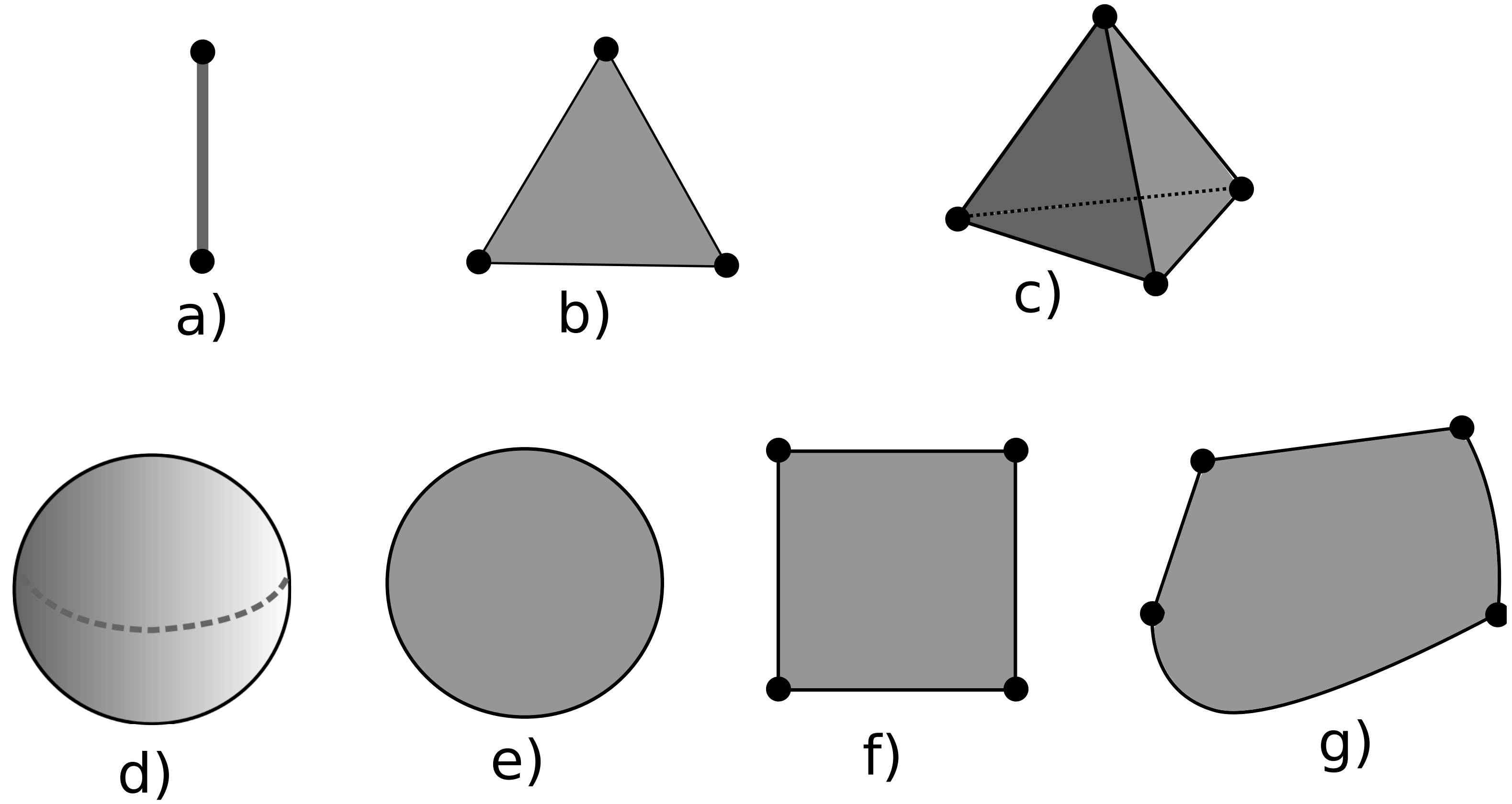


- Consequence: measurements ("effects") $\mathcal{M}$ are affine-linear:

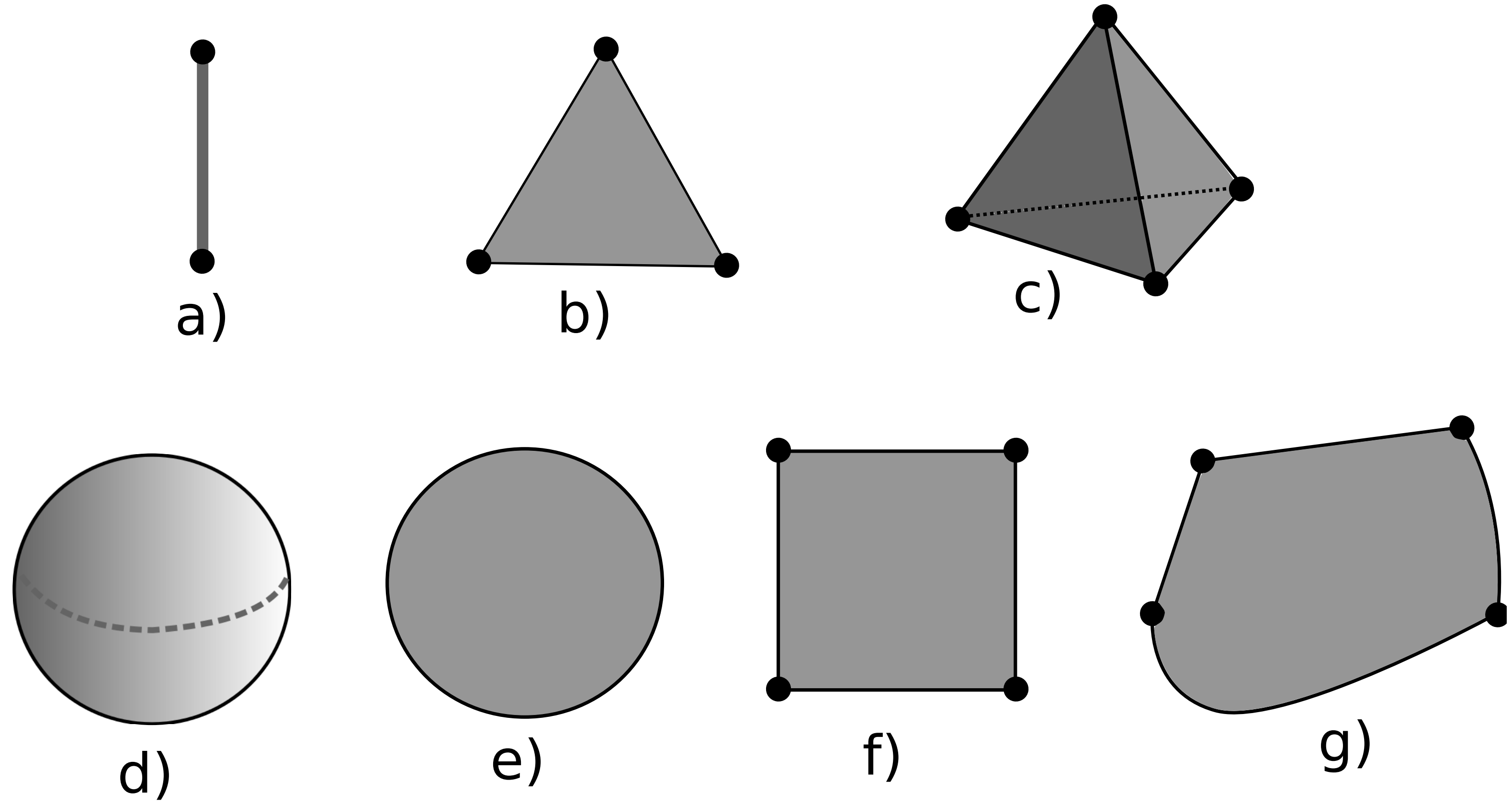
$$\mathcal{M}(p\omega + (1-p)\varphi) = p\mathcal{M}(\omega) + (1-p)\mathcal{M}(\varphi).$$

- State space Ω = set of all possible states ω.
  Convex, compact, finite-dimensional.
  Otherwise arbitrary.
  Extremal points are called pure, others mixed.

Some examples:



Some examples:

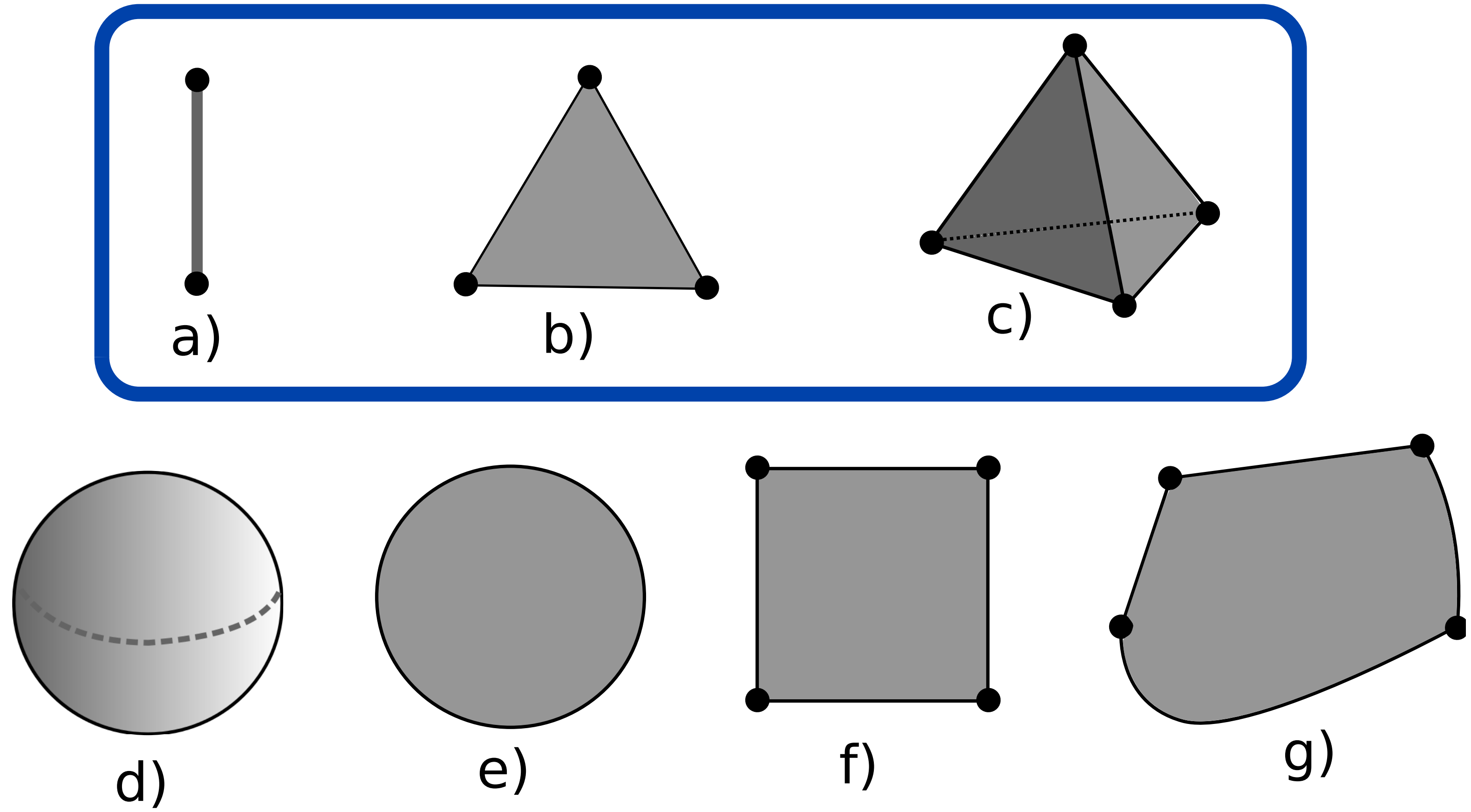


- Classical n-level system:

$$\Omega = \{\omega = (p_1, \ldots, p_n) \mid p_i \geq 0, \; \sum_i p_i = 1\}.$$

n pure states: $\omega_1 = (1, 0, \ldots, 0), \ldots, \omega_n = (0, \ldots, 0, 1).$

Some examples:



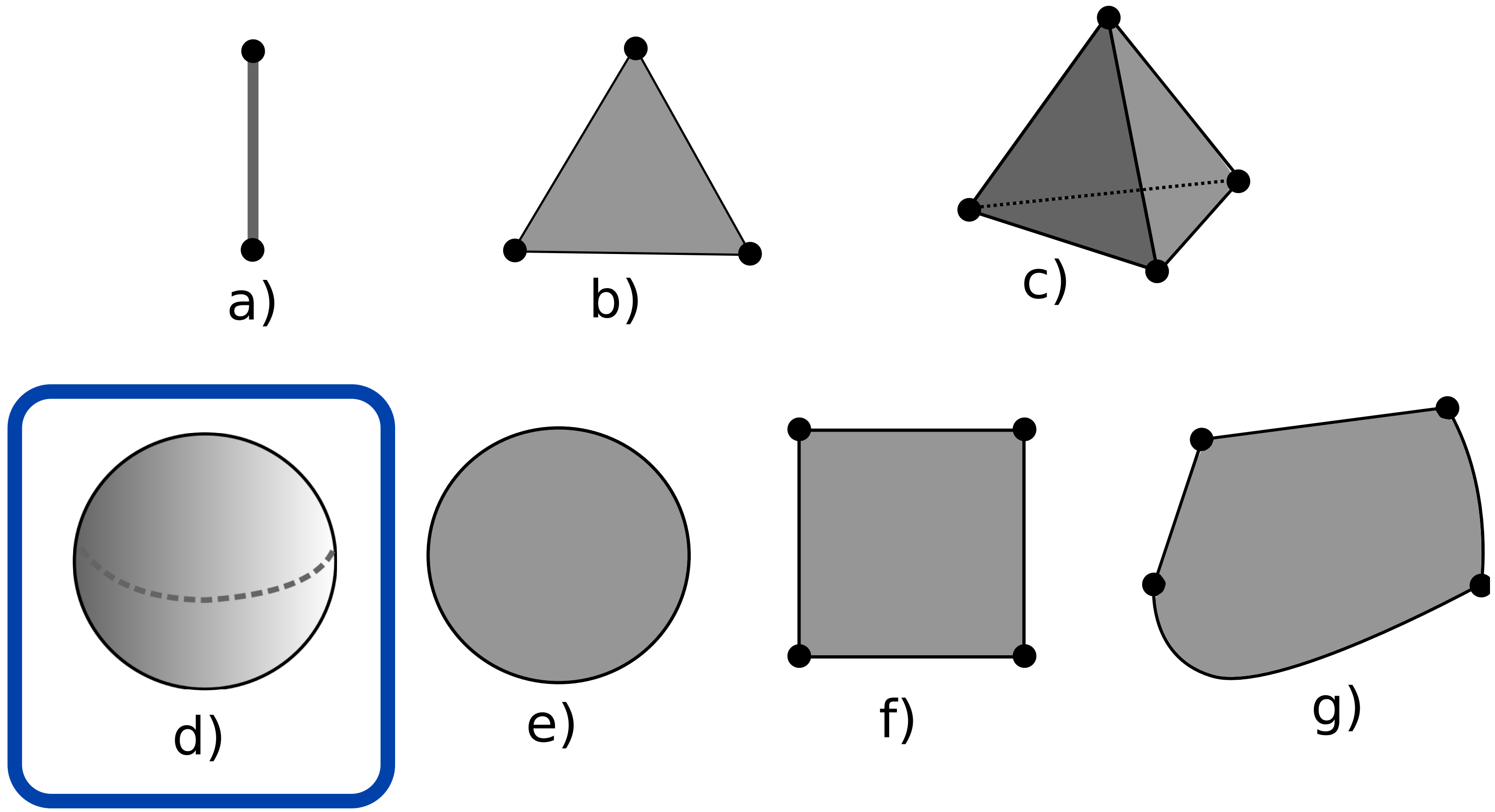
- Classical n-level system:

$$\Omega = \{\omega = (p_1, \ldots, p_n) \mid p_i \geq 0, \ \sum_i p_i = 1\}.$$

n pure states: $\omega_1 = (1, 0, \ldots, 0), \ldots, \omega_n = (0, \ldots, 0, 1).$

a), b), c): classical 2-, 3-, 4-level systems.

Some examples:



- Classical n-level system:

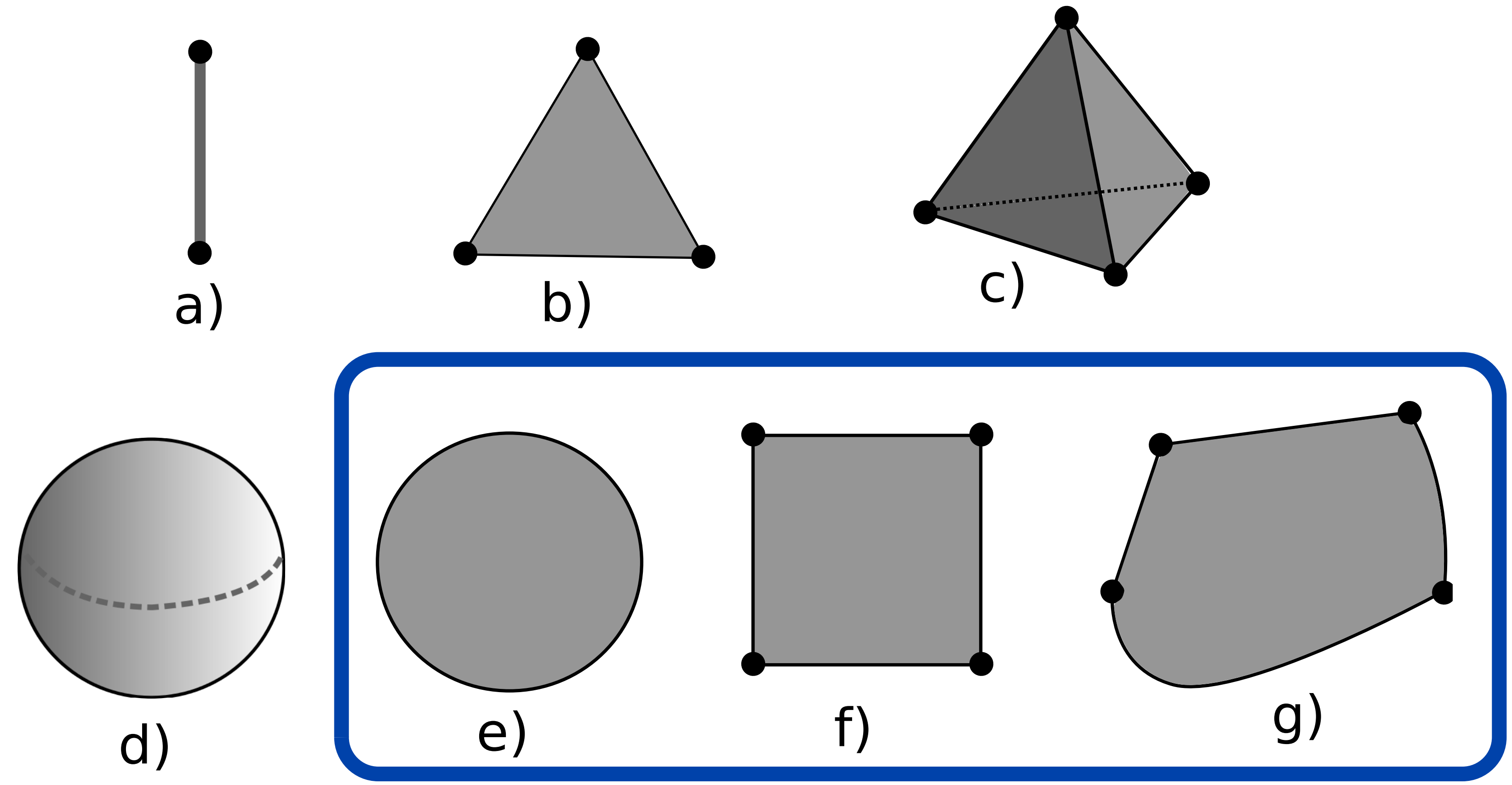
$$\Omega = \{\omega = (p_1, \ldots, p_n) \mid p_i \geq 0, \ \sum_i p_i = 1\}.$$

n pure states: $\omega_1 = (1, 0, \ldots, 0), \ldots, \omega_n = (0, \ldots, 0, 1).$

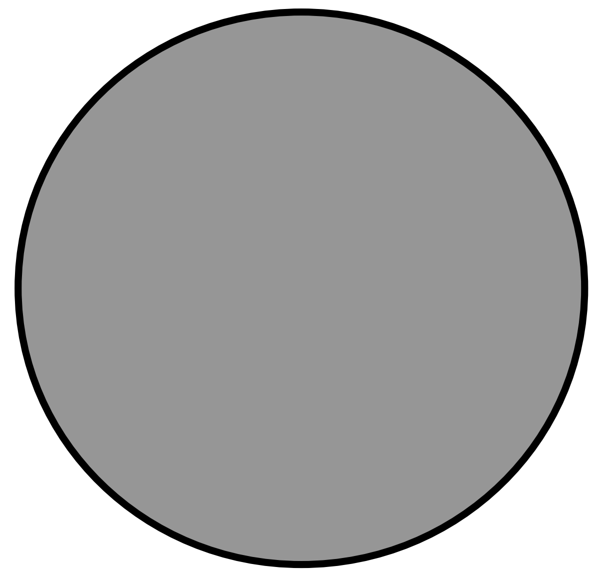
a), b), c): classical 2-, 3-, 4-level systems.

- d): quantum 2-level system (qubit)

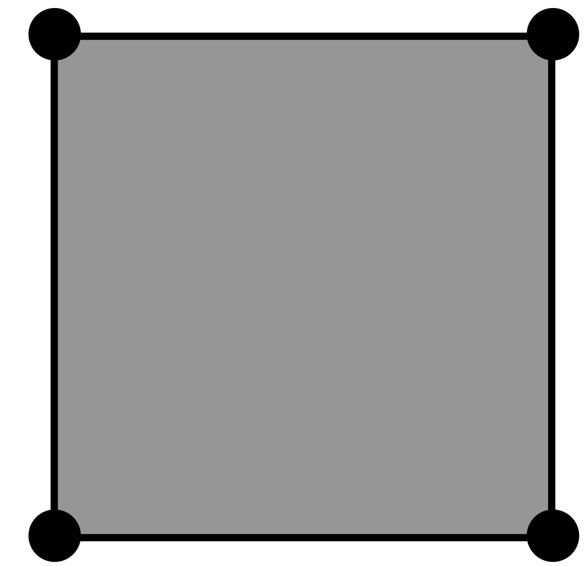
Some examples:



- e), f), g): neither classical nor quantum.

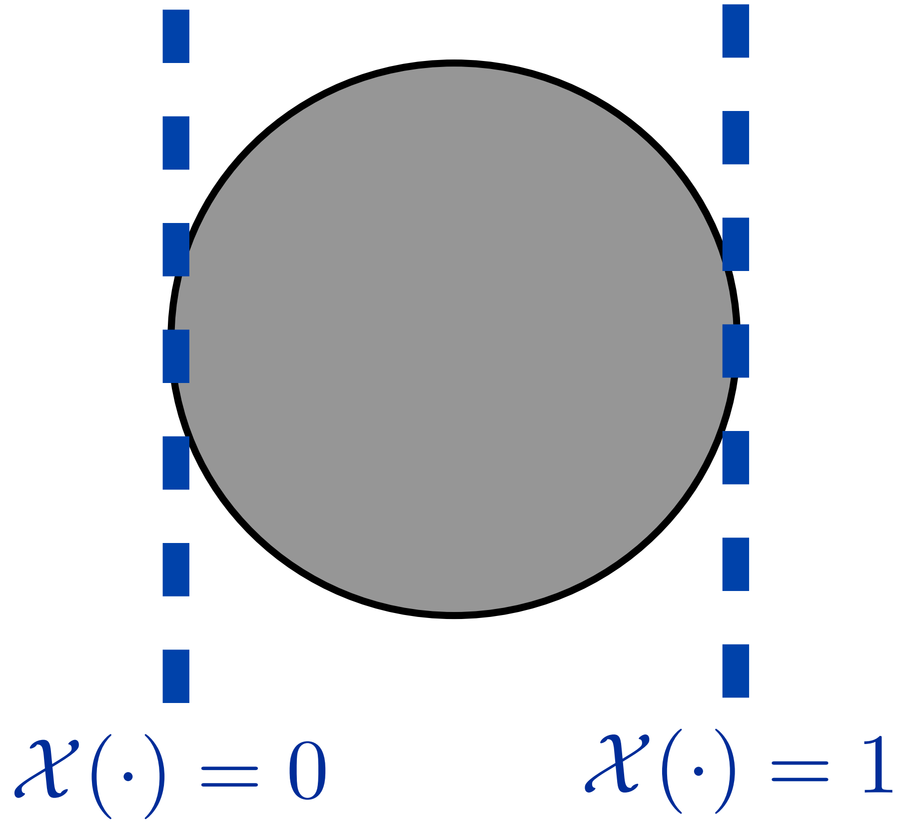


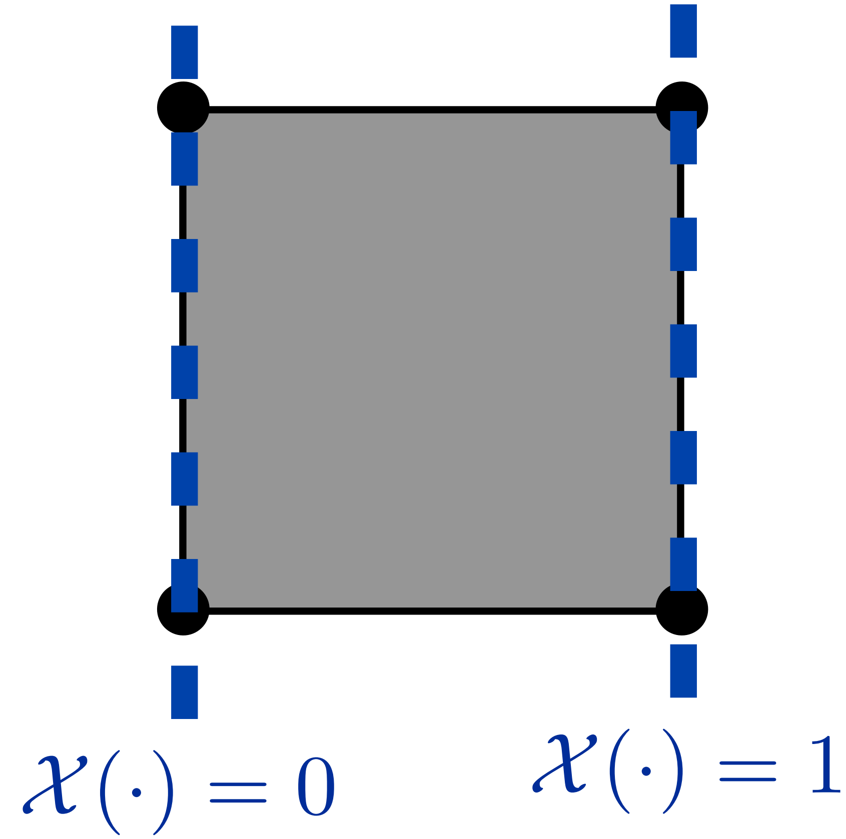
e)



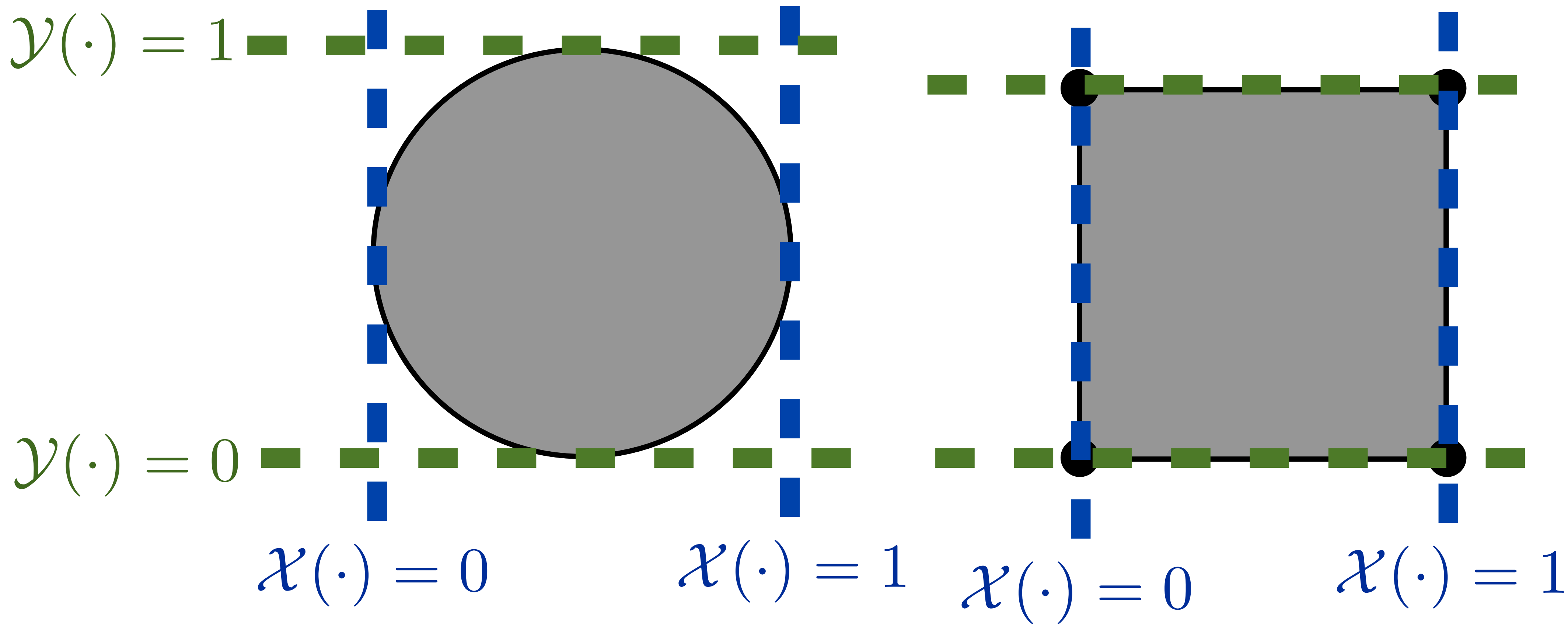
f)

- e), f), g): neither classical nor quantum.
- e), f): let χ=measurement "is spin up in x-direction?"



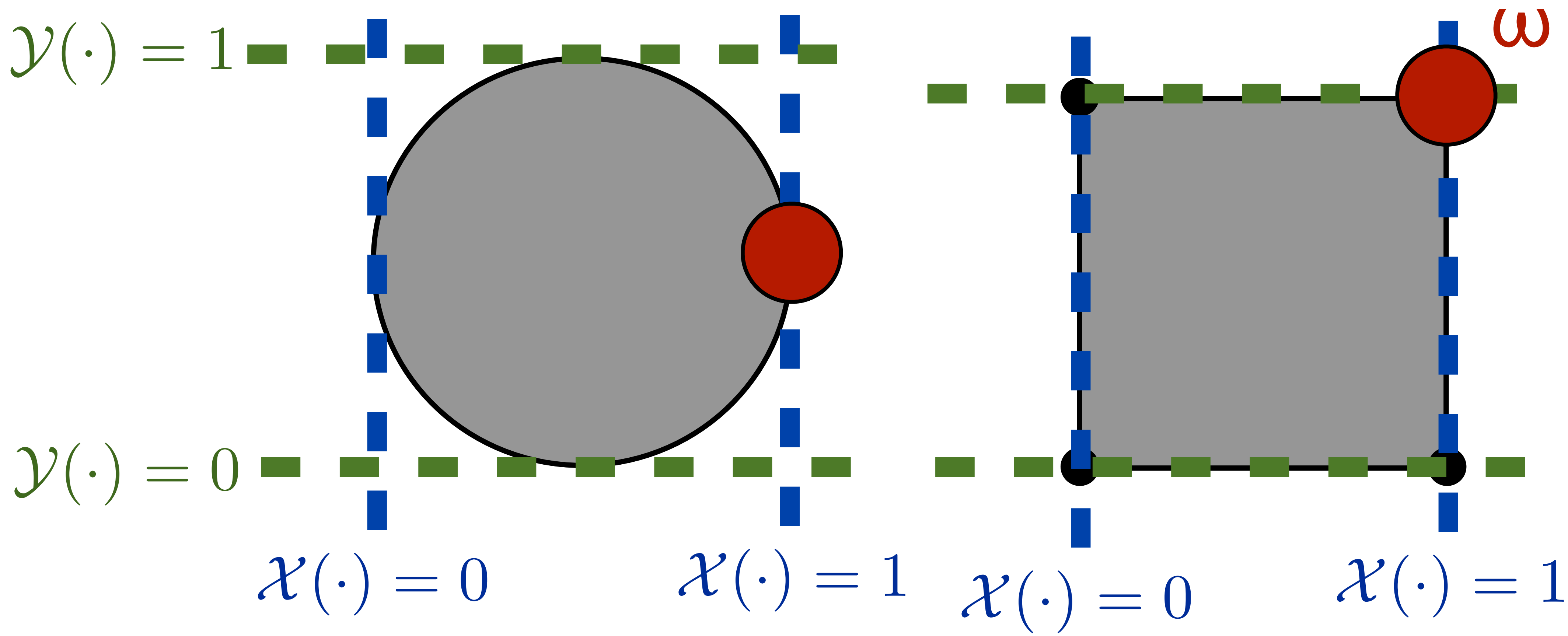


- e), f), g): neither classical nor quantum.
- e), f): let χ=measurement "is spin up in x-direction?"



- e), f), g): neither classical nor quantum.
- e), f): let χ=measurement "is spin up in x-direction?"

  analogously for Υ.

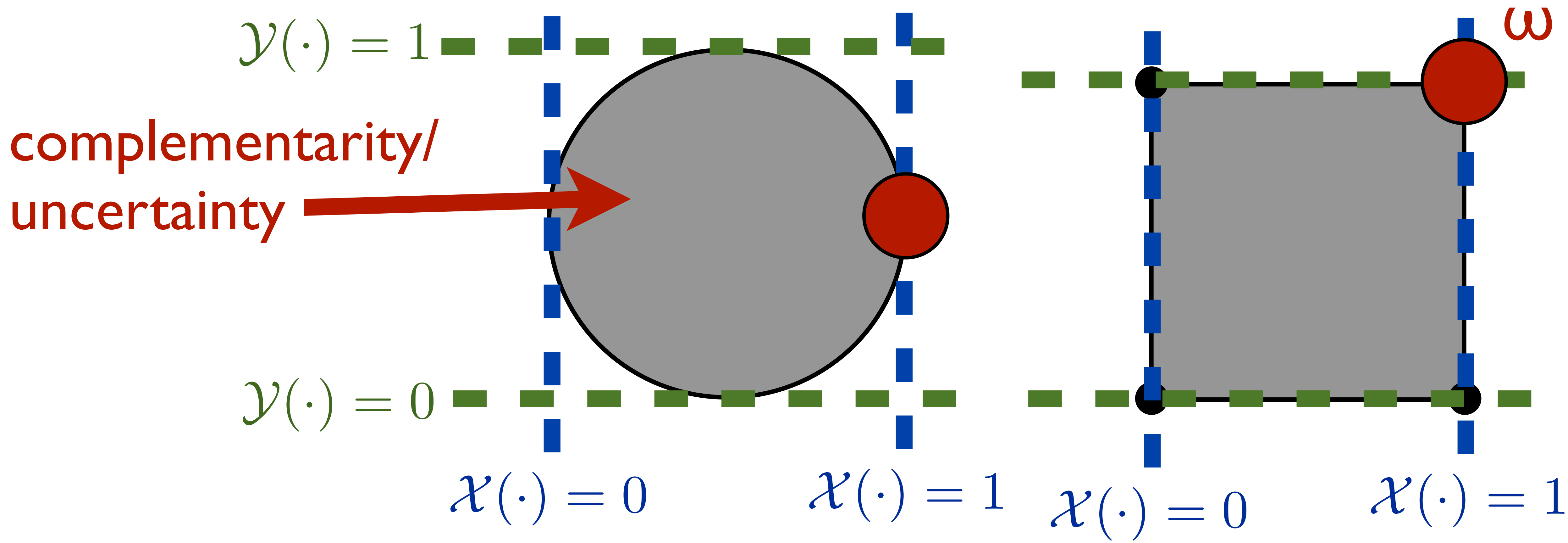


- e), f), g): neither classical nor quantum.
- e), f): let χ=measurement "is spin up in x-direction?"

  analogously for Υ.

  Square: there is a state ω with $\mathcal{X}(\omega)=\mathcal{Y}(\omega)=1.$

  Circle: if $\mathcal{X}(\omega)=1$ then necessarily $\mathcal{Y}(\omega)=1/2.$



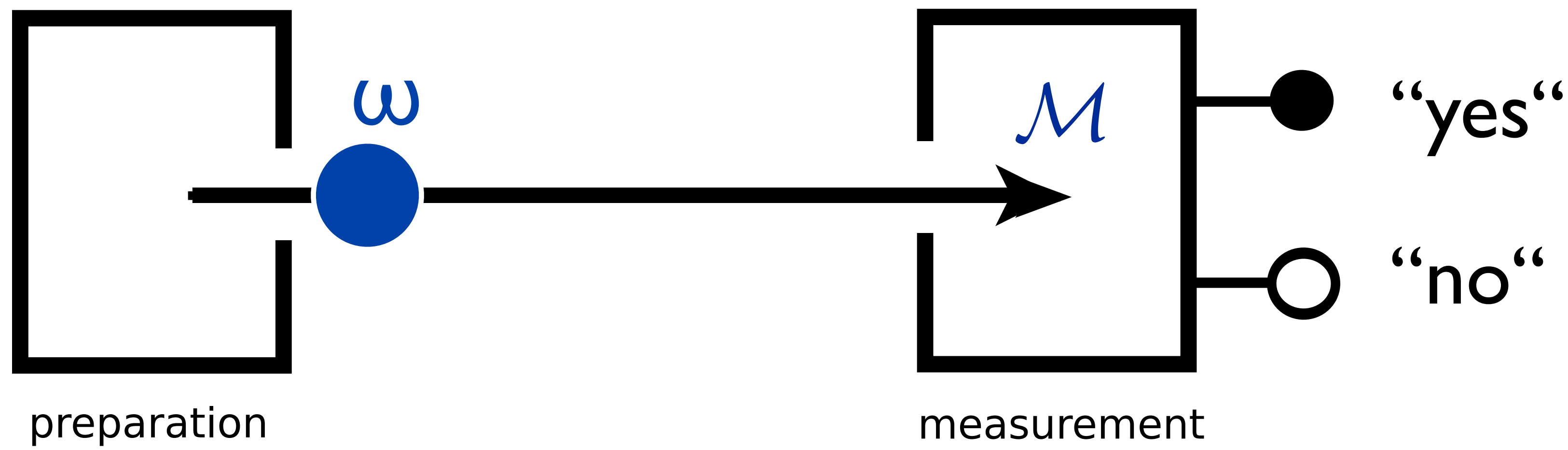
- e), f), g): neither classical nor quantum.
- e), f): let χ=measurement "is spin up in x-direction?"

  analogously for Υ.

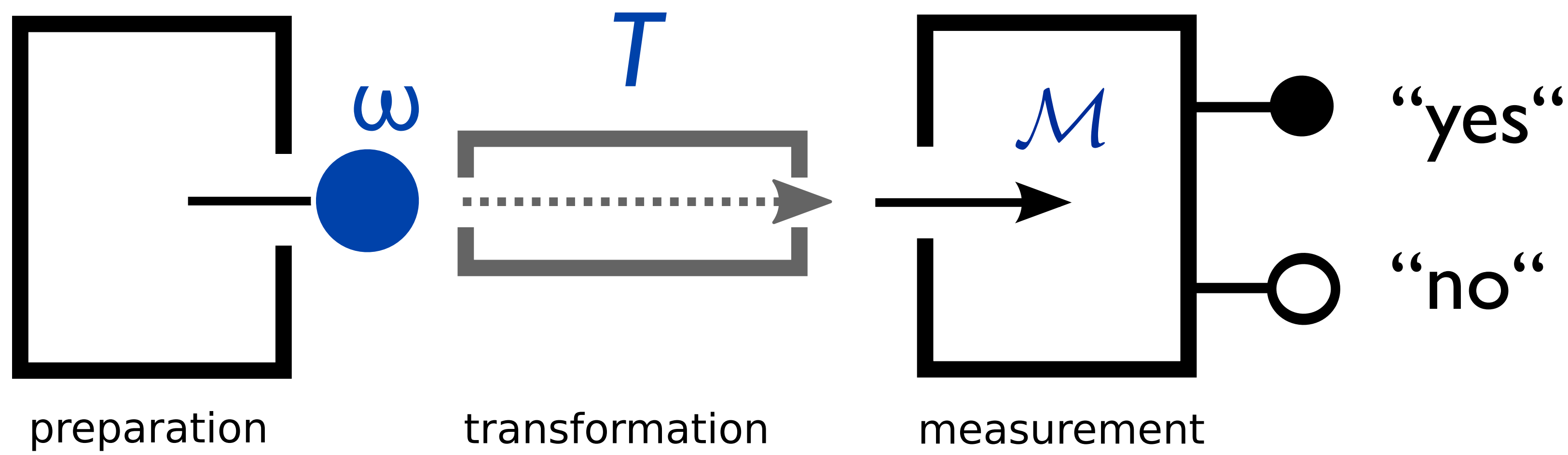
  Square: there is a state ω with $\mathcal{X}(\omega) = \mathcal{Y}(\omega) = 1.$

  Circle: if $\mathcal{X}(\omega) = 1$ then necessarily $\mathcal{Y}(\omega) = 1/2.$

2. The framework

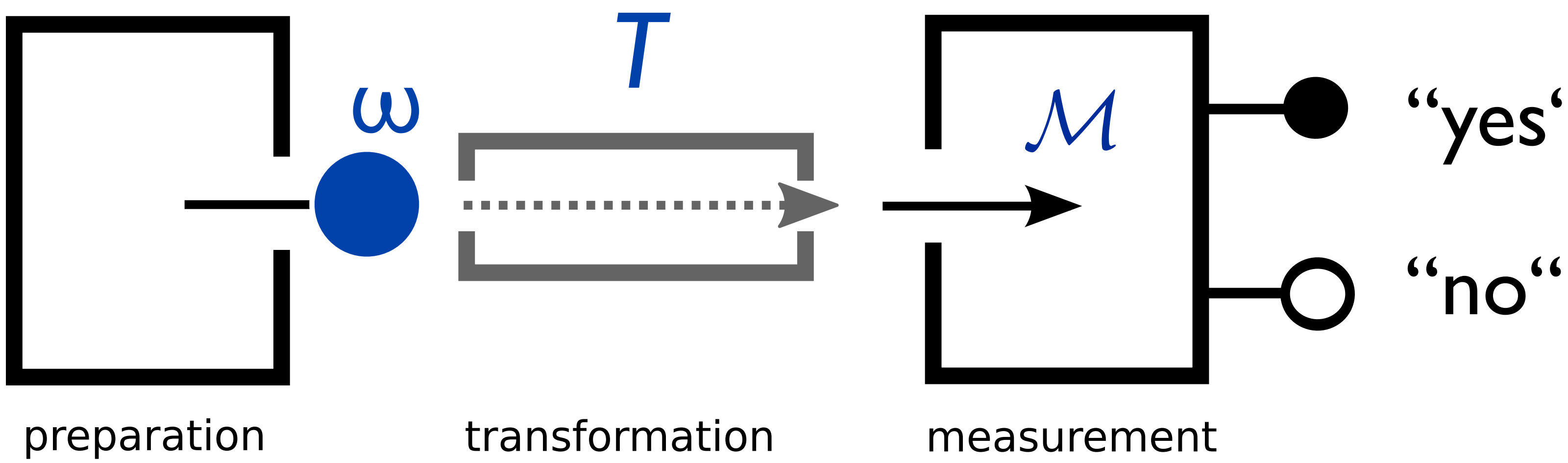


2. The framework



Transformations *T* map states to states and are linear.

2. The framework

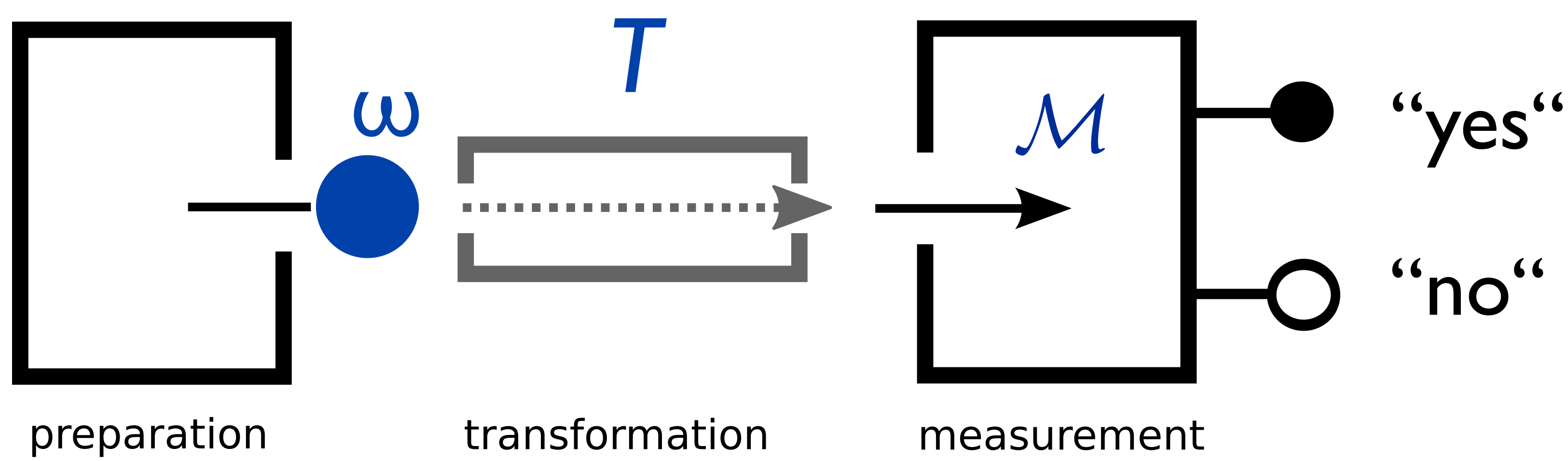


Transformations T map states to states and are linear.

- Here, only interested in reversible transformations T (i.e. invertible).
- They form a compact (maybe finite) group $\mathcal{G}$.
- In quantum theory, these are the unitaries:

$$\rho \mapsto U\rho U^\dagger .$$

2. The framework



Transformations T map states to states and are linear.

- Here, only interested in reversible transformations T (i.e. invertible).
- They form a compact (maybe finite) group $\mathcal{G}$.
- In quantum theory, these are the unitaries:

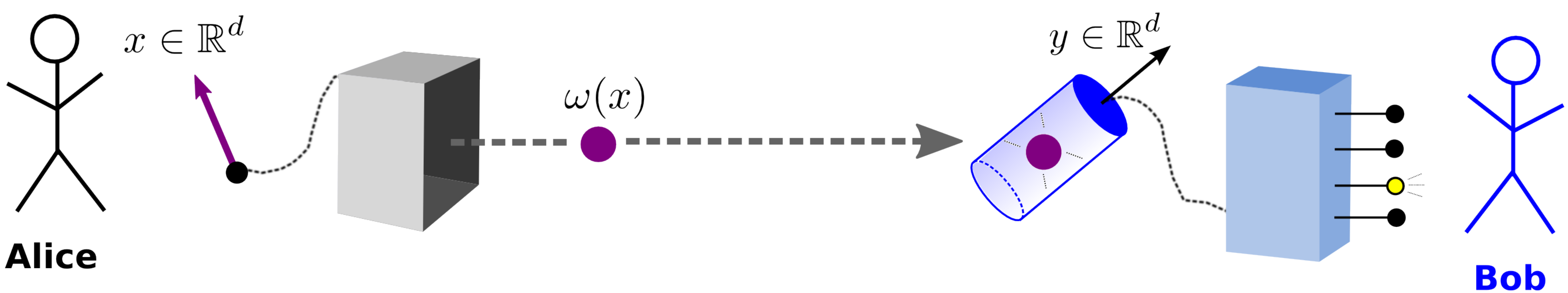
$$\rho \mapsto U\rho U^\dagger .$$

- T acts on states $\leftrightarrow$ T^* acts on measurements:

$$\mathcal{M}(T(\omega)) \equiv T^*(\mathcal{M})(\omega).$$

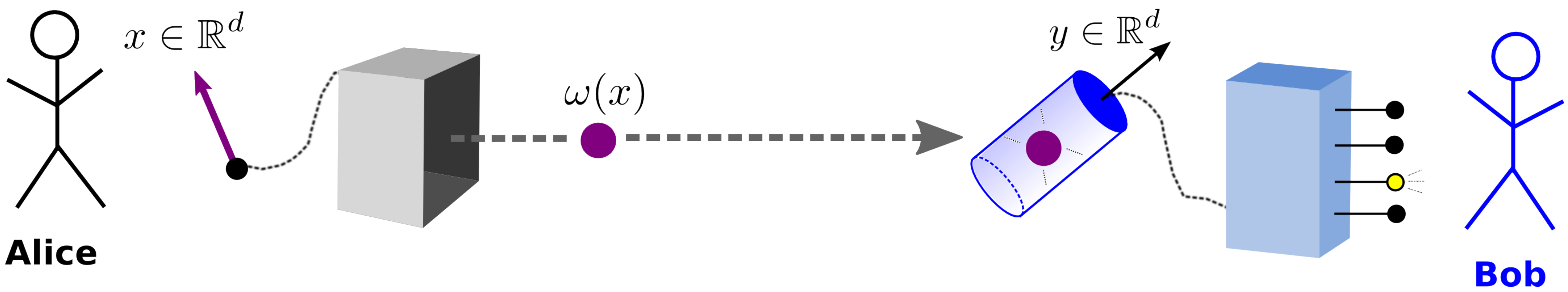
3. The task

Goal: Alice wants to send a spatial direction $x \in \mathbb{R}^d, |x| = 1,$ to **Bob**.



3. The task

Goal: Alice wants to send a spatial direction $x \in \mathbb{R}^d, |x| = 1,$ to Bob.

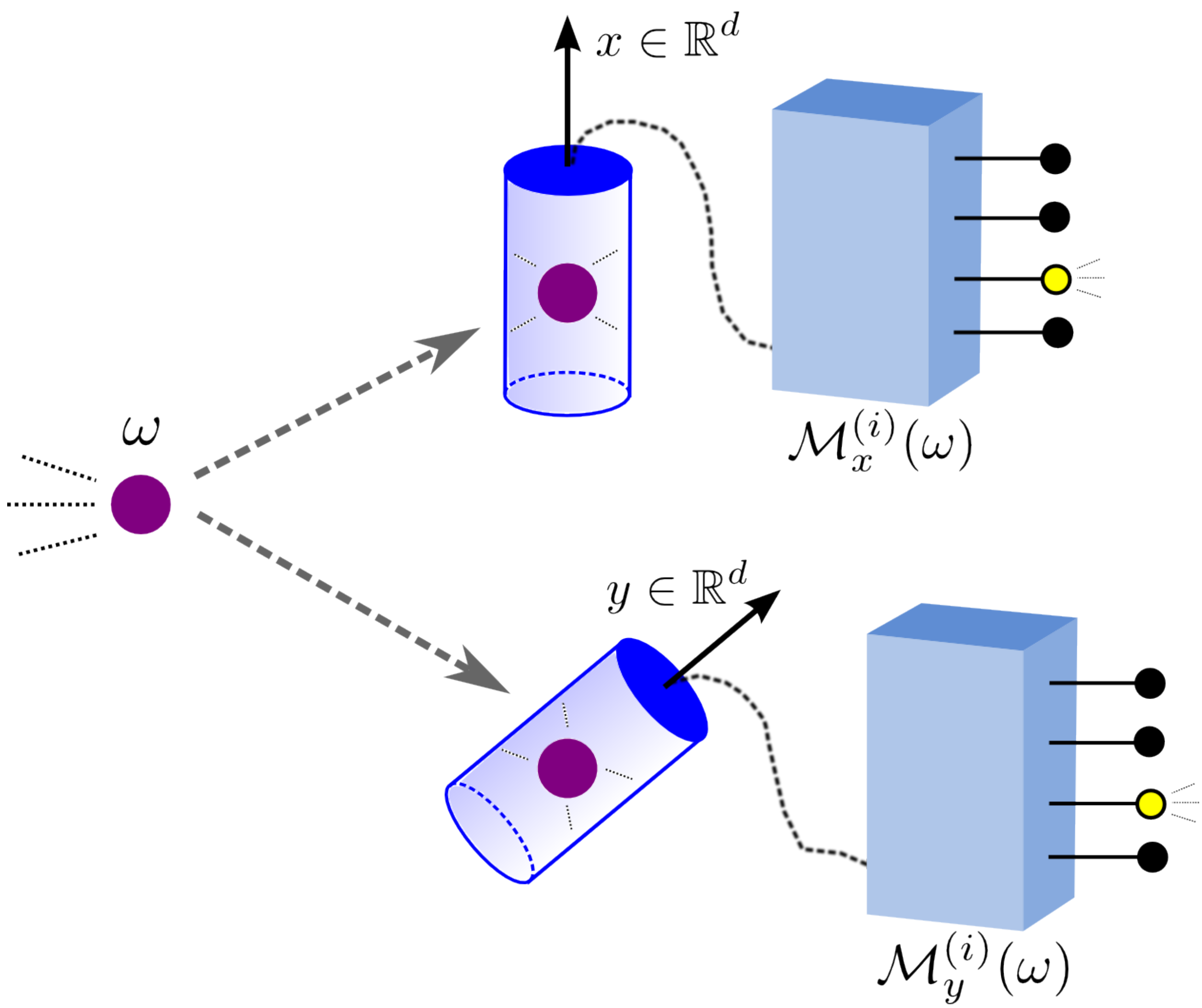


Background:

- Alice and Bob live in *d*-dimensional space.
- There is a well-defined way to transport a vector from Alice to Bob.
- Alice and Bob don‘t share a common coordinate system: can‘t just tell coordinates on the phone!

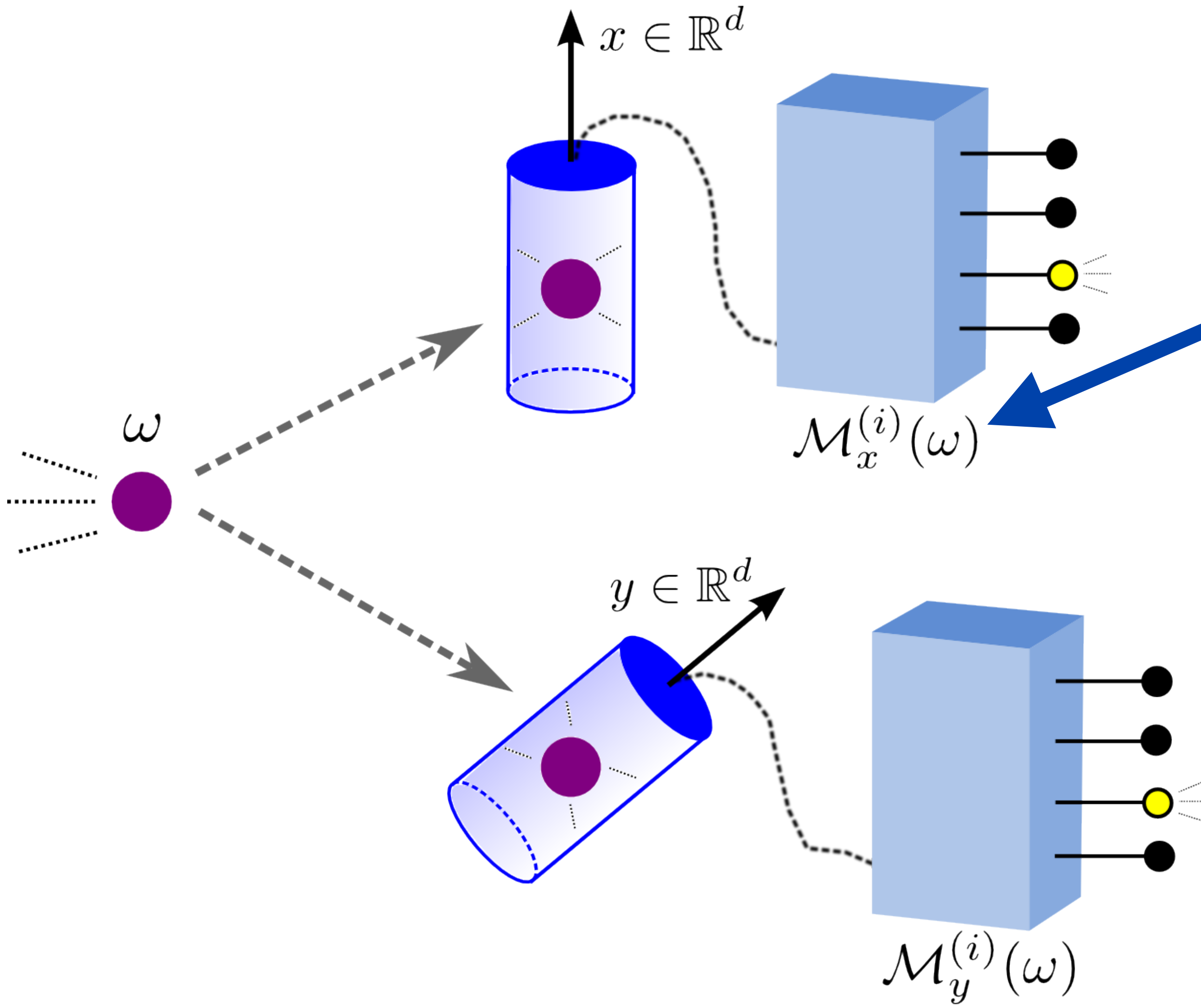
3. The task

Main tool: Bob holds a measurement device that is affected by rotations.



3. The task

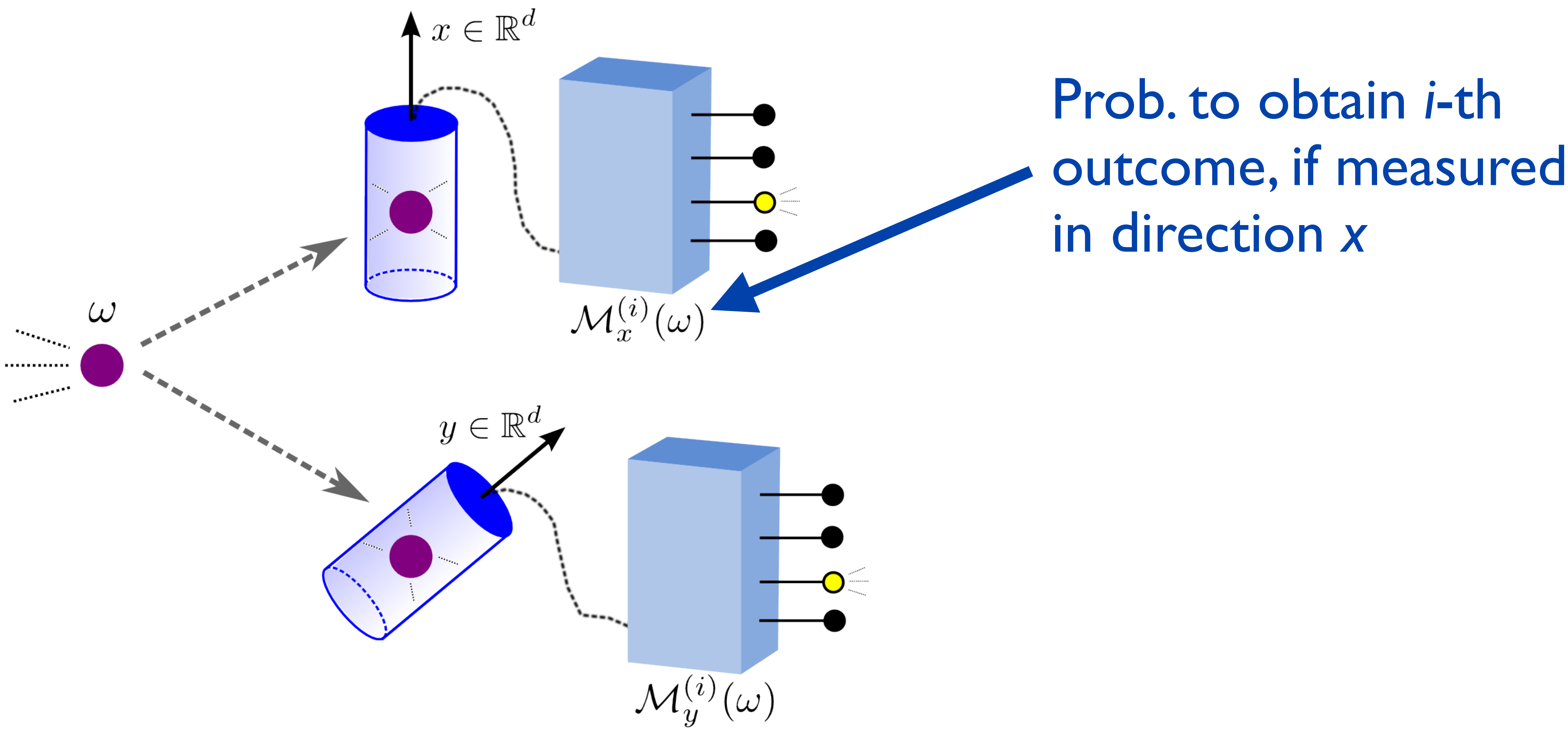
Main tool: Bob holds a measurement device that is affected by rotations.



Prob. to obtain *i*-th outcome, if measured in direction *x*

3. The task

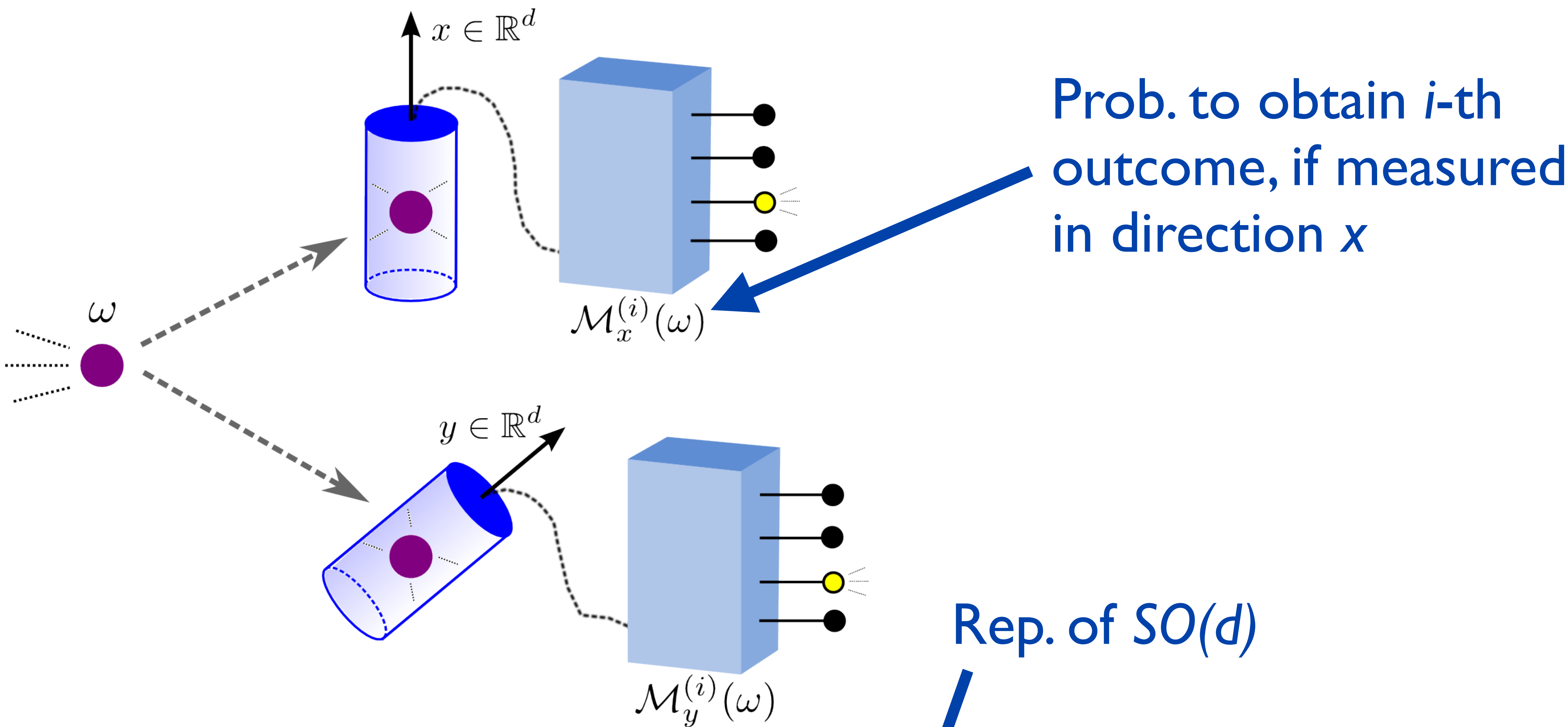
Main tool: Bob holds a measurement device that is affected by rotations.



"Heisenberg picture": Rotation R takes $\mathcal{M}_x^{(i)}$ to $\mathcal{M}_{Rx}^{(i)}$.

3. The task

Main tool: Bob holds a measurement device that is affected by rotations.

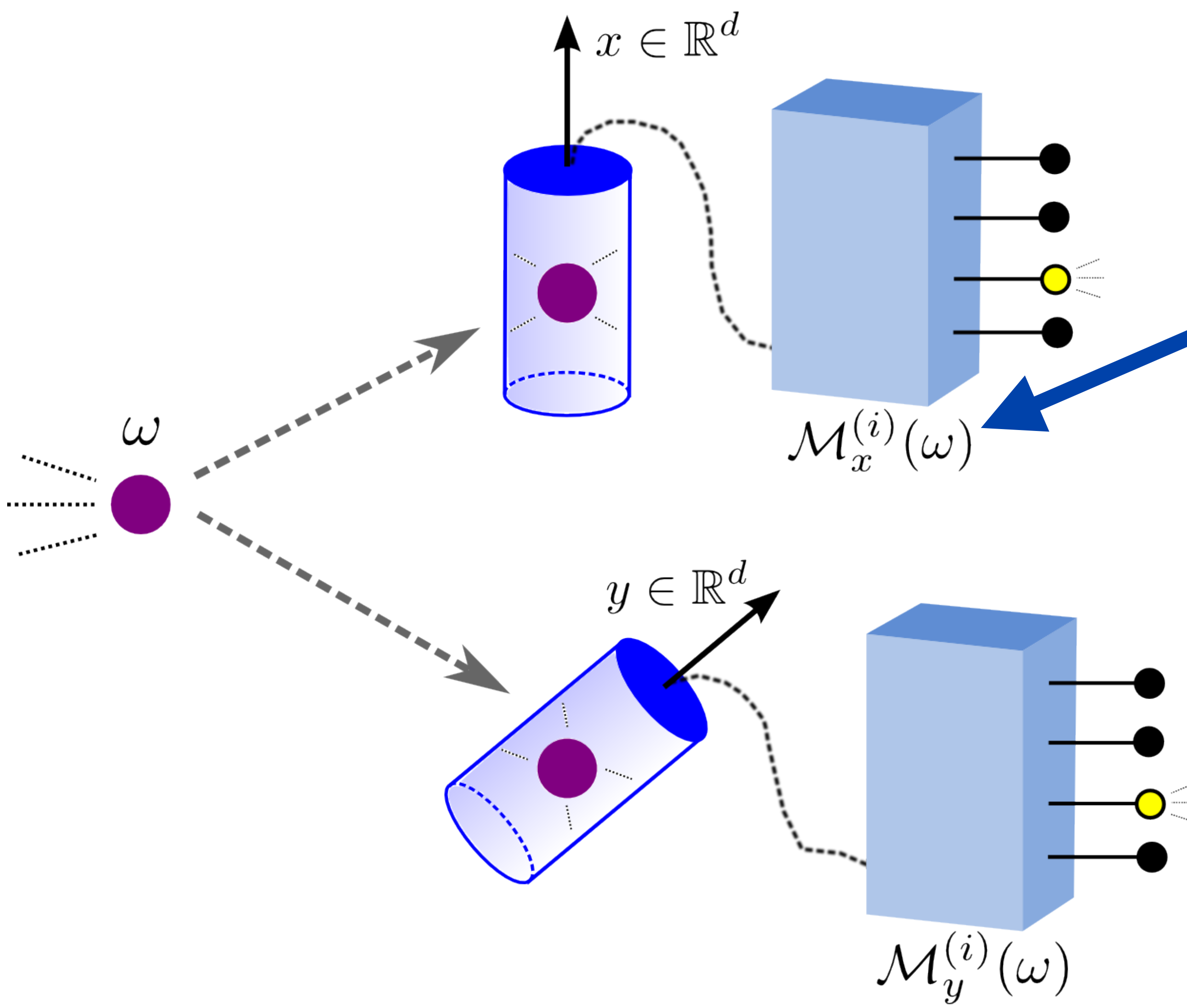


"Heisenberg picture": Rotation *R* takes $\mathcal{M}_x^{(i)}$ to $\mathcal{M}_{Rx}^{(i)}$.

"Schrödinger picture": Rotation *R* takes ω to $G_{R^{-1}}\omega$ (reversible transf.!)

3. The task

Main tool: Bob holds a measurement device that is affected by rotations.



Prob. to obtain *i*-th outcome, if measured in direction *x*

$$\mathcal{M}_x^{(i)}(\omega) \neq \mathcal{M}_{Rx}^{(i)}(\omega)$$

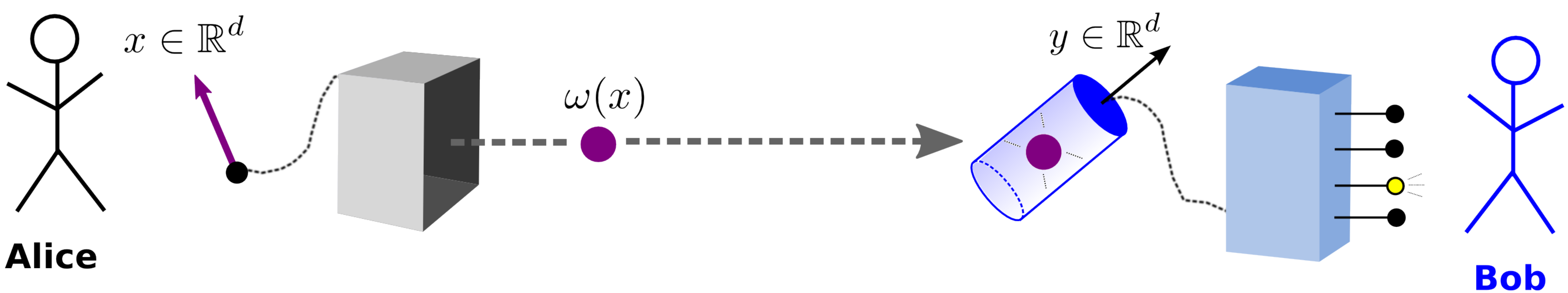
Rep. of *SO(d)*

"Heisenberg picture": Rotation *R* takes $\mathcal{M}_x^{(i)}$ to $\mathcal{M}_{Rx}^{(i)}$.

"Schrödinger picture": Rotation *R* takes ω to $G_{R^{-1}}\omega$ (reversible transf.!)

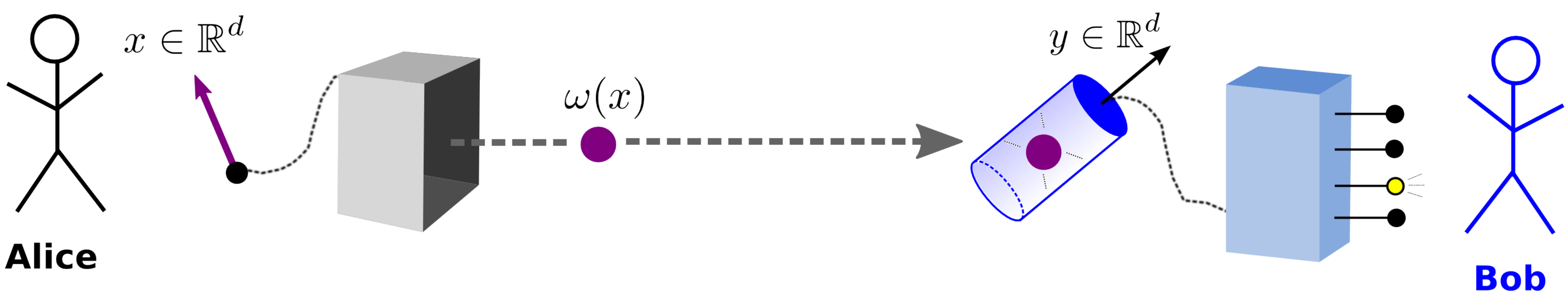
3. The task

Goal: Alice wants to send a spatial direction $x \in \mathbb{R}^d, |x| = 1,$ to Bob.



3. The task

Goal: Alice wants to send a spatial direction $x \in \mathbb{R}^d, |x| = 1,$ to Bob.

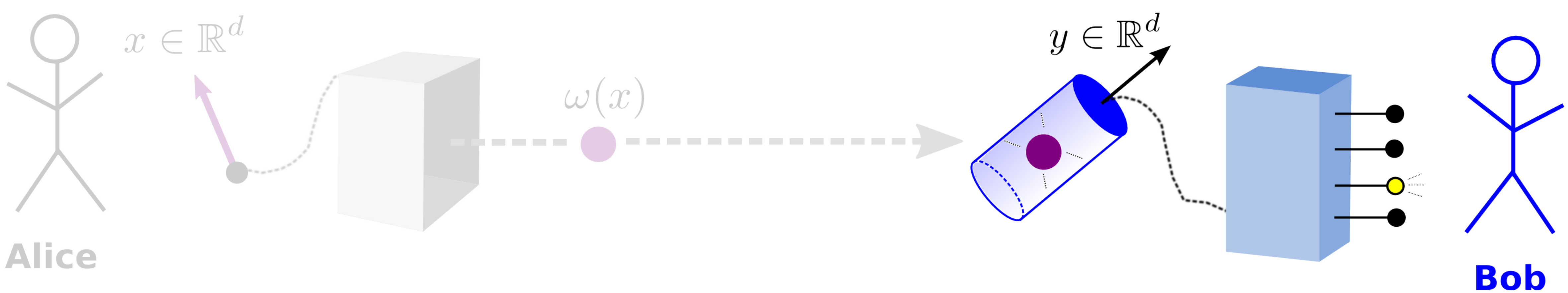


General setup:

- Alice encodes *x* into some state *ω(x)*.
- She transmits many copies of the state to Bob.
- Bob measures in different directions, getting statistics...
- ... estimating *x* in the limit of ∞ many copies.

3. The task

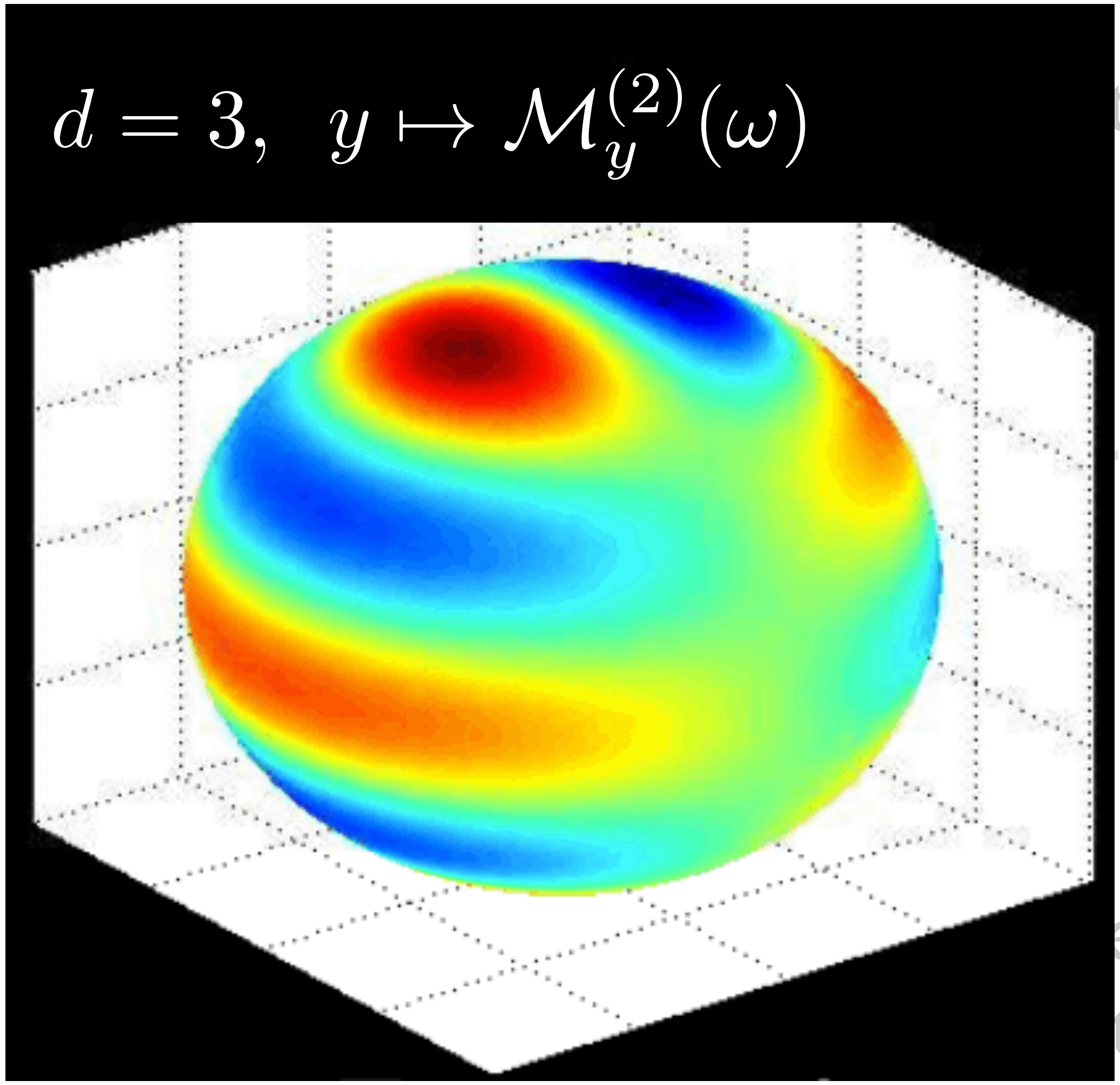
Goal: Alice wants to send a spatial direction $x \in \mathbb{R}^d, |x| = 1,$ to Bob.



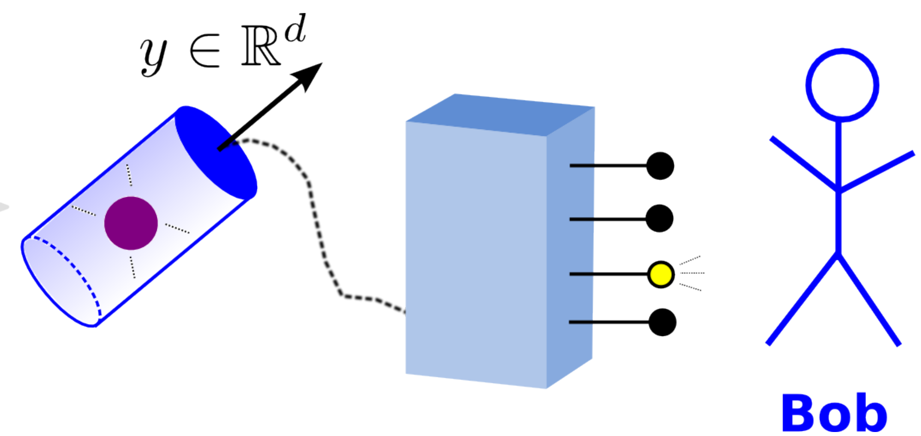
General setup:

- Alice encodes *x* into some state *ω(x)*.
- She transmits many copies of the state to Bob.
- Bob measures in different directions, getting statistics...
- ... estimating *x* in the limit of ∞ many copies.

3. The task



tial direction $x \in \mathbb{R}^d, |x| = 1,$ to Bob.

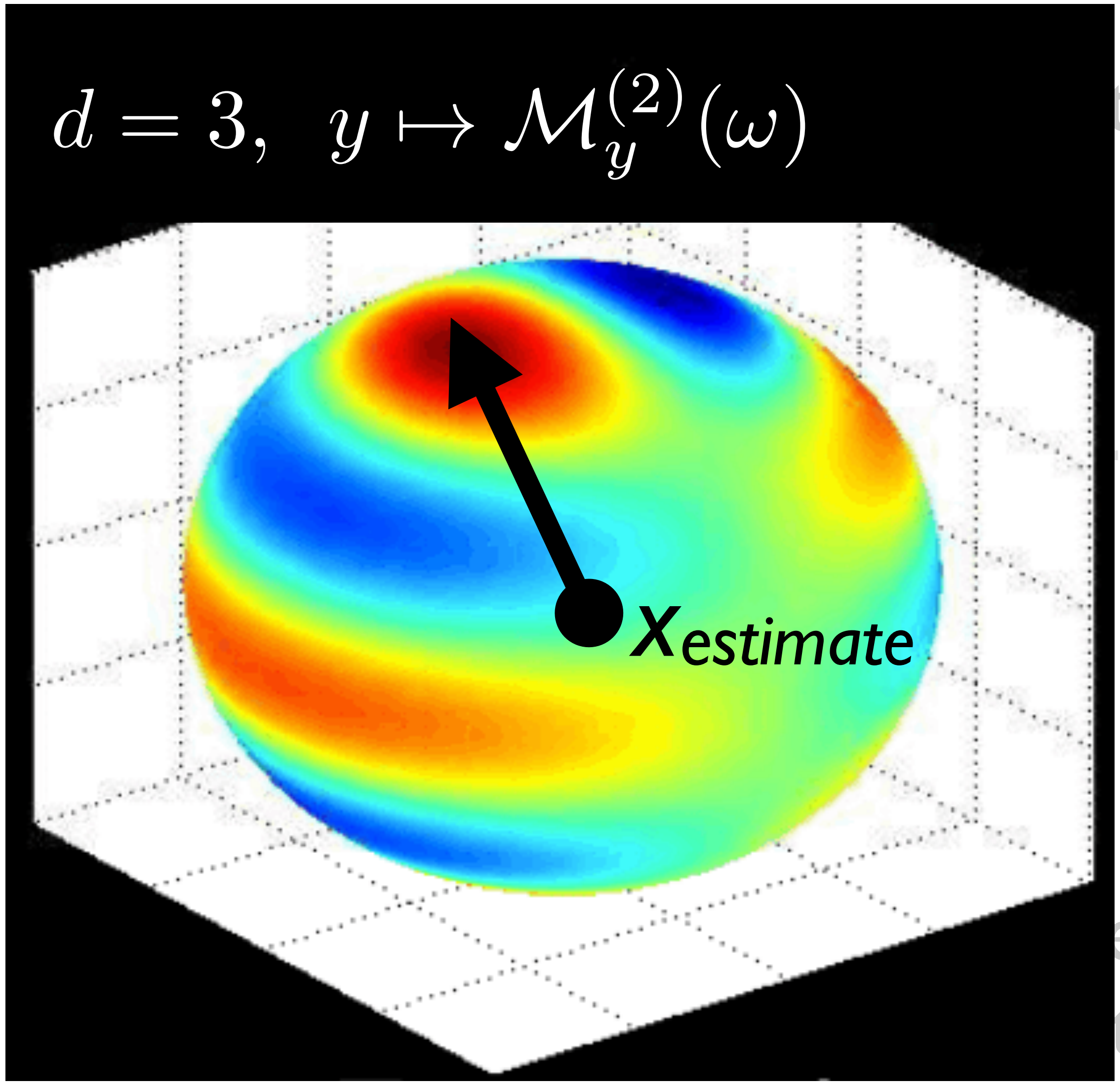


es *x* into some state *ω(x)*.

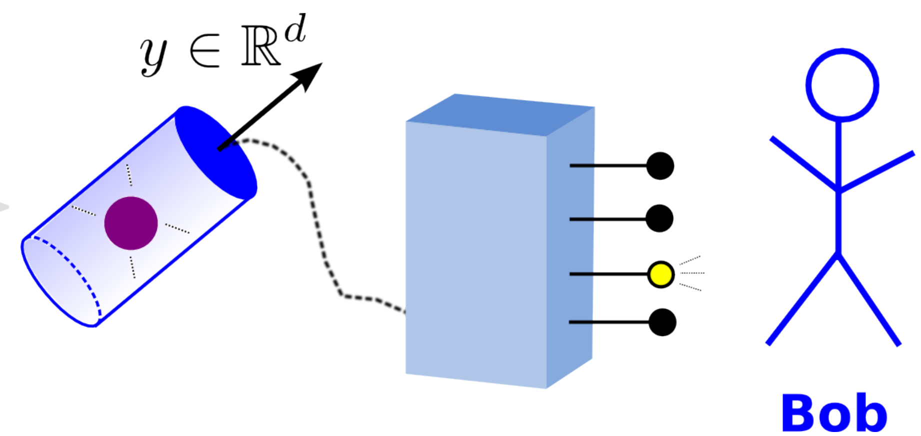
ts many copies of the state to Bob.

- Bob measures in different directions, getting statistics...
- ... estimating *x* in the limit of ∞ many copies.

3. The task



…tial direction $x \in \mathbb{R}^d, |x| = 1,$ to Bob.



…es *x* into some state *ω(x)*.

…ts many copies of the state to Bob.

- Bob measures in different directions, getting statistics...
- ... estimating *x* in the limit of ∞ many copies.

3. The task

Question:

- In what dimensions *d*, and
- for what state spaces

can this task be accomplished in a “nice“ way?

M. Müller*, Ll. Masanes

3. The task

Question:

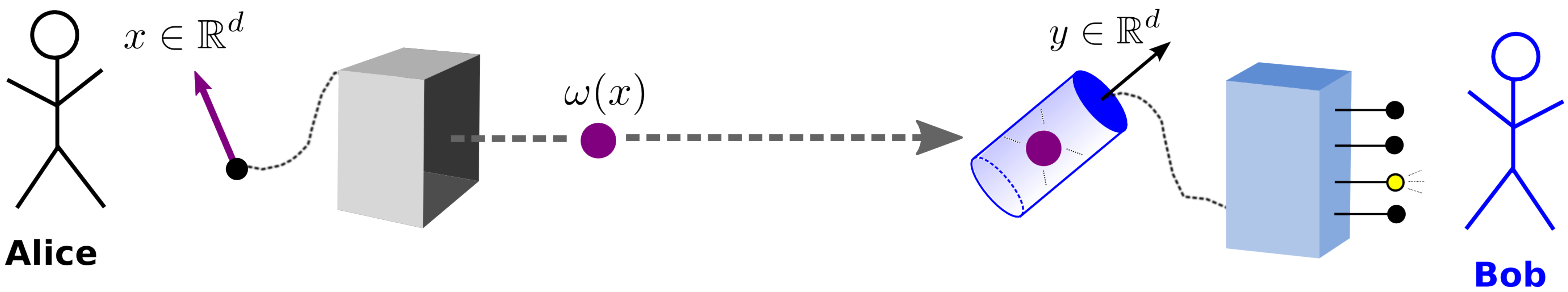
- In what dimensions *d*, and
- for what state spaces

can this task be accomplished in a "nice" way?

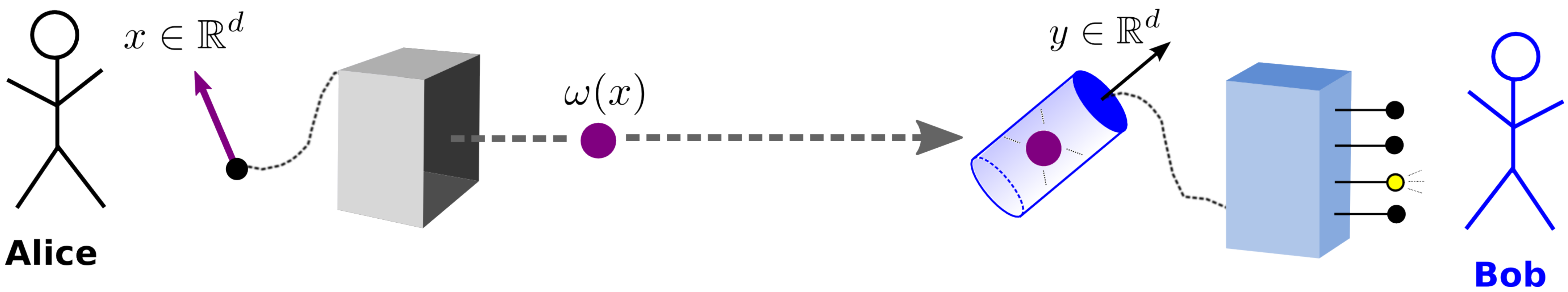
with minimal overhead

satisfying basic physical requirements

3. The task

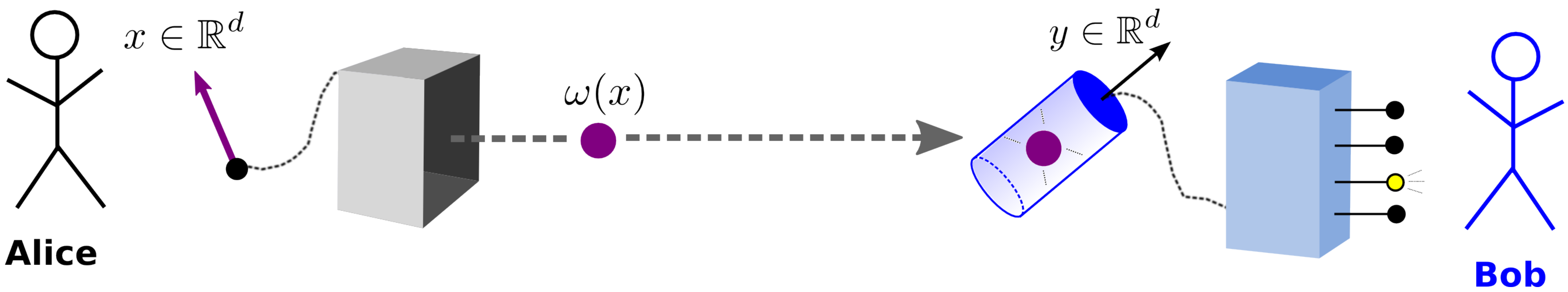


3. The task



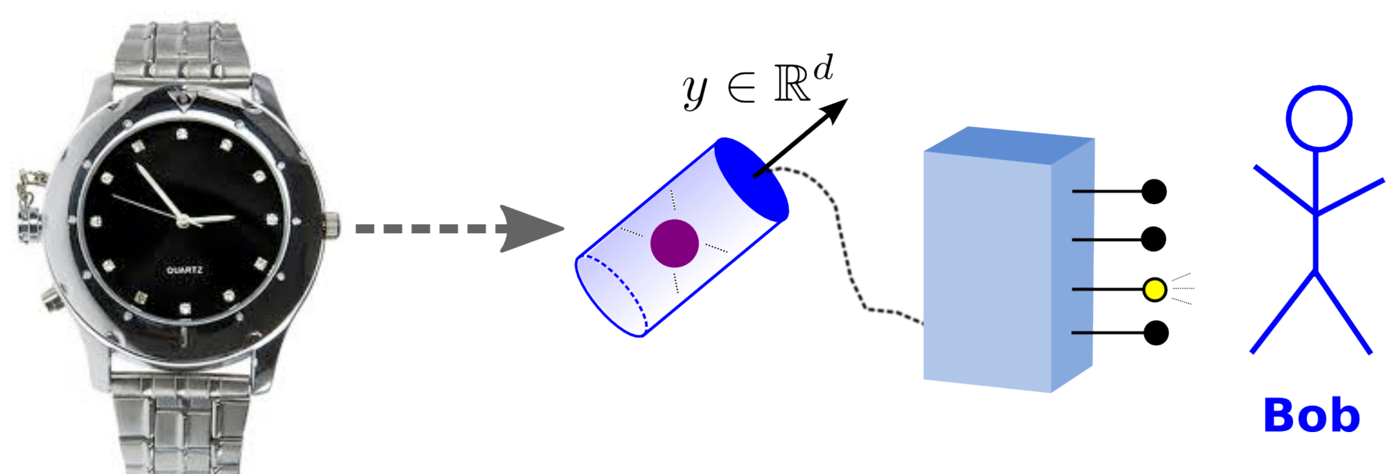
Postulate 1 (Achievability). There is a protocol which allows Alice to encode any spatial direction $x \in \mathbb{R}^d$, $|x| = 1$, into a state $\omega(x)$, such that Bob is able to retrieve x in the limit of many copies.

3. The task

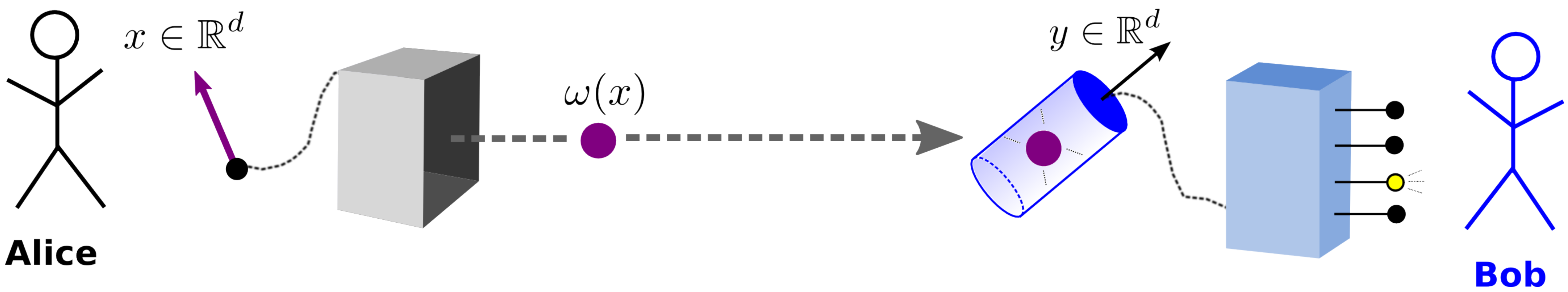


Postulate 1 (Achievability). There is a protocol which allows Alice to encode any spatial direction $x \in \mathbb{R}^d$, $|x| = 1$, into a state $\omega(x)$, such that Bob is able to retrieve x in the limit of many copies.

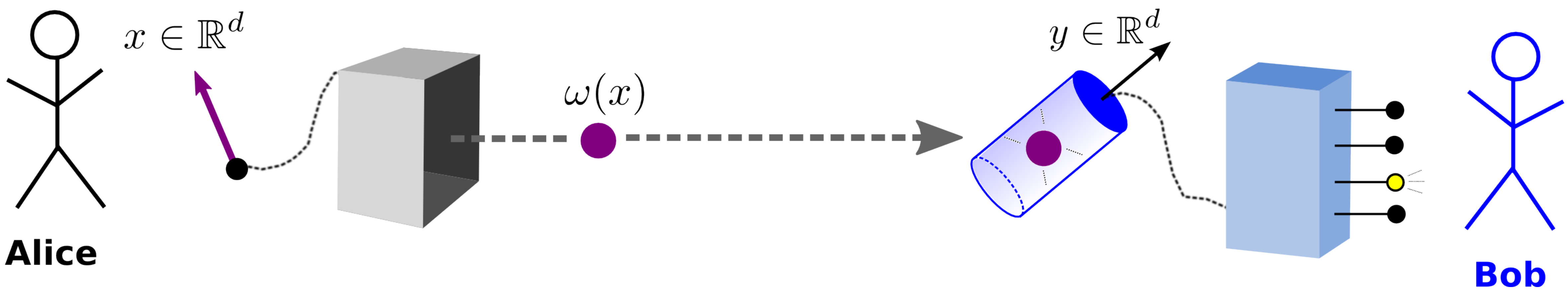
Trivial solution (d=2): Alice sends (stopped) wristwatch.



3. The task

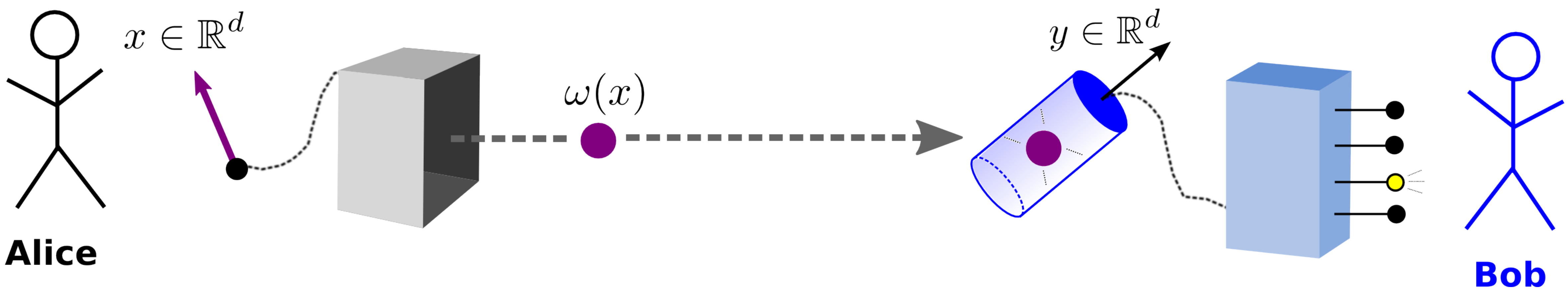


3. The task

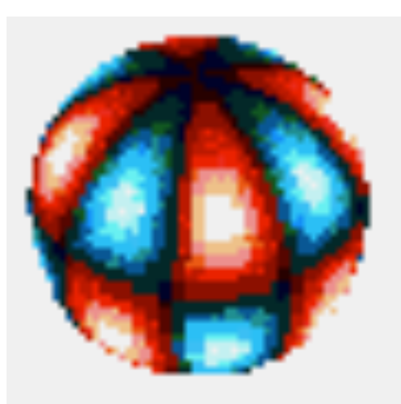


Postulate 2 (Minimality). No protocol which satisfies Postulate 1 allows Alice to encode any further information, except for adding some amount of uniform noise.

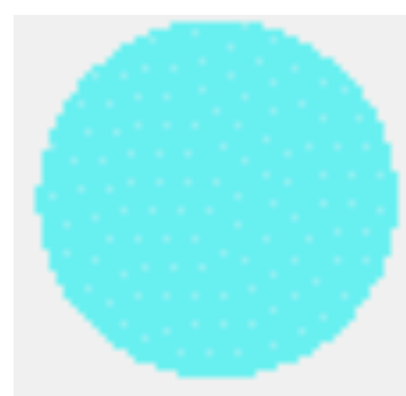
3. The task



Postulate 2 (Minimality). No protocol which satisfies Postulate 1 allows Alice to encode any further information, except for adding some amount of uniform noise.



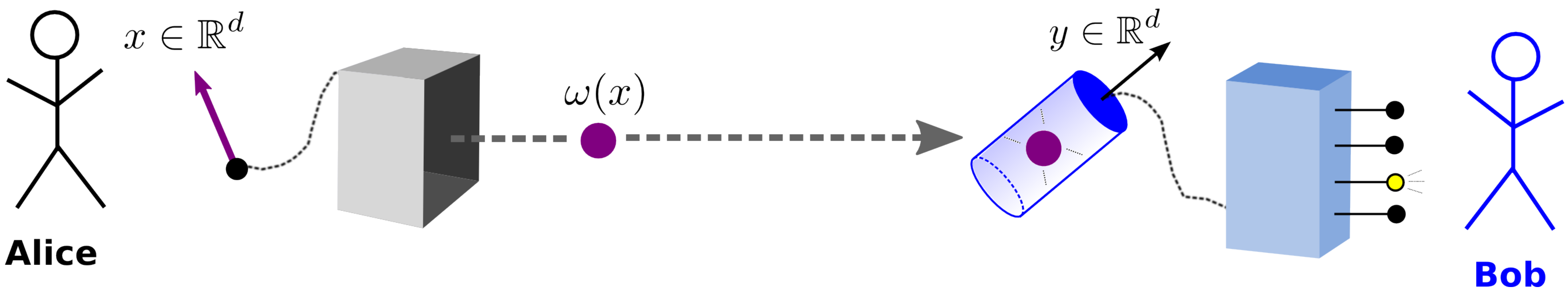
Bob's "signal"



uniform noise

signal+noise

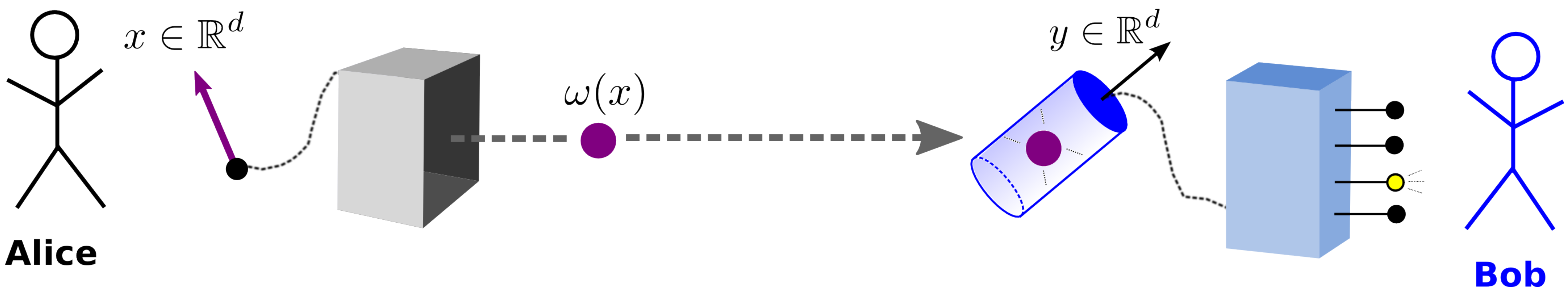
3. The task



Postulate 2 (Minimality). No protocol which satisfies Postulate 1 allows Alice to encode any further information, except for adding some amount of uniform noise.

ω, ω' encode same vector $x \in \mathbb{R}^d \Rightarrow$
$\omega' = (1-\lambda)\omega + \lambda\mu$ or vice versa

3. The task

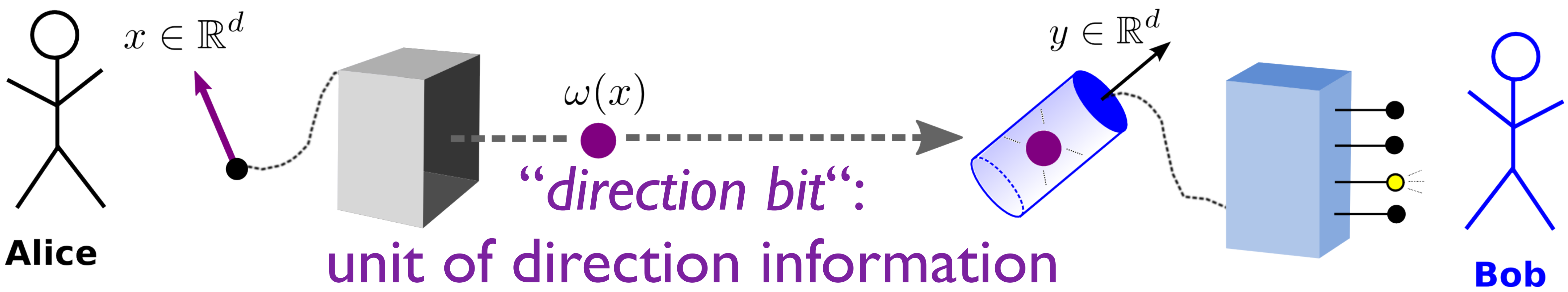


Postulate 2 (Minimality). No protocol which satisfies Postulate 1 allows Alice to encode any further information, except for adding some amount of uniform noise.

ω, ω' encode same vector $x \in \mathbb{R}^d \Rightarrow$
$\omega' = (1-\lambda)\omega + \lambda\mu$ or vice versa

"uniform noise" state: $\mathcal{M}_x^{(i)}(\mu) = \mathcal{M}_y^{(i)}(\mu) \ \forall x, y.$

3. The task



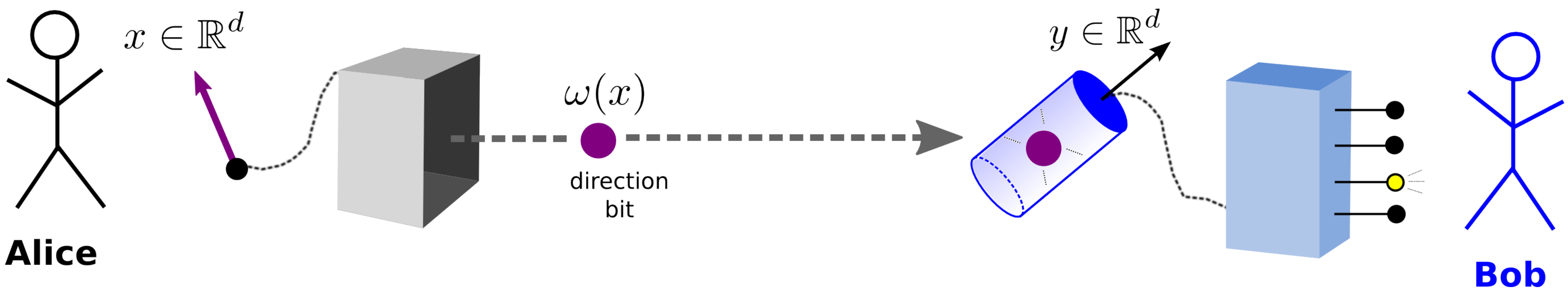
"direction bit":
unit of direction information

Postulate 2 (Minimality). No protocol which satisfies Postulate 1 allows Alice to encode any further information, except for adding some amount of uniform noise.

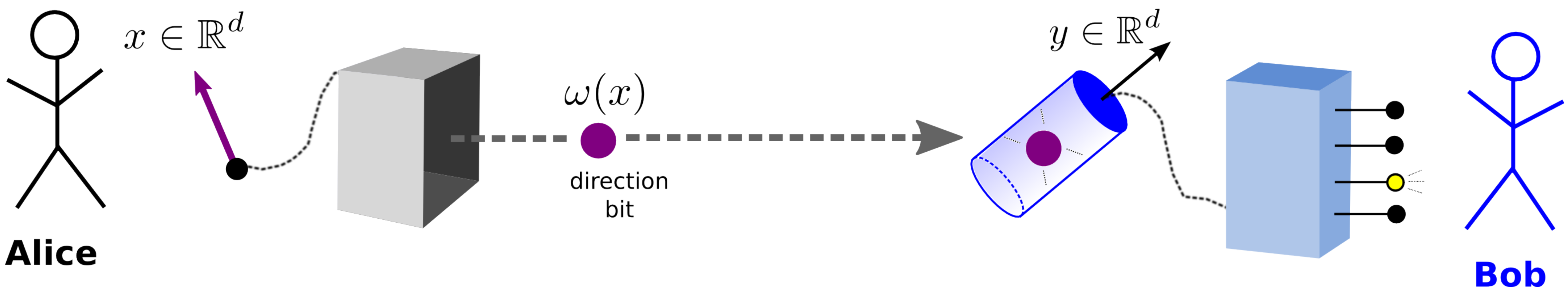
ω, ω' encode same vector $x \in \mathbb{R}^d \Rightarrow$
$\omega' = (1-\lambda)\omega + \lambda\mu$ or vice versa

"uniform noise" state: $\mathcal{M}_x^{(i)}(\mu) = \mathcal{M}_y^{(i)}(\mu) \;\forall x, y.$

3. The task



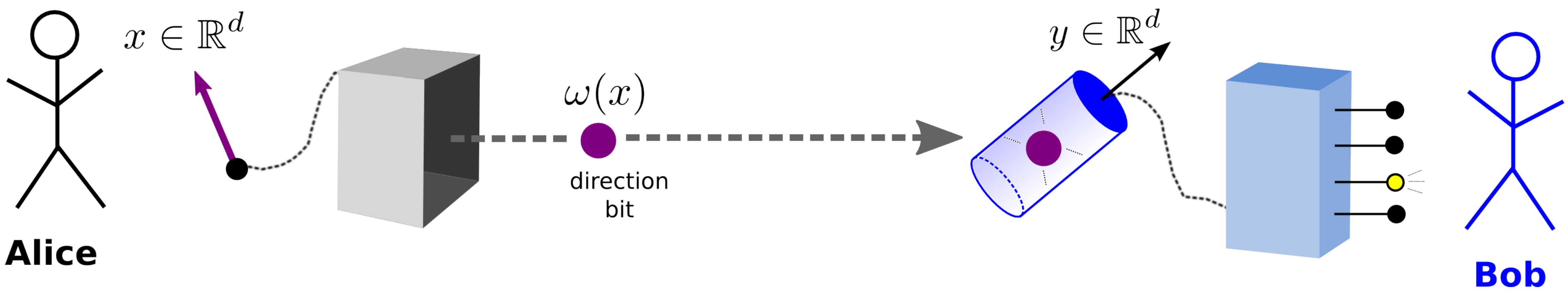
3. The task



IF a state space allows any protocol satisfying Postulates 1 and 2,

THEN the following standard protocol works, too (for some *i*):

3. The task

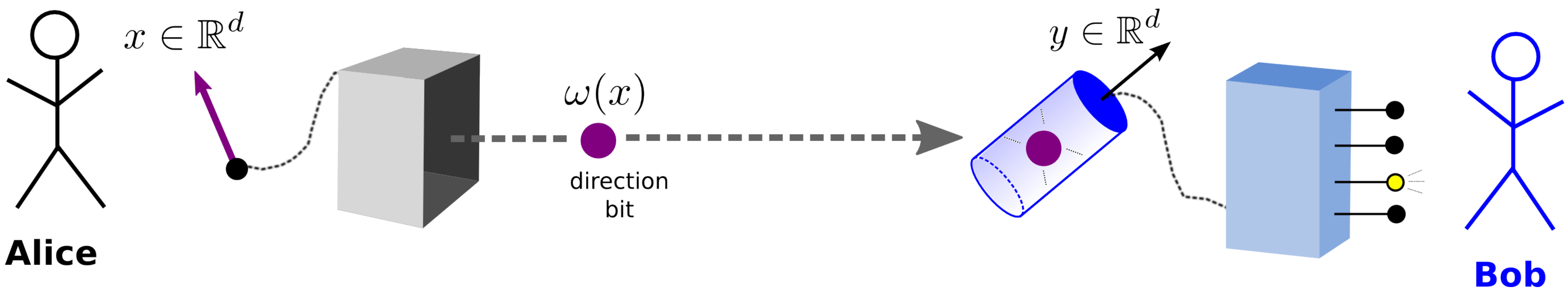


IF a state space allows any protocol satisfying Postulates 1 and 2,

THEN the following standard protocol works, too (for some *i*):

- Alice encodes *x* in some clever state ω, and
- Bob guesses the *y* where $L_y(\omega) := \mathcal{M}_y^{(i)}(\omega) - \mathcal{M}_{-y}^{(i)}(\omega)$ is maximal.

3. The task



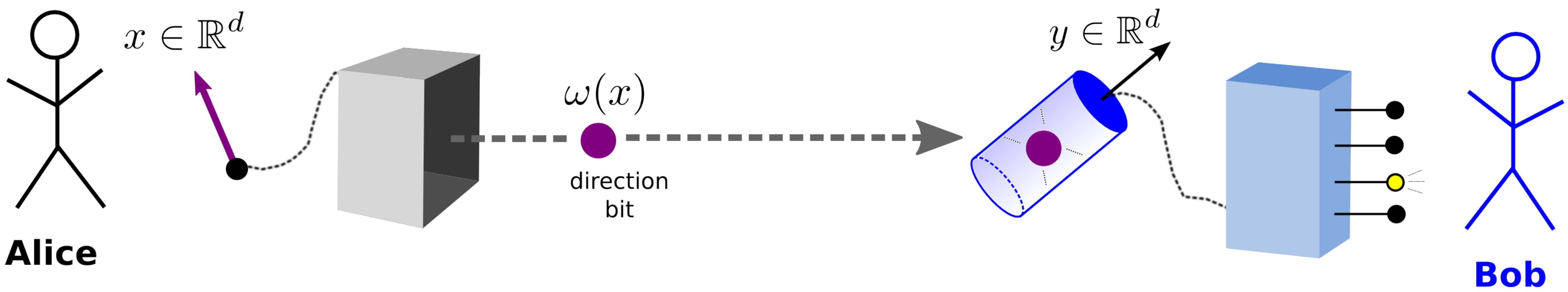
IF a state space allows any protocol satisfying Postulates 1 and 2,

THEN the following standard protocol works, too (for some *i*):

- Alice encodes *x* in some clever state ω, and
- Bob guesses the *y* where $L_y(\omega) := \mathcal{M}_y^{(i)}(\omega) - \mathcal{M}_{-y}^{(i)}(\omega)$ is maximal.

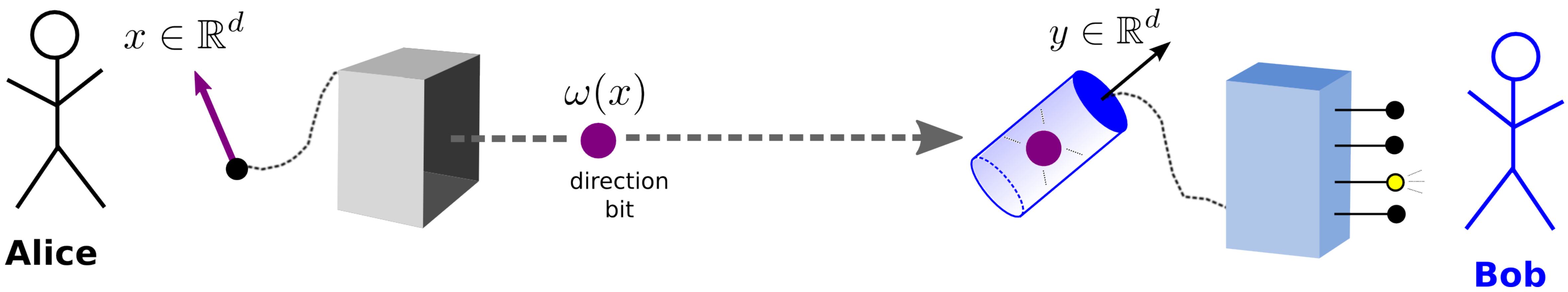
looks like spin-1/2-expectation value!

3. The task



- Alice encodes *x* in some clever state ω, and
- Bob guesses the *y* where $L_y(\omega) := \mathcal{M}_y^{(i)}(\omega) - \mathcal{M}_{-y}^{(i)}(\omega)$ is maximal.

3. The task

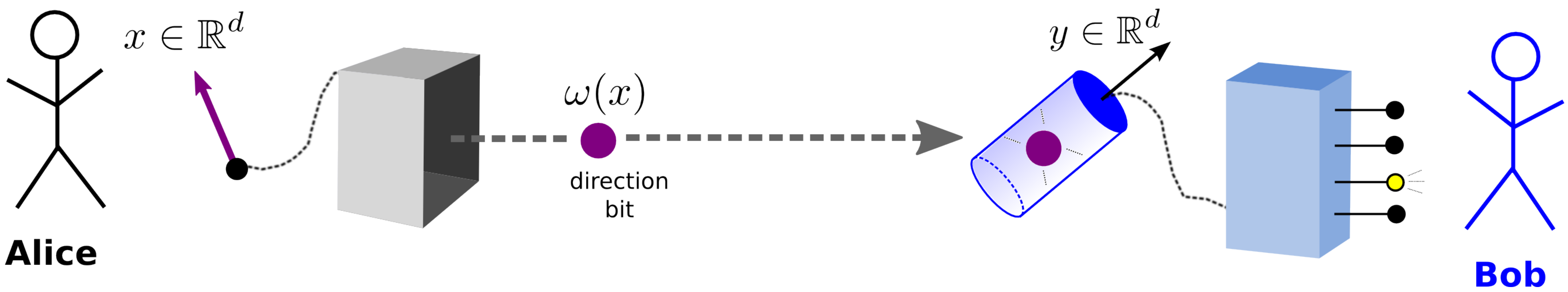


If ω and ω' have same maximizer *y* and same $L_y(\omega) = L_y(\omega')$
⇒ both encode same state *x*=*y* and are equally noisy

Postulate 2 ⇒ ω= ω'.

- Alice encodes *x* in some clever state ω, and
- Bob guesses the *y* where $L_y(\omega) := \mathcal{M}_y^{(i)}(\omega) - \mathcal{M}_{-y}^{(i)}(\omega)$ is maximal.

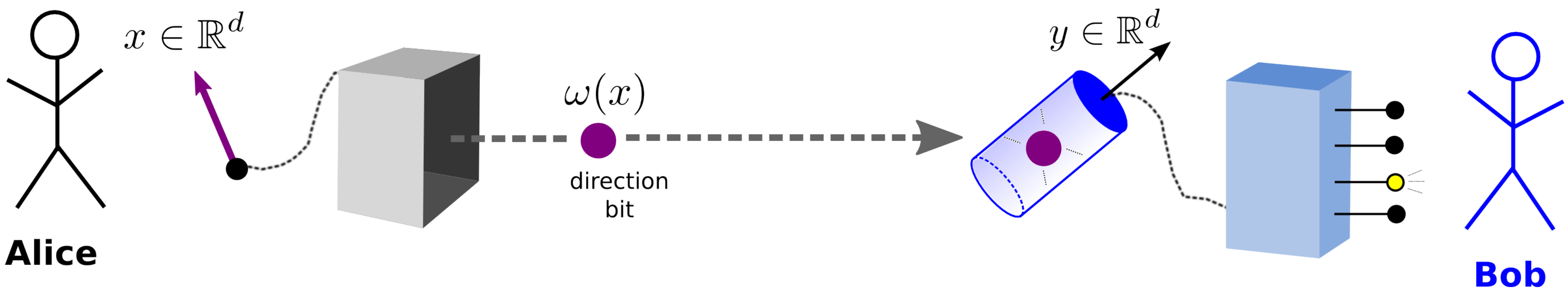
3. The task



If ω and ω‘ have same maximizer *y* and same $L_y(\omega) = L_y(\omega')$
⇒ both encode same state *x*=*y* and are equally noisy

Postulate 2 ⇒ ω= ω‘.

3. The task



If ω and ω‘ have same maximizer *y* and same $L_y(\omega) = L_y(\omega')$
⇒ both encode same state *x*=*y* and are equally noisy

Postulate 2 ⇒ ω= ω‘.

Consequence: Every state ω can be written in the form

$$\omega = \lambda \omega_x + (1-\lambda)\mu,$$

where ω_x is a pure state encoding some direction x, and μ is the maximally mixed state $\mu = \int_{R\in SO(d)} G_R \omega_x \, dR$.

3. The task

Consequence: Every state ω can be written in the form

$$\omega = \lambda \omega_x + (1-\lambda)\mu,$$

where ω_x is a pure state encoding some direction x, and μ is the maximally mixed state $\mu = \int_{R \in SO(d)} G_R \omega_x \, dR$.

3. The task

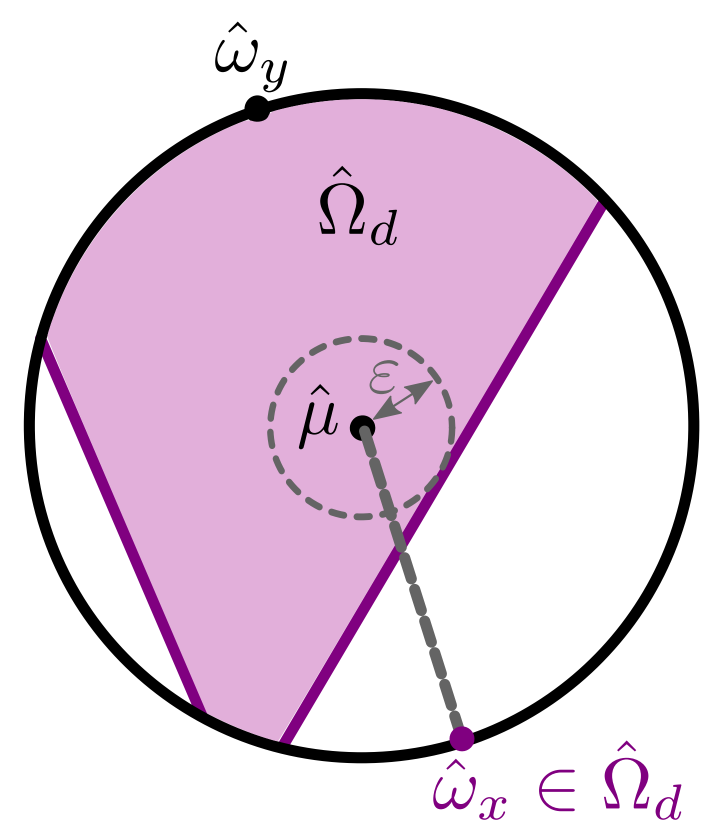
Consequence: Every state ω can be written in the form

$$\omega = \lambda\omega_x + (1-\lambda)\mu,$$

where ω_x is a pure state encoding some direction x, and μ is the maximally mixed state $\mu = \int_{R\in SO(d)} G_R\omega_x\, dR$.

All ω_x, ω_y connected by rotations G_R and $G_R\,\mu=\mu$.

Group rep. th. $\Rightarrow$ can find inner product such that $\|\omega_x - \mu\| = 1\ \forall x.$



3. The task

Consequence: Every state ω can be written in the form

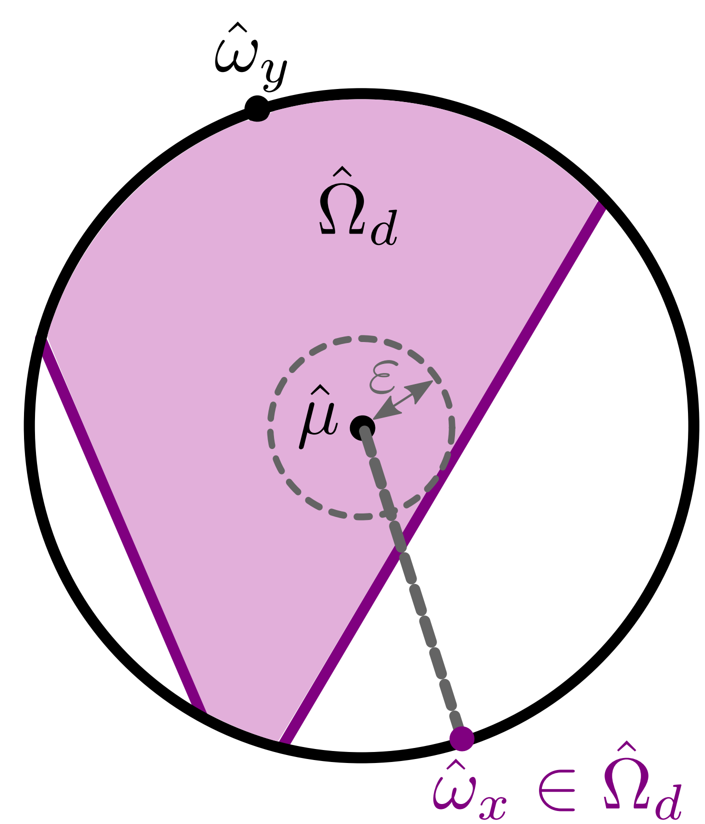
$$\omega = \lambda\omega_x + (1-\lambda)\mu,$$

where ω_x is a pure state encoding some direction x, and μ is the maximally mixed state $\mu = \int_{R\in SO(d)} G_R \omega_x \, dR$.

All ω_x, ω_y connected by rotations G_R and $G_R\,\mu=\mu$.

Group rep. th. ⇒ can find inner product such that $\|\omega_x - \mu\| = 1 \;\; \forall x.$

Only possible if state space is the full ball.



3. The task

Theorem 1: The state space of a direction bit is a d-dimensional unit ball, and it holds $d \neq 2$.

3. The task

Theorem 1: The state space of a direction bit is a d-dimensional unit ball, and it holds $d \neq 2$.

bit (not trit)

complementarity etc.!

3. The task

Theorem 1: The state space of a direction bit is a d-dimensional unit ball, and it holds $d \neq 2$.

3. The task

Theorem 1: The state space of a direction bit is a d-dimensional unit ball, and it holds $d \neq 2$.

[Not all analogs of spin measurements may be possible -- *noisy ball.*]

3. The task

Theorem 1: The state space of a direction bit is a d-dimensional unit ball, and it holds $d \neq 2$.

[Not all analogs of spin measurements may be possible -- *noisy ball.*]

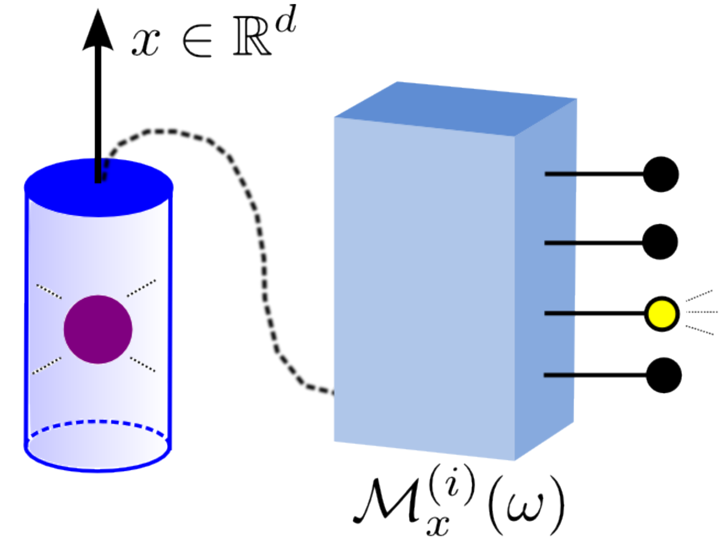
In our world: *d=3*, state space=qubit, Bob holds Stern-Gerlach device.

Why *d=3?* → *need two more postulates.*

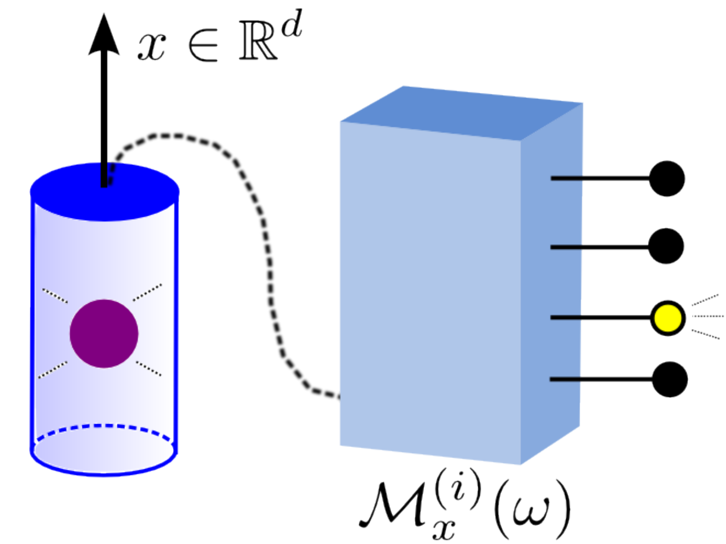
M. Müller*, Ll. Masanes

3. The task

So far: due to symmetry, measurements characterized by vector $x \in \mathbb{R}^d, \quad |x| = 1.$



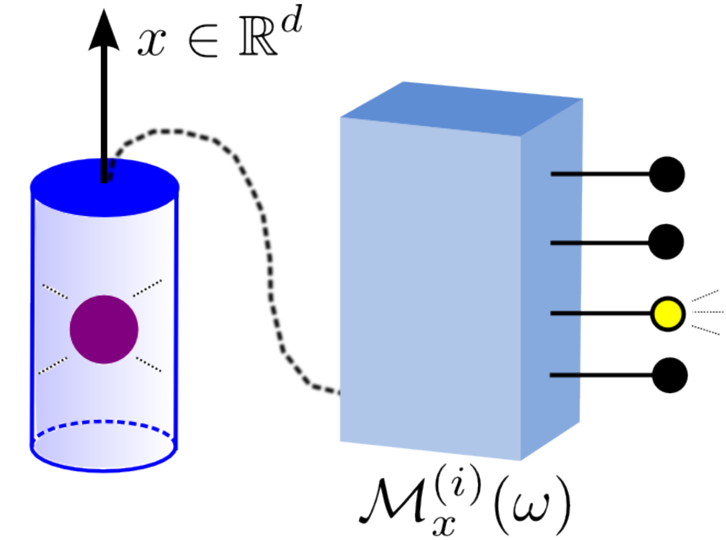
3. The task



So far: due to symmetry, measurements characterized by vector $x \in \mathbb{R}^d, \quad |x| = 1.$

For $d \geq 3$:what if device does not have this symmetry?
Orientation characterized by matrix $X \in SO(d)$.

3. The task

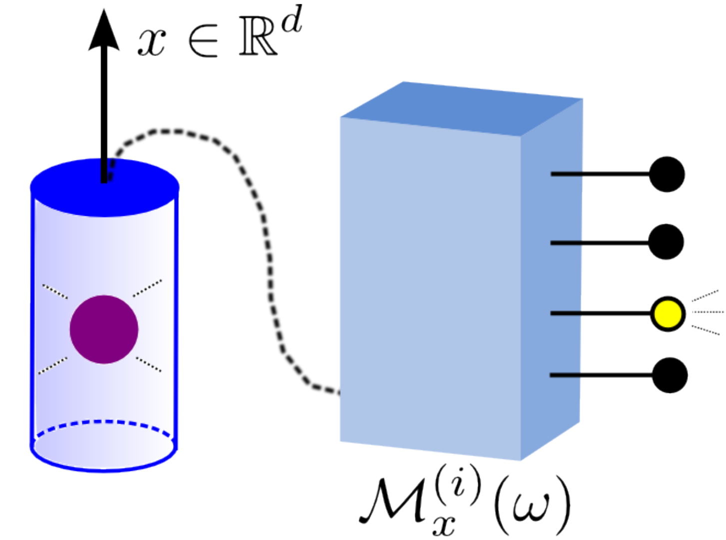


So far: due to symmetry, measurements characterized by vector $x \in \mathbb{R}^d, \quad |x| = 1.$

For $d \geq 3$:what if device does not have this symmetry?
Orientation characterized by matrix $X \in SO(d)$.

Theorem: The analogs of Postulates 1+2 (for "orientation" instead of "direction") do not have any solution.

3. The task



So far: due to symmetry, measurements characterized by vector $x \in \mathbb{R}^d, \quad |x| = 1.$

For $d \geq 3$:what if device does not have this symmetry?
Orientation characterized by matrix $X \in SO(d)$.

Theorem: The analogs of Postulates 1+2 (for "orientation" instead of "direction") do not have any solution.

Proof: State space would again be a unit ball. Pure states: $\{\omega_X\}_{X \in SO(d)}$
But SO(*d*) is not simply connected, and the sphere is.

4. Postulates 3+4

Two direction bits should be able to interact via some continuous reversible time evolution:

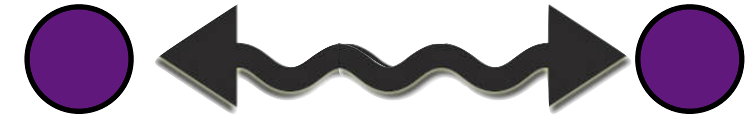
4. Postulates 3+4

Two direction bits should be able to interact via some continuous reversible time evolution:

<u>Postulate 3</u> (interaction):
On the joint state space of two direction bits *A* and *B*, there is a continuous one-parameter group of transformations $\{T_t^{AB}\}_{t\in\mathbb{R}}$ which is not a product of local transformations, $T_t^{AB} \neq T_t^A T_t^B$.

Otherwise no correlation, never!

4. Postulates 3+4



Basic assumptions on composite state space *AB*:

- Contains "product states" $\omega^A \omega^B$.
- Allows for "product measurements" $\mathcal{M}^A \mathcal{M}^B$:

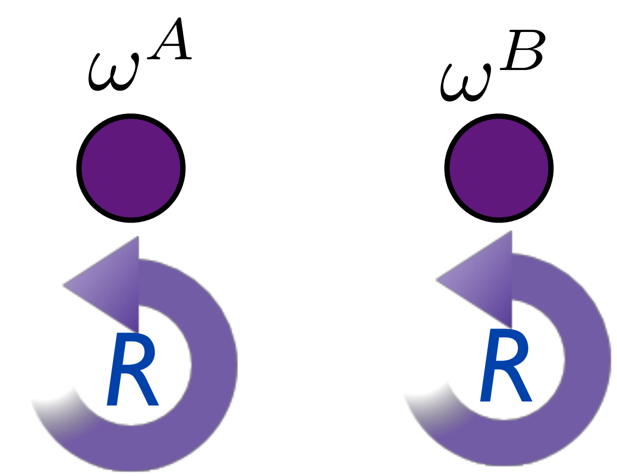
$$\mathcal{M}^A \mathcal{M}^B(\omega^A \omega^B) = \mathcal{M}^A(\omega^A) \cdot \mathcal{M}^B(\omega^B).$$

4. Postulates 3+4

Basic assumptions on composite state space *AB*:

- Contains "product states" $\omega^A\omega^B$.
- Allows for "product measurements" $\mathcal{M}^A\mathcal{M}^B$:

$$\mathcal{M}^A\mathcal{M}^B(\omega^A\omega^B) = \mathcal{M}^A(\omega^A)\cdot\mathcal{M}^B(\omega^B).$$



Given $R \in SO(d)$, we want a unique way to specify the global rotation on the composite system.

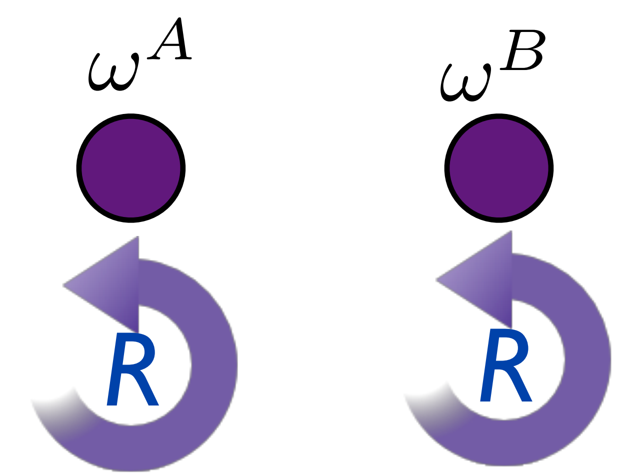
4. Postulates 3+4

Basic assumptions on composite state space *AB*:

- Contains "product states" $\omega^A\omega^B$.
- Allows for "product measurements" $\mathcal{M}^A\mathcal{M}^B$:

$$\mathcal{M}^A\mathcal{M}^B(\omega^A\omega^B)=\mathcal{M}^A(\omega^A)\cdot\mathcal{M}^B(\omega^B).$$

Given $R\in SO(d)$, we want a unique way to specify the global rotation on the composite system.



<u>Postulate 4</u> (global coordinate transf.):
Given any rotation *R*, there is a unique linear map on AB which acts as *R* on both subsystems individually.

4. Postulates 3+4

<u>Postulate 4</u> (global coordinate transf.):
Given any rotation *R*, there is a unique linear map on AB which acts as *R* on both subsystems individually.

4. Postulates 3+4

Postulate 4 (global coordinate transf.):
Given any rotation *R*, there is a unique linear map on AB which acts as *R* on both subsystems individually.

- We know what happens locally: $\omega^A \mapsto G_R \omega^A$.
- Thus, it's clear for product states: $\omega^A \omega^B \mapsto (G_R \omega^A)(G_R \omega^B)$.

4. Postulates 3+4

<u>Postulate 4</u> (global coordinate transf.):
Given any rotation *R*, there is a unique linear map on AB which acts as *R* on both subsystems individually.

- We know what happens locally: $\omega^A \mapsto G_R \omega^A$.
- Thus, it's clear for product states: $\omega^A \omega^B \mapsto (G_R \omega^A)(G_R \omega^B)$.
- $\Rightarrow$ Postulate 4 is equivalent to: product states span all of AB.

"Local tomography"

4. Postulates 3+4

Postulate 4 (global coordinate transf.):
Given any rotation *R*, there is a unique linear map on AB which acts as *R* on both subsystems individually.

- We know what happens locally: $\omega^A \mapsto G_R \omega^A$.
- Thus, it's clear for product states: $\omega^A \omega^B \mapsto (G_R \omega^A)(G_R \omega^B)$.
- $\Rightarrow$ Postulate 4 is equivalent to: product states span all of AB.

"Local tomography"

$\omega^A \omega^B = \omega^A \otimes \omega^B$. $\qquad \omega^{AB} \mapsto G_R \otimes G_R(\omega^{AB})$.

Theorem: From Postulates 1-4 it follows that $d=3$.

Theorem: From Postulates 1-4 it follows that *d*=*3*.

Proof idea *(Ll. Masanes, MM, D. Pérez-García, R. Augusiak, arXiv:1111.4060)*

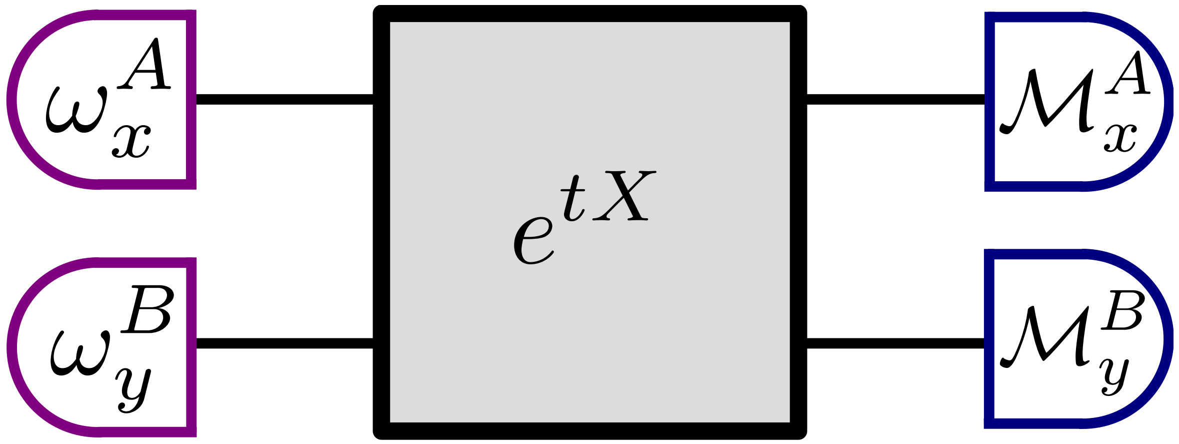
- Consider global Lie group $\mathcal{G}^{AB}$ generated by $\{T_t^{AB}\}_{t\in\mathbb{R}}$ and $G^A \otimes G^B$.

Theorem: From Postulates 1-4 it follows that *d*=3.

<u>Proof idea</u> *(Ll. Masanes, MM, D. Pérez-García, R. Augusiak, arXiv:1111.4060)*

- Consider global Lie group $\mathcal{G}^{AB}$ generated by $\{T_t^{AB}\}_{t\in\mathbb{R}}$ and $G^A \otimes G^B$.
- Global Lie algebra element $X \in \mathfrak{g}^{AB}$, then

$$\mathcal{M}_x \otimes \mathcal{M}_y \left(e^{tX}(\omega_x \otimes \omega_y)\right) \in [0,1].$$

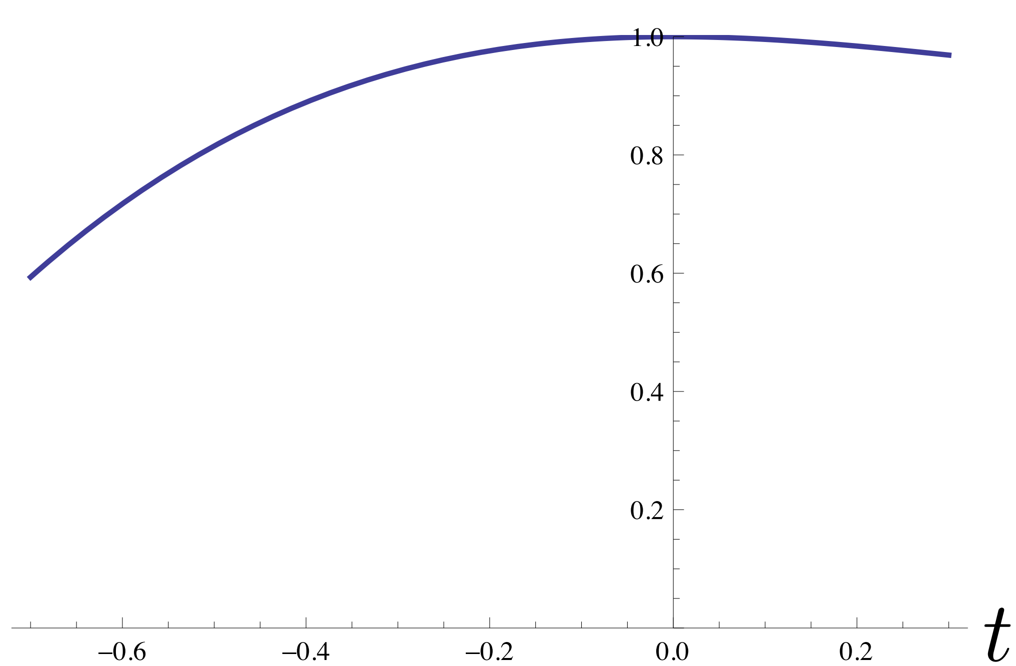


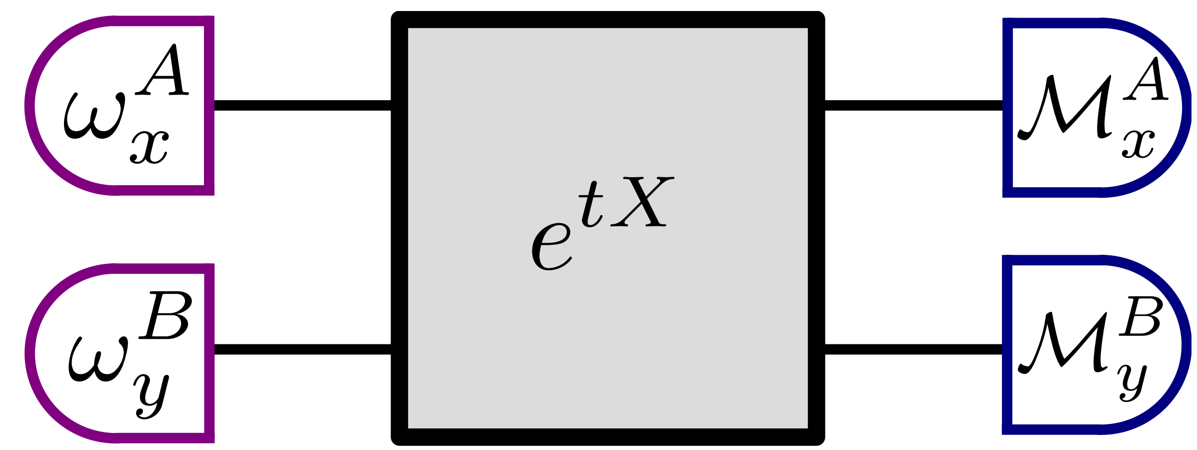
Theorem: From Postulates 1-4 it follows that d=3.

Proof idea *(Ll. Masanes, MM, D. Pérez-García, R. Augusiak, arXiv:1111.4060)*

- Consider global Lie group $\mathcal{G}^{AB}$ generated by $\{T_t^{AB}\}_{t\in\mathbb{R}}$ and $G^A\otimes G^B$.
- Global Lie algebra element $X\in\mathfrak{g}^{AB}$, then

$$\mathcal{M}_x\otimes\mathcal{M}_y\left(e^{tX}(\omega_x\otimes\omega_y)\right)\in[0,1].$$





- But this equals 1 for $t=0$, thus

$$\mathcal{M}_x\otimes\mathcal{M}_y\,X\,\omega_x\otimes\omega_y=0,$$

$$\mathcal{M}_x\otimes\mathcal{M}_y\,X^2\,\omega_x\otimes\omega_y\leq 0.$$

Theorem: From Postulates 1-4 it follows that $d=3$.

<u>Proof idea</u> *(Ll. Masanes, MM, D. Pérez-García, R. Augusiak, arXiv:1111.4060)*

Theorem: From Postulates 1-4 it follows that *d*=*3*.

Proof idea *(Ll. Masanes, MM, D. Pérez-García, R. Augusiak, arXiv:1111.4060)*

- We get several constraints on $X \in \mathfrak{g}^{AB}$:

$$\begin{aligned} \mathcal{M}_x \otimes \mathcal{M}_y X \omega_x \otimes \omega_y &= 0, \\ \mathcal{M}_x \otimes \mathcal{M}_y X^2 \omega_x \otimes \omega_y &\leq 0, \qquad \ldots \end{aligned}$$

Theorem: From Postulates 1-4 it follows that *d*=3.

<u>Proof idea</u> *(Ll. Masanes, MM, D. Pérez-García, R. Augusiak, arXiv:1111.4060)*

- We get several constraints on $X \in \mathfrak{g}^{AB}$:

$$\begin{aligned} \mathcal{M}_x \otimes \mathcal{M}_y X \omega_x \otimes \omega_y &= 0, \\ \mathcal{M}_x \otimes \mathcal{M}_y X^2 \omega_x \otimes \omega_y &\leq 0, \qquad \ldots \end{aligned}$$

- If $d \neq 3$, the only X satisfying them all are of the form $X = X^A + X^B$ with local rotation generators X^A, X^B.
  These generate non-interacting dynamics.

Theorem: From Postulates 1-4 it follows that d=3.

Proof idea *(Ll. Masanes, MM, D. Pérez-García, R. Augusiak, arXiv:1111.4060)*

- We get several constraints on $X \in \mathfrak{g}^{AB}$:

$$\begin{aligned} \mathcal{M}_x \otimes \mathcal{M}_y X \omega_x \otimes \omega_y &= 0, \\ \mathcal{M}_x \otimes \mathcal{M}_y X^2 \omega_x \otimes \omega_y &\leq 0, \qquad \ldots \end{aligned}$$

- If $d \neq 3$, the only X satisfying them all are of the form $X = X^A + X^B$ with local rotation generators X^A, X^B.
  These generate non-interacting dynamics.

- For $d \geq 3$, evaluating constraints involves integrals like

$$X \mapsto \int_{SO(d-1)} G^A \otimes \mathbf{1}^B X (G^A)^{-1} \otimes \mathbf{1}^B \, dG^A.$$

This behaves very differently if *SO(d-1)* is Abelian, i.e. iff *d*=3. □

Theorem: From Postulates 1-4, it follows that the state space of two direction bits is 2-qubit quantum state space (i.e. the set of 4x4 density matrices), and time evolution is given by a one-parameter group of unitaries,

$$\rho \mapsto U(t)\rho U(t)^\dagger.$$

Theorem: From Postulates 1-4, it follows that the state space of two direction bits is 2-qubit quantum state space (i.e. the set of 4x4 density matrices), and time evolution is given by a one-parameter group of unitaries,

$$\rho \mapsto U(t)\rho U(t)^\dagger.$$

Proof idea *(G. de la Torre, Ll. Masanes, A. J. Short, MM, arXiv:1110.5482)*

Theorem: From Postulates 1-4, it follows that the state space of two direction bits is 2-qubit quantum state space (i.e. the set of 4x4 density matrices), and time evolution is given by a one-parameter group of unitaries,

$$\rho \mapsto U(t)\rho U(t)^\dagger.$$

Proof idea *(G. de la Torre, Ll. Masanes, A. J. Short, MM, arXiv:1110.5482)*

- We have *d*=*3*. Embed the *3*-ball in the unit trace matrices of $\mathbb{C}^{2\times 2}_{s.a.}$

$$(r_1, r_2, r_3) \mapsto \begin{pmatrix} \frac{1}{2} + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & \frac{1}{2} - r_3 \end{pmatrix}.$$

- Thus, global states will be unit trace matrices in $\mathbb{C}^{2\times 2}_{s.a.} \otimes \mathbb{C}^{2\times 2}_{s.a.} = \mathbb{C}^{4\times 4}_{s.a.}$

Theorem: From Postulates 1-4, it follows that the state space of two direction bits is 2-qubit quantum state space (i.e. the set of 4x4 density matrices), and time evolution is given by a one-parameter group of unitaries,

$$\rho \mapsto U(t)\rho U(t)^\dagger.$$

Proof idea *(G. de la Torre, Ll. Masanes, A. J. Short, MM, arXiv:1110.5482)*

- We have *d=3*. Embed the 3-ball in the unit trace matrices of $\mathbb{C}^{2\times 2}_{s.a.}$

$$(r_1, r_2, r_3) \mapsto \begin{pmatrix} \frac{1}{2}+r_3 & r_1 - ir_2 \\ r_1 + ir_2 & \frac{1}{2}-r_3 \end{pmatrix}.$$

- Thus, global states will be unit trace matrices in $\mathbb{C}^{2\times 2}_{s.a.} \otimes \mathbb{C}^{2\times 2}_{s.a.} = \mathbb{C}^{4\times 4}_{s.a.}$
- Now some $X \neq X^A + X^B$ satisfy constraints. But they all generate maps of the form $e^{tX}(\rho) = U\rho U^\dagger$ with $U \in SU(4)$.

Theorem: From Postulates 1-4, it follows that the state space of two direction bits is 2-qubit quantum state space (i.e. the set of 4x4 density matrices), and time evolution is given by a one-parameter group of unitaries,

$$\rho \mapsto U(t)\rho U(t)^\dagger.$$

Proof idea *(G. de la Torre, Ll. Masanes, A. J. Short, MM, arXiv:1110.5482)*

Theorem: From Postulates 1-4, it follows that the state space of two direction bits is 2-qubit quantum state space (i.e. the set of 4x4 density matrices), and time evolution is given by a one-parameter group of unitaries,

$$\rho \mapsto U(t)\rho U(t)^\dagger.$$

Proof idea *(G. de la Torre, Ll. Masanes, A. J. Short, MM, arXiv:1110.5482)*

- We have at least one entangling unitary (Postulate 3) and all local unitaries (rotations). This generates all unitaries!

Theorem: From Postulates 1-4, it follows that the state space of two direction bits is 2-qubit quantum state space (i.e. the set of 4x4 density matrices), and time evolution is given by a one-parameter group of unitaries,

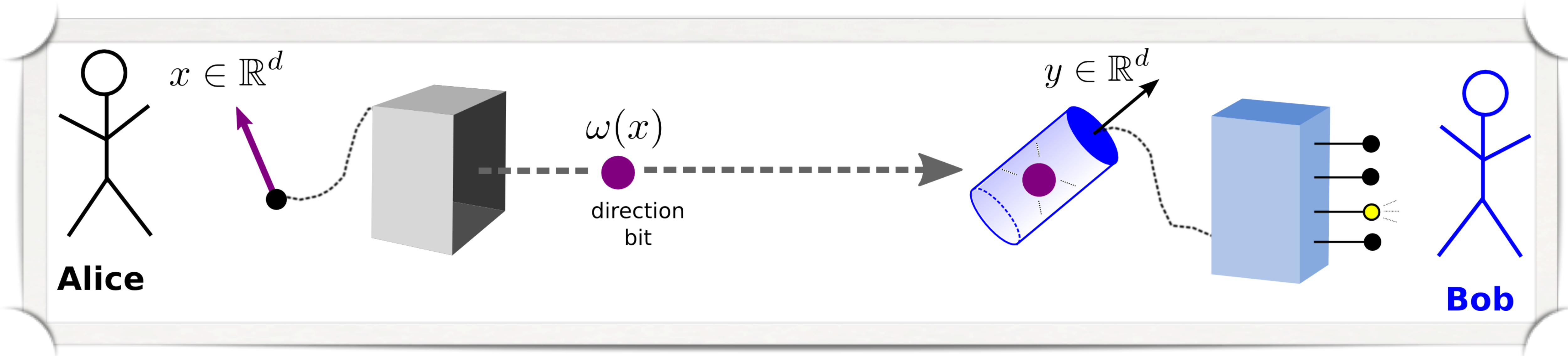
$$\rho \mapsto U(t)\rho U(t)^\dagger.$$

Proof idea *(G. de la Torre, Ll. Masanes, A. J. Short, MM, arXiv:1110.5482)*

- We have at least one entangling unitary (Postulate 3) and all local unitaries (rotations). This generates all unitaries!
- But these generate all 4-level quantum states.
- If there were additional states, these would generate negative probabilities. □

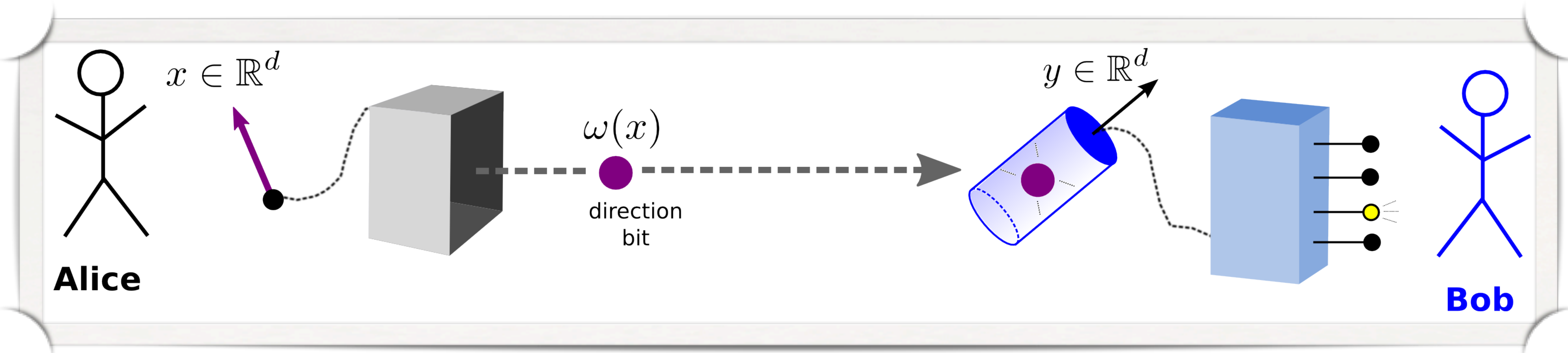
5. Conclusions

Attempt to clarify relationship between geometry and the qubit (still clumsy - first step only):



5. Conclusions

Attempt to clarify relationship between geometry and the qubit (still clumsy - first step only):



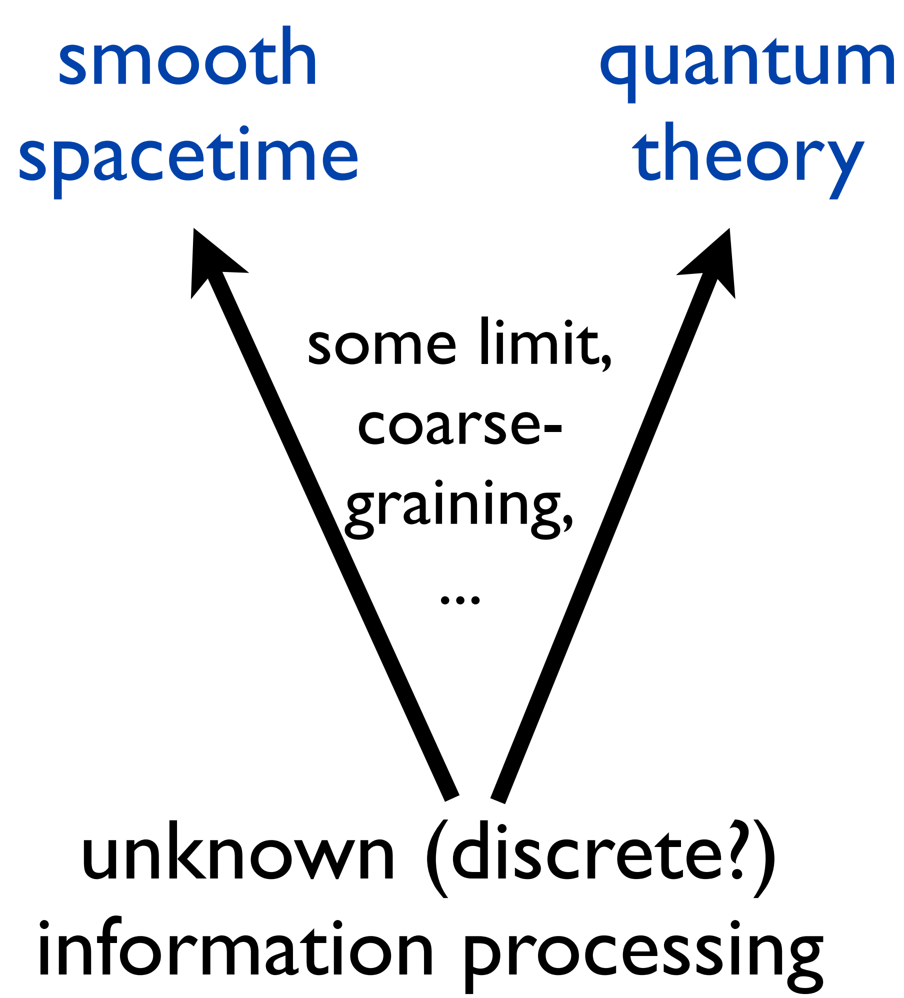
- Start with *d* spatial dimensions, not assuming quantum theory.
- Four "information-theoretic" postulates on the relation between spatial geometry (rotations) and probability
- Proof that these determine *d*=3 and quantum theory on 2 bits.

5. Conclusions

What does this tell us? Some speculation...

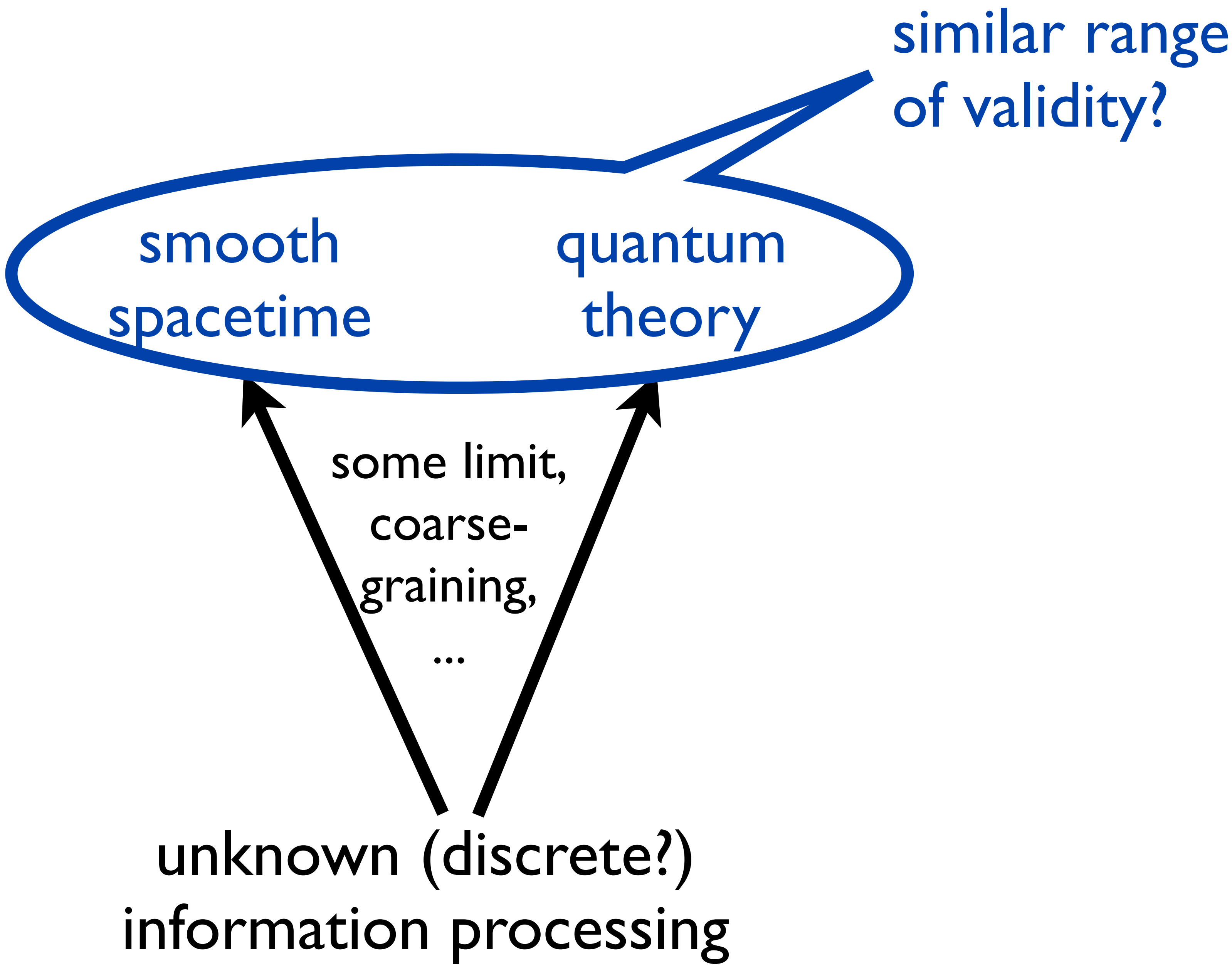
5. Conclusions

What does this tell us? Some speculation...



5. Conclusions

What does this tell us? Some speculation...



5. Conclusions

ruling out *d≠3*:

Ll. Masanes, MM, D. Pérez-García, R. Augusiak, arXiv:1111.4060

d=3 implies quantum theory:

G. de la Torre, Ll. Masanes, T. Short, MM, Phys. Rev. Lett. **109**, 090403 (2012)
arXiv:1110.5482

this talk:

MM, Ll. Masanes, arXiv:1206.0630

Thank you!