# Undecidability in quantum measurements

Markus Müller

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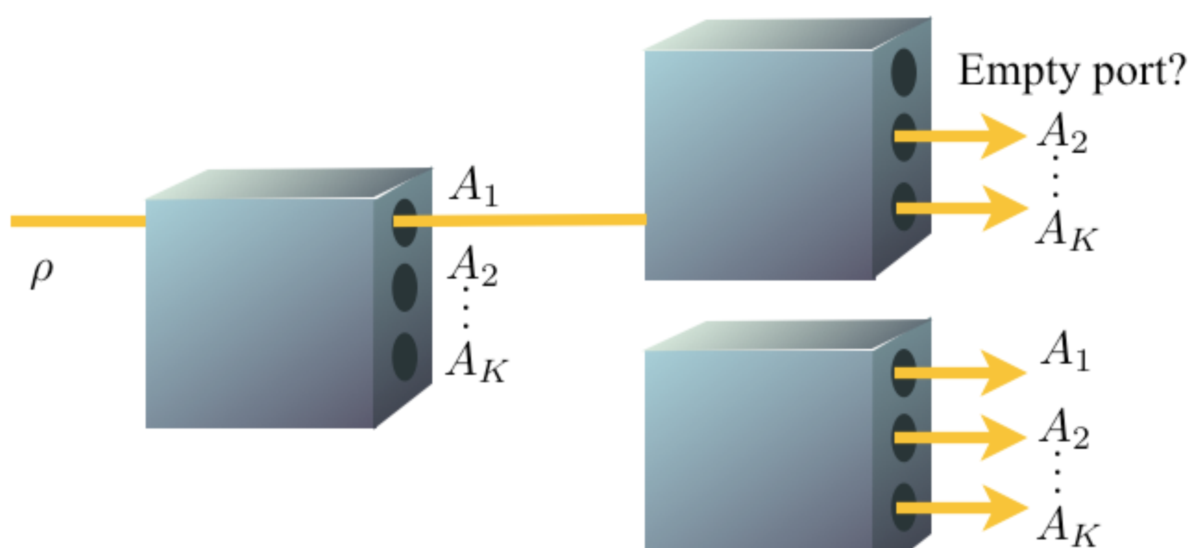
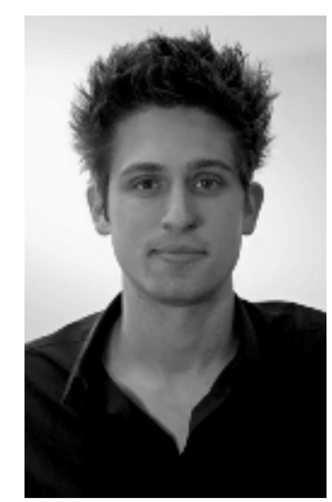


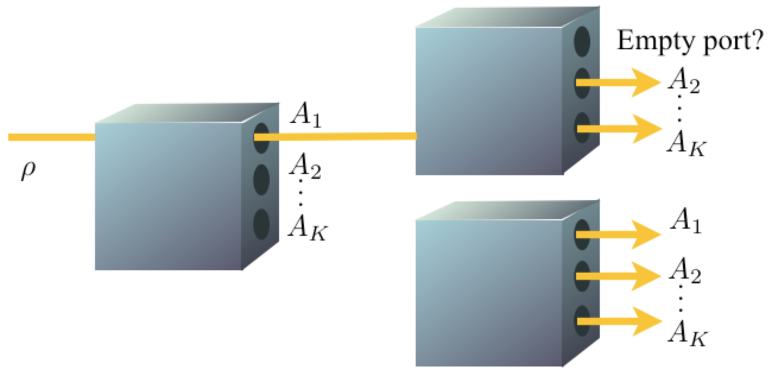

Joint work with Jens Eisert & Christian Gogolin (FU Berlin)

arXiv: 1111.3965



# Outline

1. Motivation / undecidability in general
2. The "measurement occurrence problem"




3. Undecidability of the quantum problem
4. Decidability of the classical problem
5. Outlook

# 1. Motivation / undecidability in general

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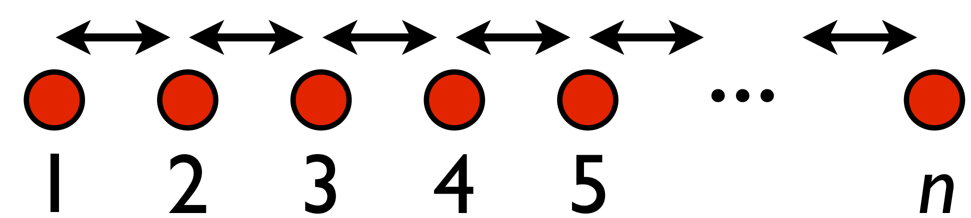
- Quantum complexity theory. Example: The 2-local Hamiltonian problem. Given $a < b$, and

$$H = \sum_{j=1}^{r} H_j ,$$




where all $H_j$ act on at most 2 qubits, $r, \|H_j\| \leq poly(n)$, decide if the smallest eigenvalue is $< a$ or $> b$.
This problem is QMA-complete.

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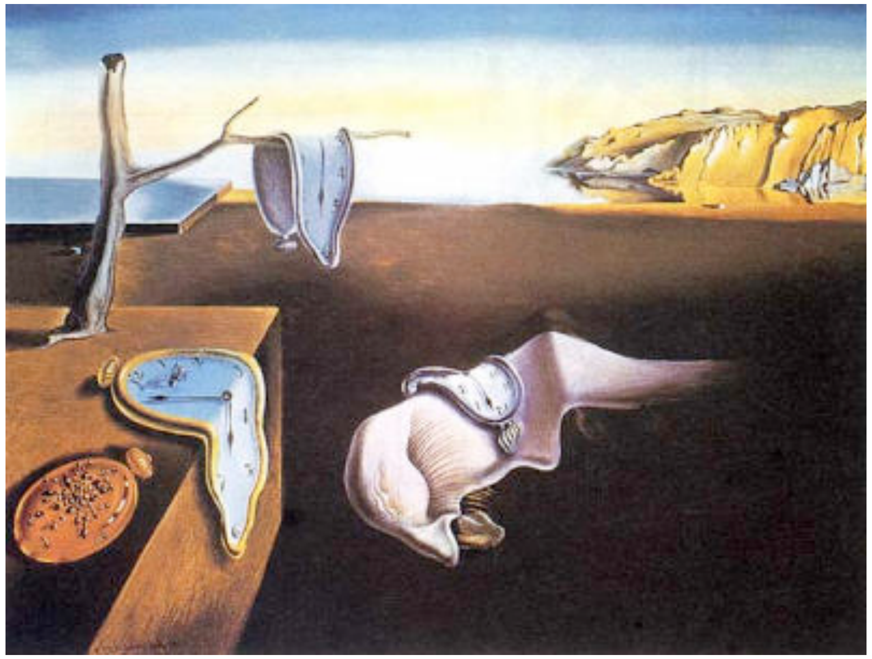

No algorithm will solve the problem... ever!

Origin: the Halting Problem.

# 1. Motivation / undecidability in general


Fix a universal Turing machine which takes natural numbers $x \in \mathbb{N}$ as input.

Halting problem: Given input $x \in \mathbb{N}$, will the TM eventually halt on that input, or will it run forever?

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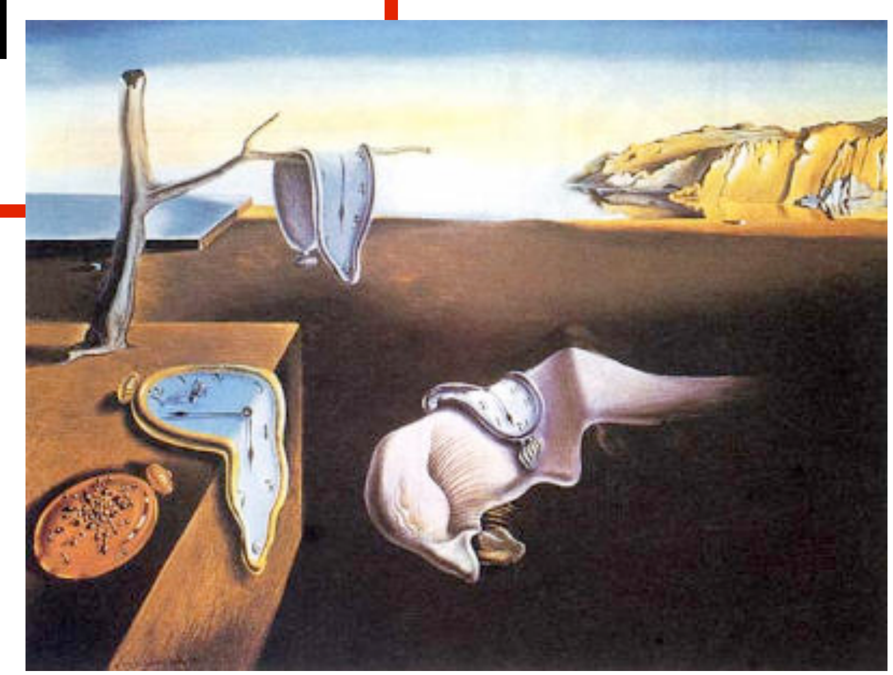
Fix a universal Turing machine which takes natural numbers $x \in \mathbb{N}$ as input.

Halting problem: Given input $x \in \mathbb{N}$, will the TM eventually halt on that input, or will it run forever?

Alan Turing 1936: The halting problem is undecidable. That is, there is no single algorithm which, for every input *x*, decides in finite time whether the TM halts on input *x* or not.



# 1. Motivation / undecidability in general

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} = (a_{ij})$$

Matrix mortality problem: Given some finite set of integer matrices $\{M_1, \ldots, M_k\}$, is there any finite matrix product $M_{i_1} M_{i_2} \ldots M_{i_n}$ which equals the zero matrix?

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Paterson 1970; Halava, Harju 2001: The matrix mortality problem is undecidable, even for eight 3x3 integer matrices.

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Inspiration: Michael M. Wolf, Toby S. Cubitt, David Perez-Garcia, *Are problems in Quantum Information Theory (un)decidable?*, arXiv:1111.5425

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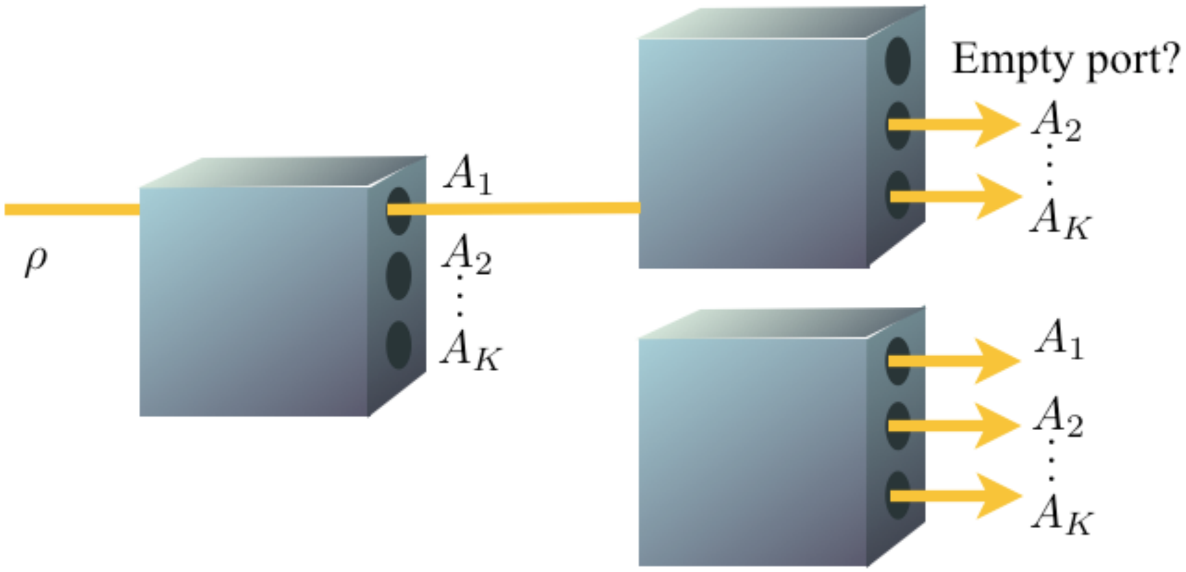
Earlier works in similar spirit:

- V. Blondel, E. Jeandel, P. Koiran, and N. Portier, *Decidable and undecidable problems about quantum automata*, SIAM J Comp. **34**, 1464-1473 (2005).
- H. Derksen, E. Jeandel, and P. Koiran, *Quantum automata and algebraic groups*, J. Symb. Comp. **39**, 357-371 (2005)
- M. Hirvensalo, *Various aspects of finite quantum automata*, Developments of Language Theory, vol. 5257, Lecture Notes in Computer Science, Springer (2008).

# 2. The "measurement occurrence problem"

## The Setting

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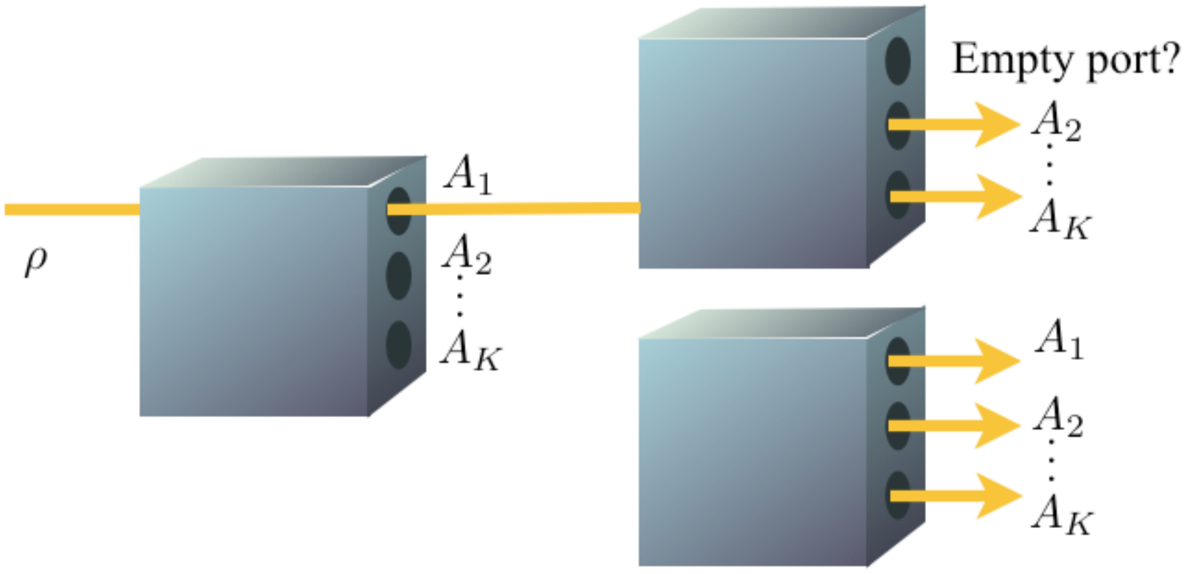


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- **Input:** quantum state $\rho \in \mathbb{C}^{d\times d}, \;\; \rho \geq 0, \;\; \mathrm{Tr}\rho = 1.$
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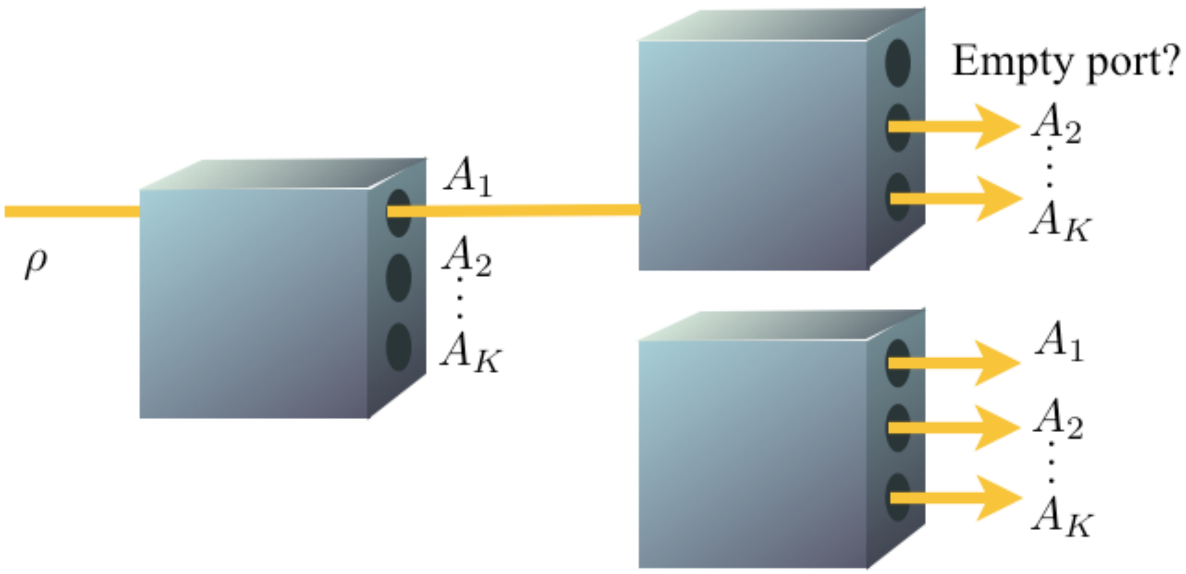


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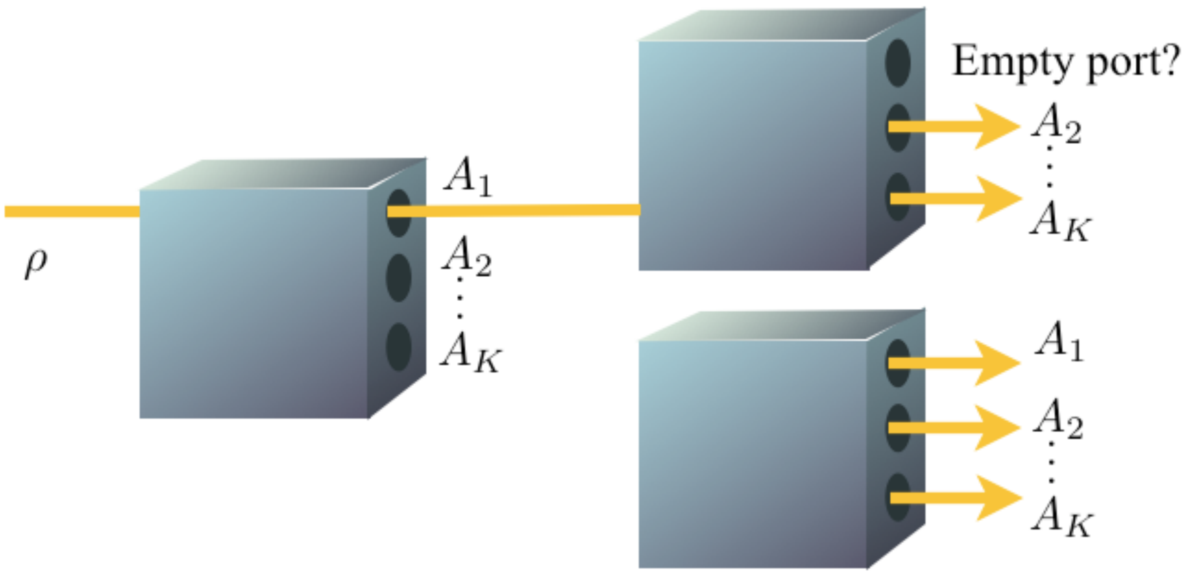


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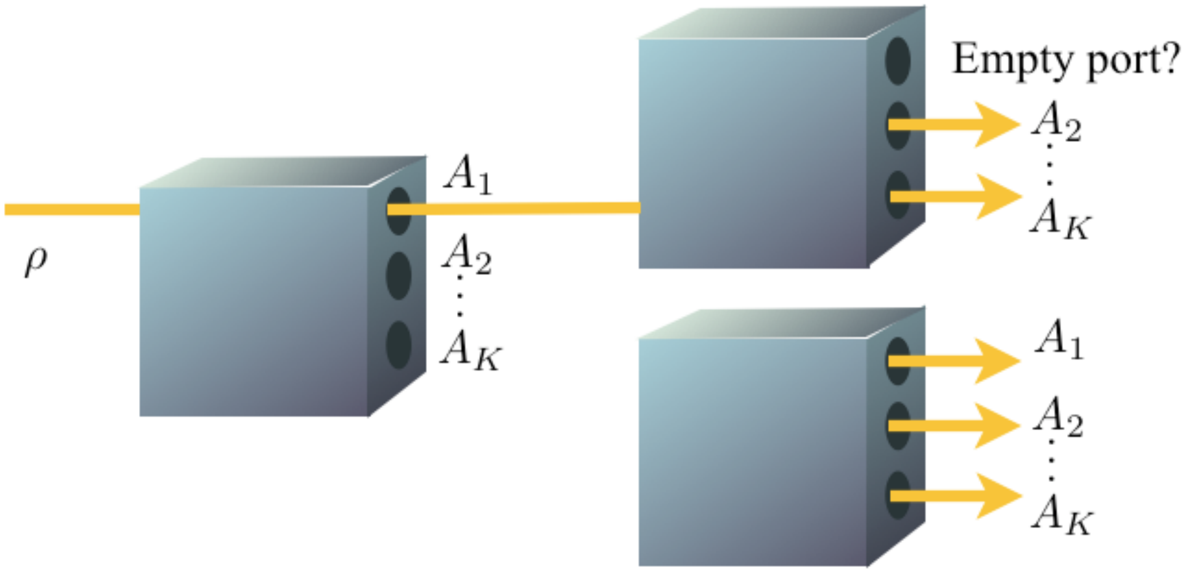


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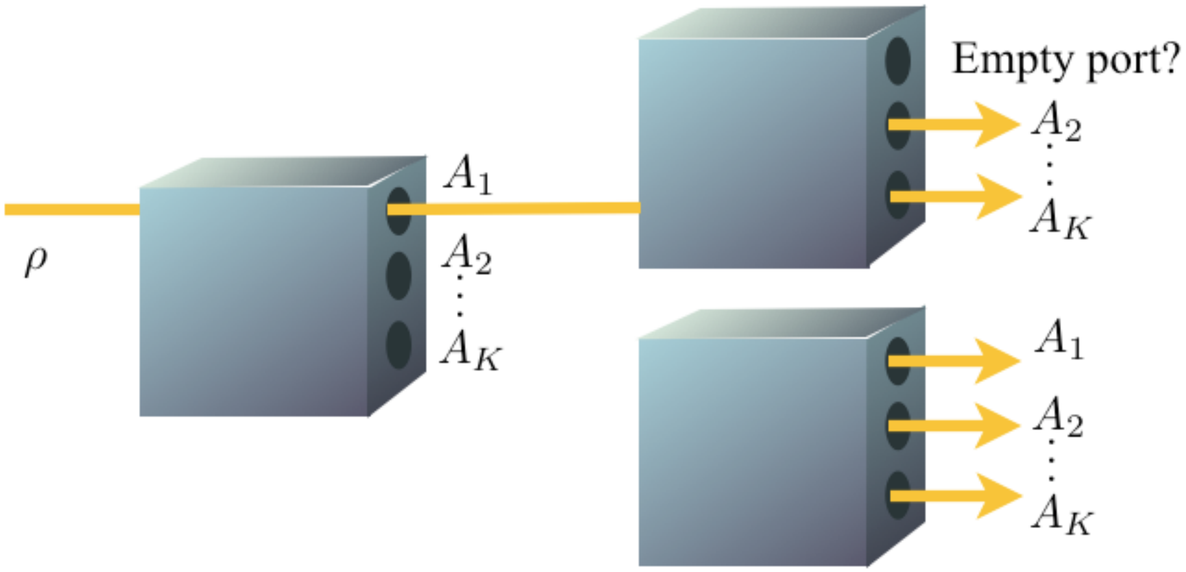


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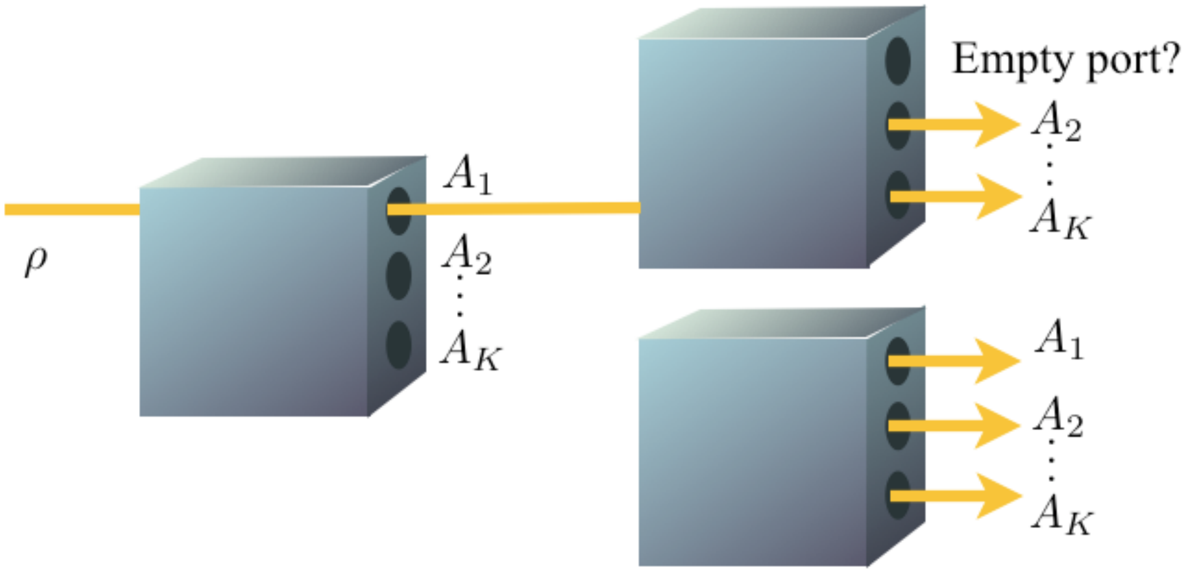


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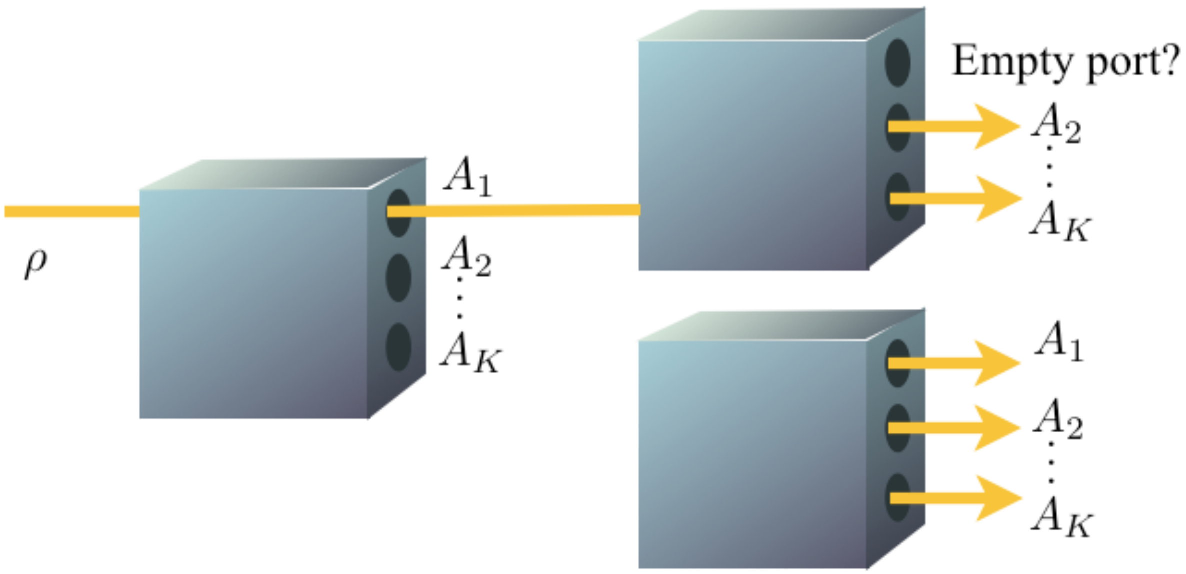


Quantum Measurement occurrence problem (QMOP):
Given a description of a quantum measurement device in terms of $K$ Kraus operators $A_1, \ldots, A_K \in \mathbb{Q}^{d\times d}$, decide whether there is any finite sequence $j_1, \ldots, j_n$ which can never be observed, even if the input state has full rank.

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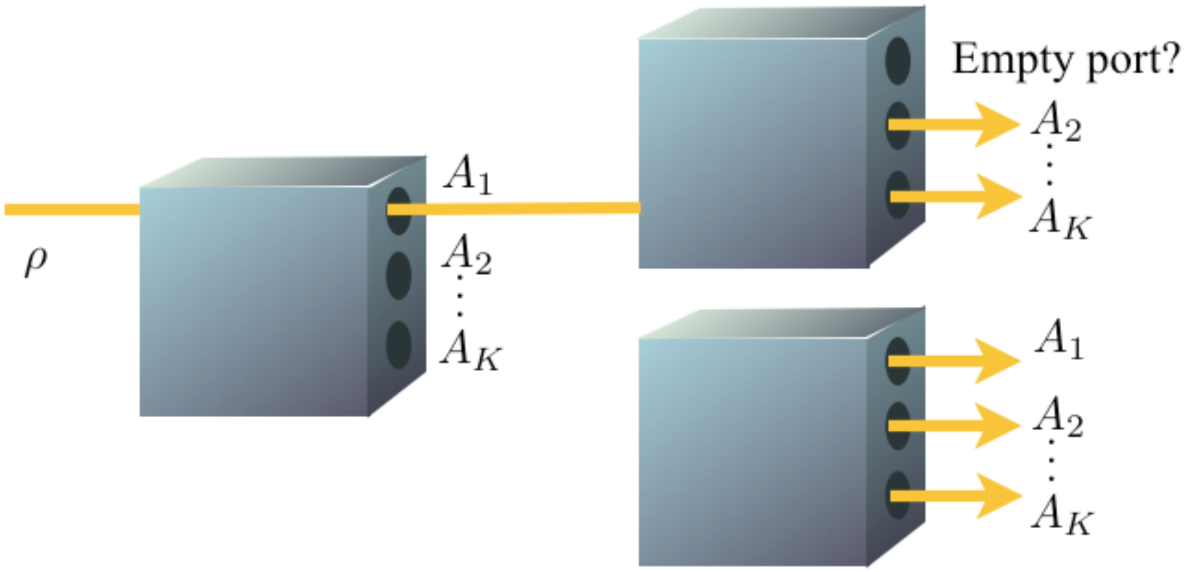


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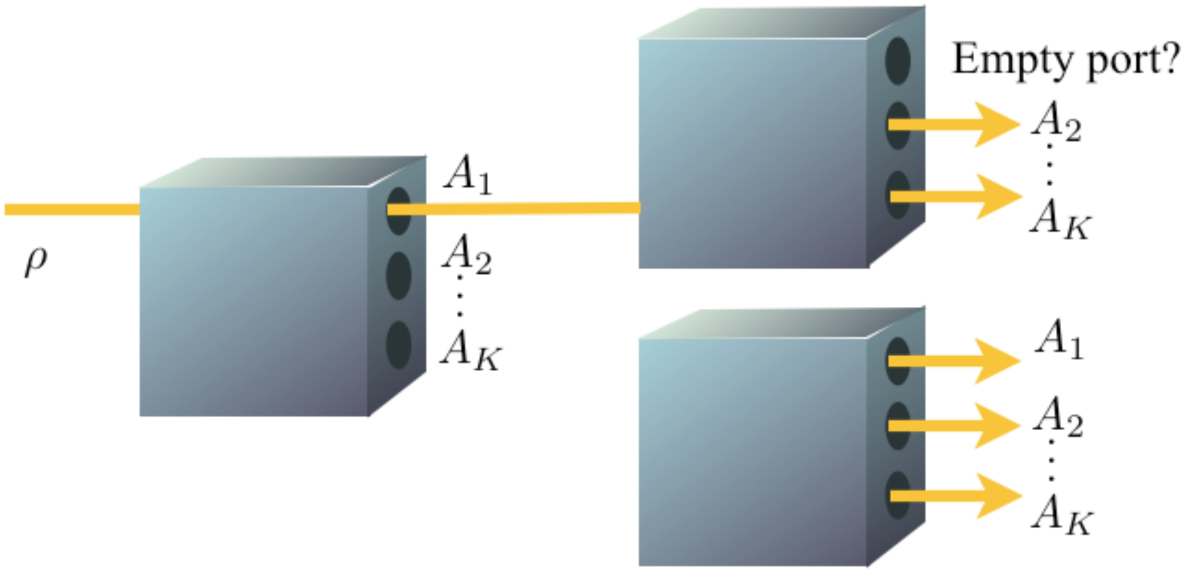


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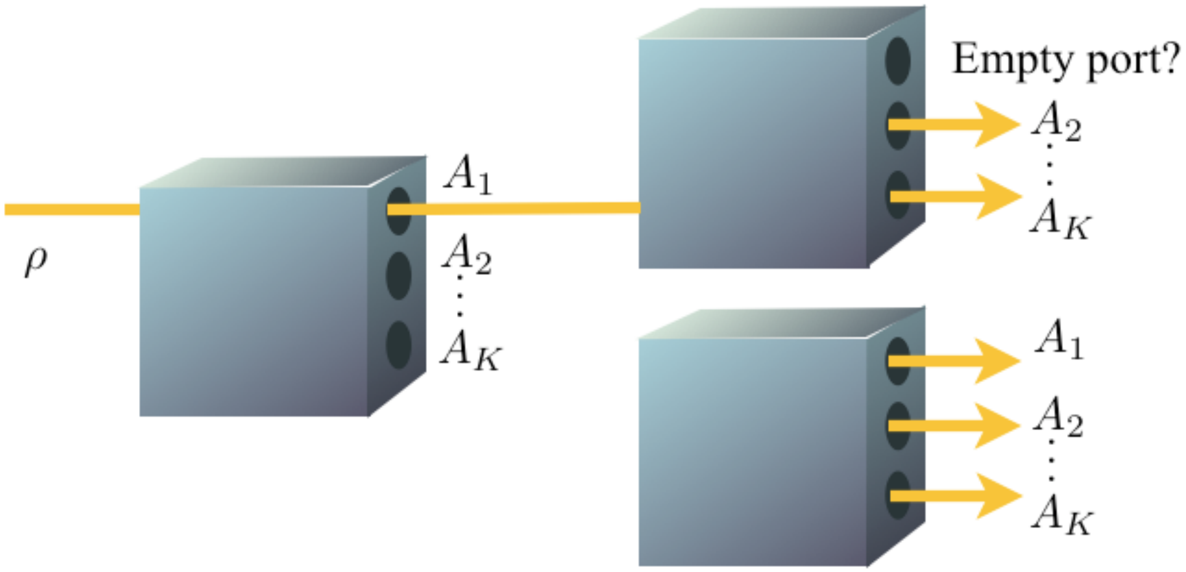


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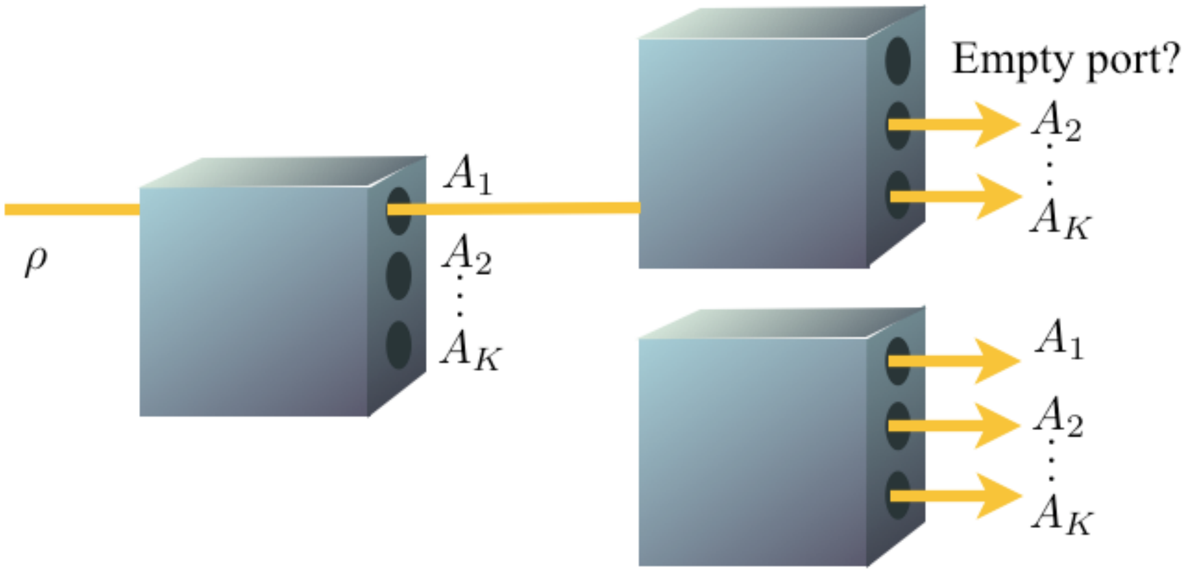


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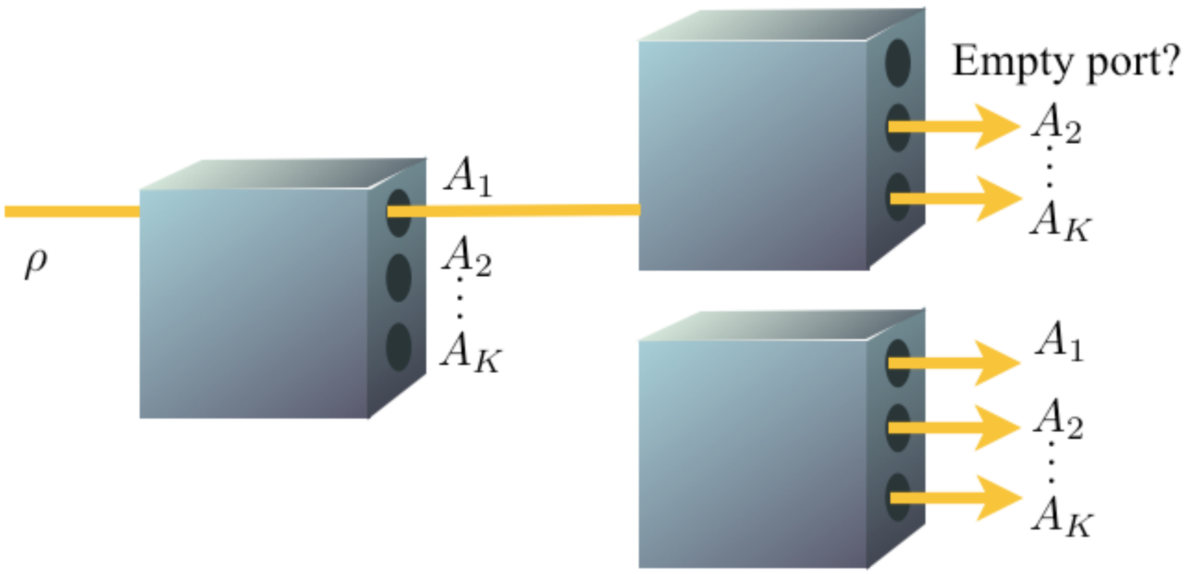


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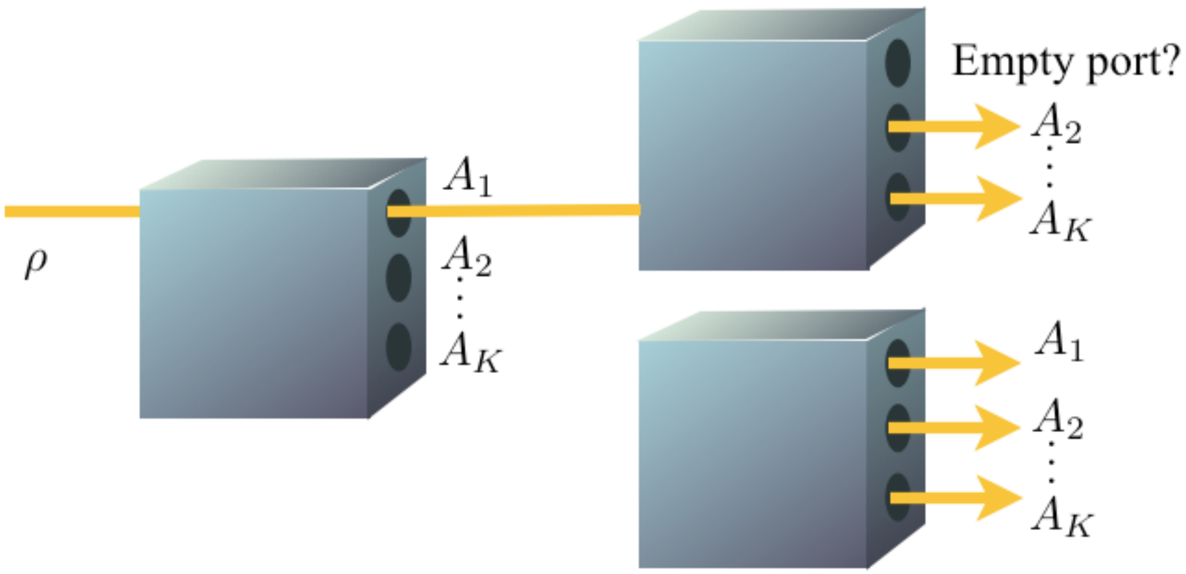


Classical measurement occurrence problem (CMOP):
Given a description of a measurement device in terms of $K$ substochastic matrices $Q_1, \ldots, Q_K \in \mathbb{Q}^{d\times d}$, decide whether there is any finite sequence $j_1, \ldots, j_n$ which can never be observed, regardless of the input state.

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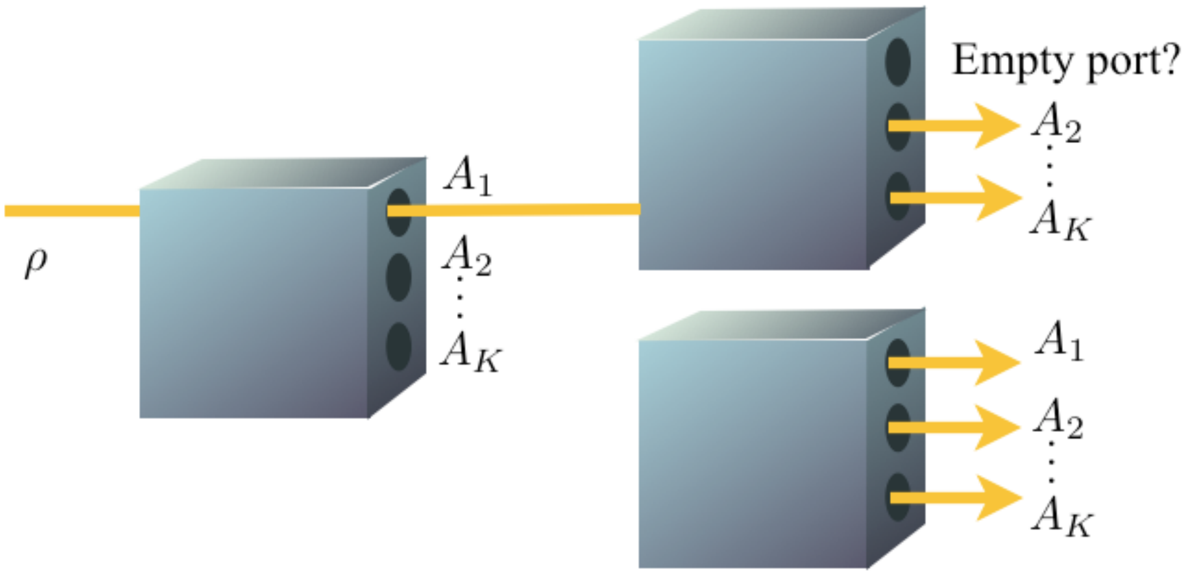


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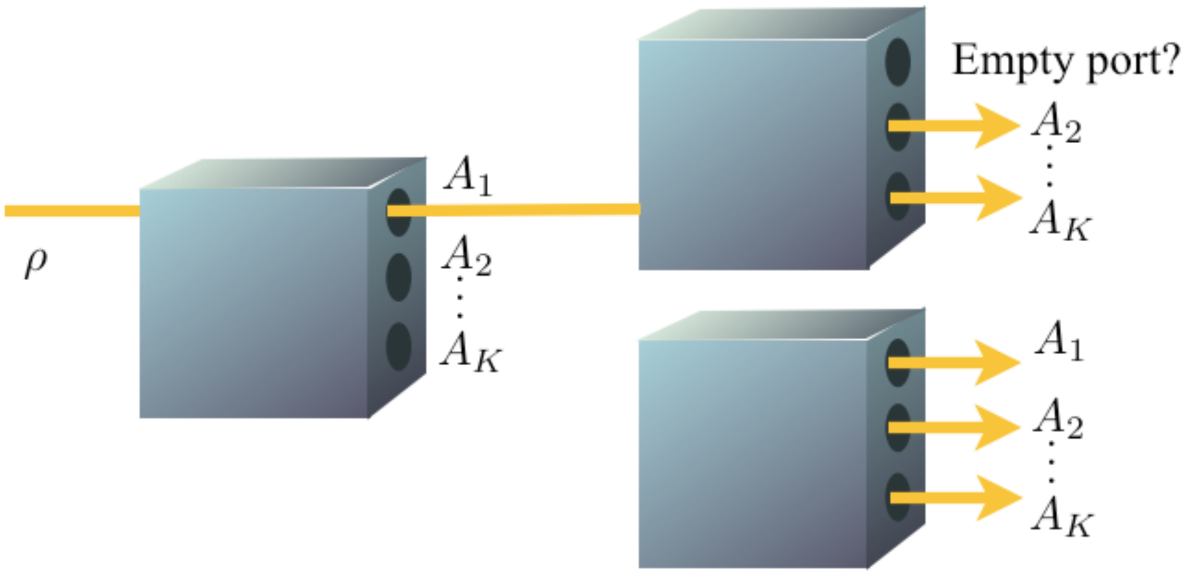


$$\mathrm{Prob}(j_1, \ldots, j_n) = \mathrm{Tr}(A_{j_n} \ldots A_{j_1} \rho A_{j_1}^\dagger \ldots A_{j_n}^\dagger) = 0$$

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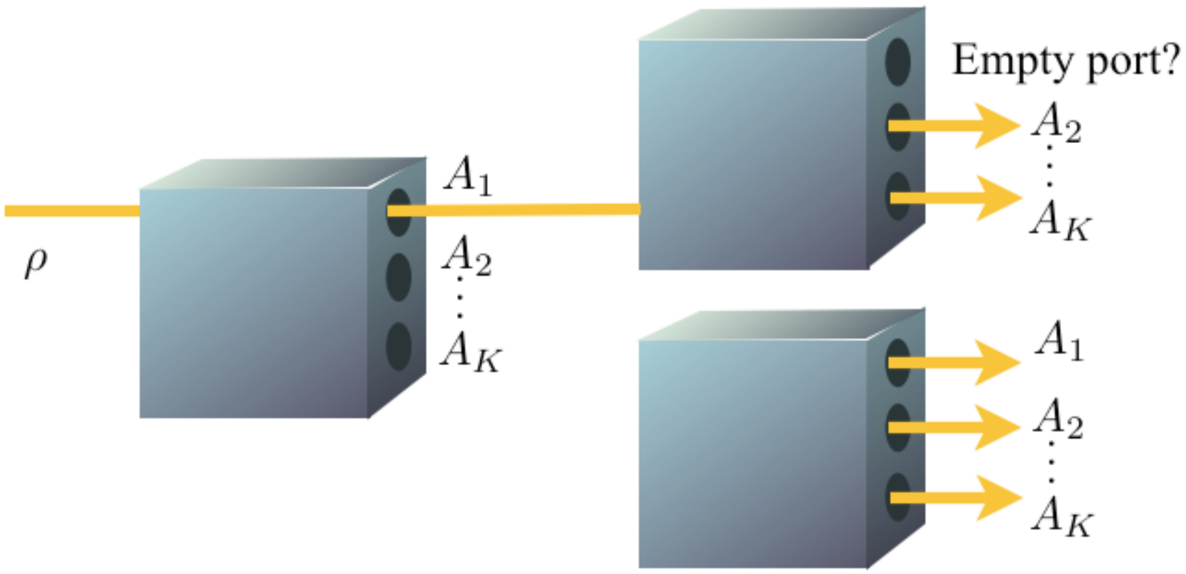


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Undecidability of MMP $\Rightarrow$ undecidability of QMOP ?

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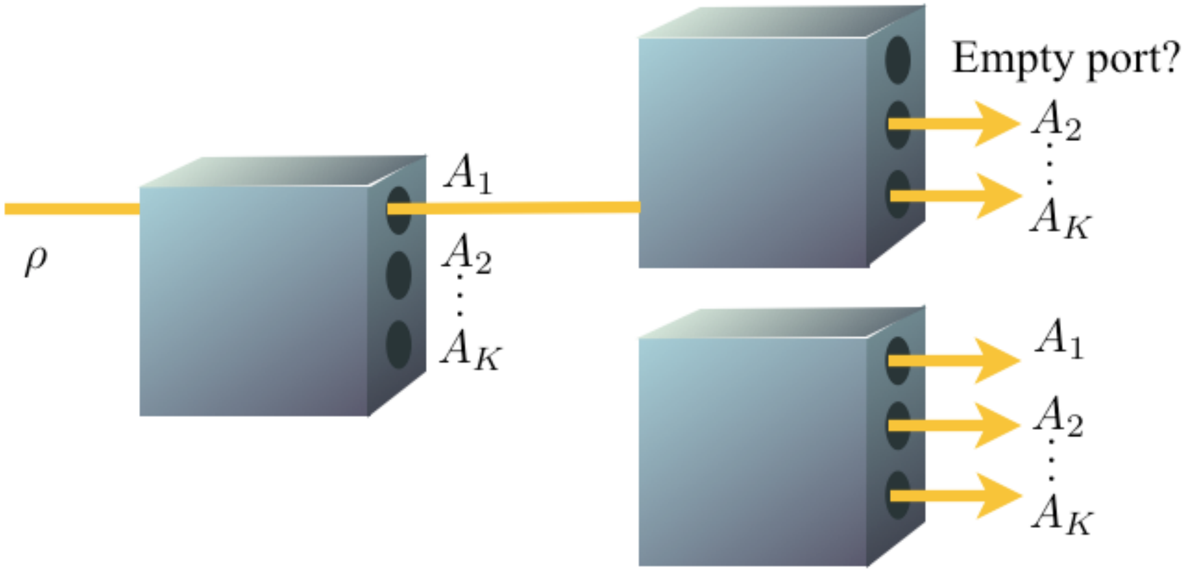


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Not quite! Normalization $\sum_j A_j^\dagger A_j = 1$ gives additional information.

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## Encoding MMP-instances into QMOP:

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- MMP undecidable already for eight integer 3x3 matrices.
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- First, add some more matrices:

$$P_1 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, P_2 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, P_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix},$$

$$M_{8+j} = M_j P_1, \quad M_{16+j} = M_j P_2, \quad M_{24+j} = M_j P_3.$$

$$\Rightarrow \sum_{j=1}^{32} M_j^\dagger M_j = \begin{pmatrix} 4T_{11} & 0 & 0 \\ 0 & 4T_{22} & 0 \\ 0 & 0 & 4T_{33} \end{pmatrix}.$$

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$$+ \underbrace{\begin{pmatrix} ? & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & ? \end{pmatrix}}_{M_{35}^\dagger M_{35}} + \underbrace{\begin{pmatrix} ? & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & ? \end{pmatrix}}_{M_{36}^\dagger M_{36}} = \begin{pmatrix} c^2 & & \\ & c^2 & \\ & & c^2 \end{pmatrix}$$

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Lagrange: $c^2 - 4T_{ii}$ can be written as sum of four integer squares!

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Encoding MMP-instances into QMOP:

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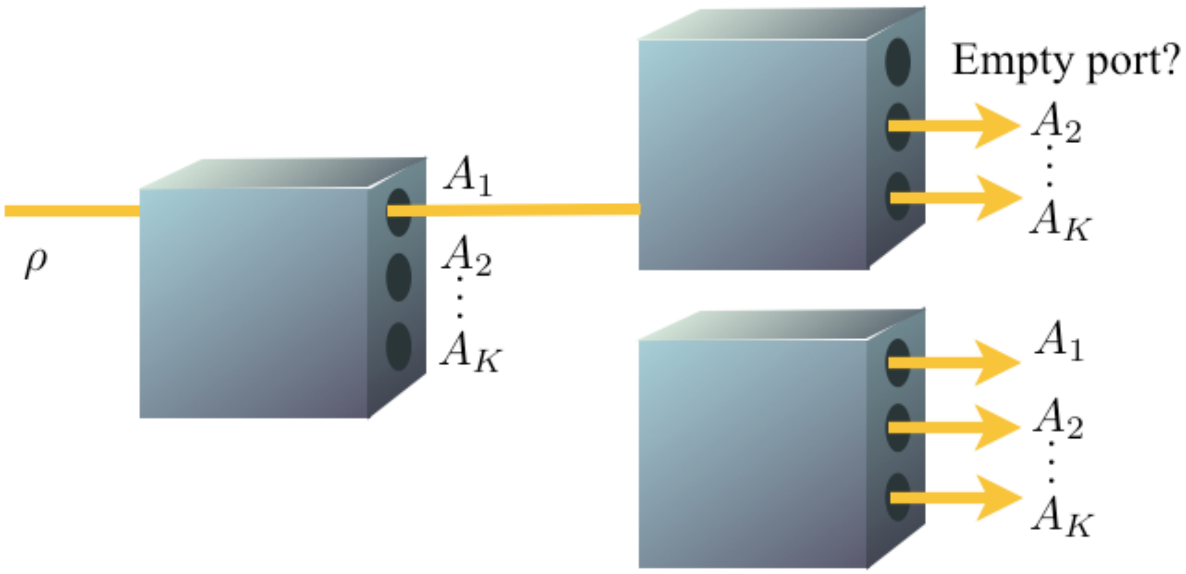


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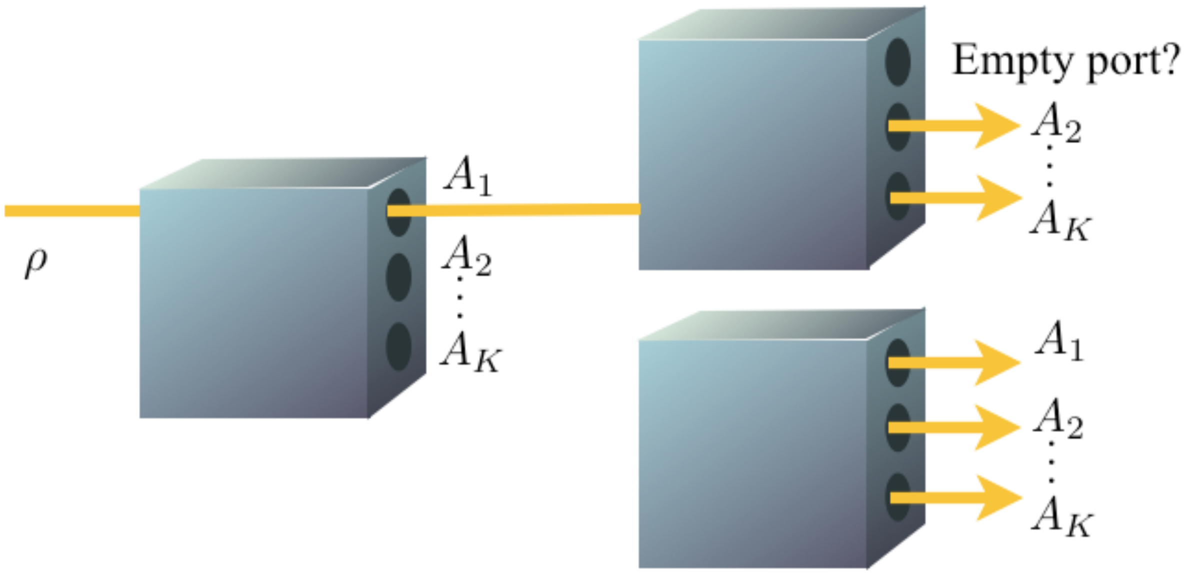


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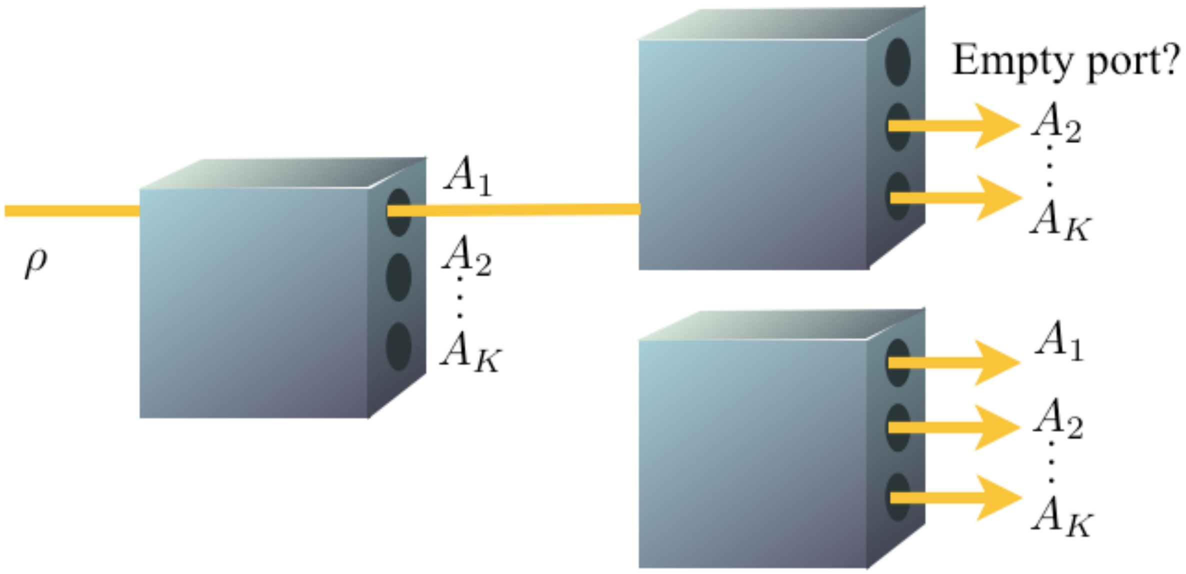


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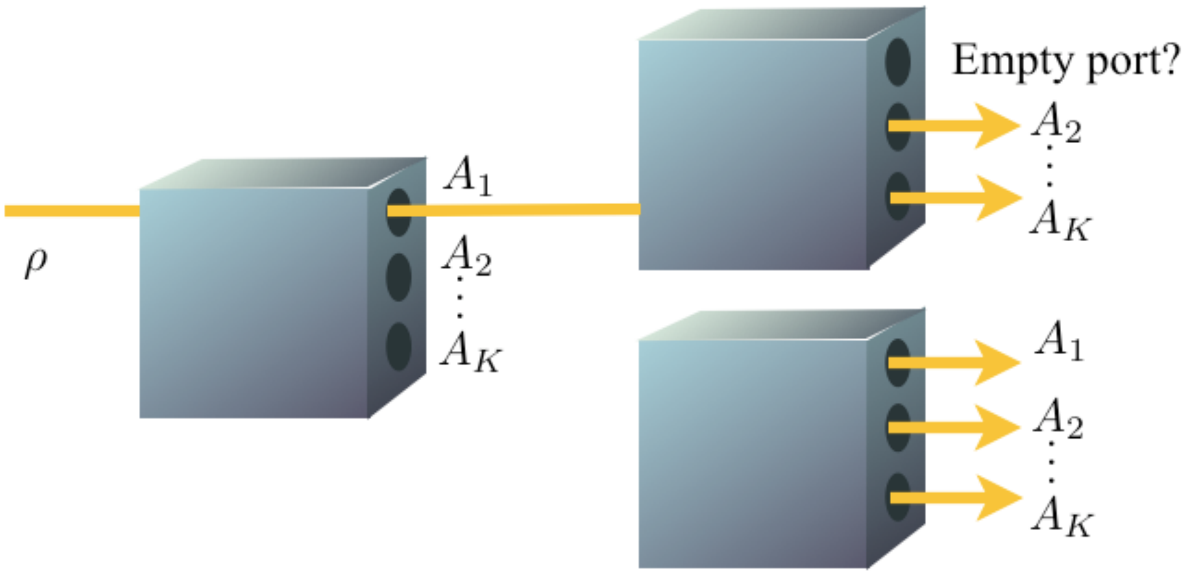


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|---|---|---|

**-decidable-**

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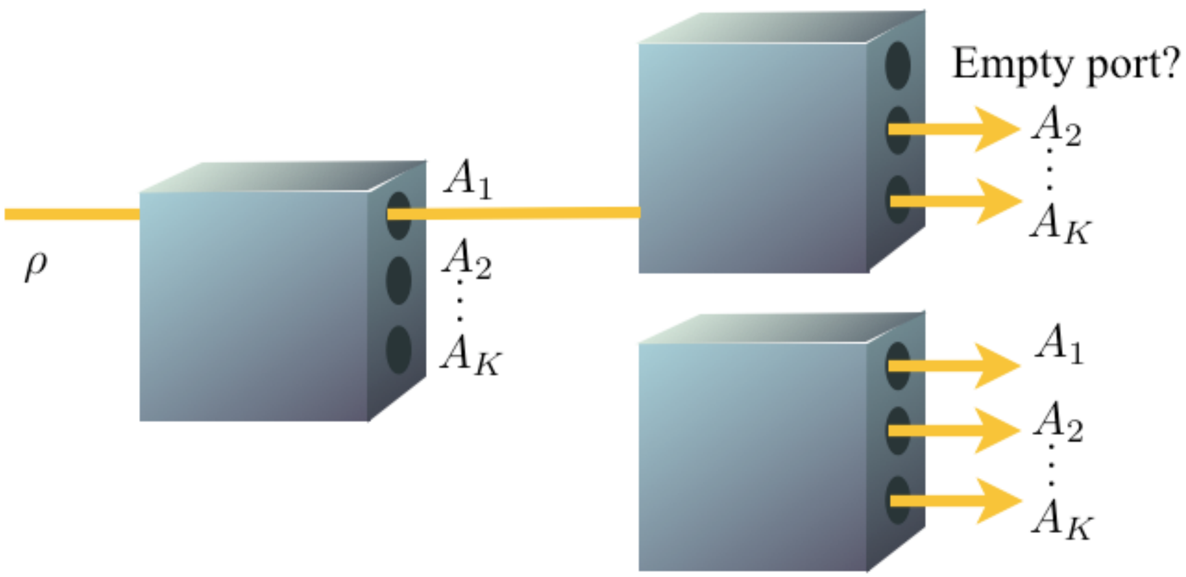


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# Summary: quantum cs. classical MOP

Quantum MOP

↕

MMP

↕

Destructive interference

undecidable

Classical MOP

↕

$MMP_{\geq 0}$

↕

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Are further natural quantum problems undecidable?

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Paradigm of a non-computable number: Chaitin's Omega.
Let *U* be a prefix-free universal Turing machine. Set

$$\Omega := \sum_{p:\ U \text{ halts on input } p} 2^{-\ell(p)} \leq 1.$$

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- There is an algorithm which, on input $n$, computes an approximation $\Omega_n$ such that $\Omega_n \leq \Omega_{n+1}$ and $\lim_{n\to\infty} \Omega_n = \Omega.$
- But: There is *no* algorithm which, on input $n$, computes an approximation $\Omega'_n$ such that $|\Omega - \Omega'_n| < 1/n.$ $\Omega$ is not computable.

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where $\chi(\mathcal{M}) = \max_{p_i,\varphi_i}\left[S\left(\mathcal{M}\left(\sum_i p_i|\varphi_i\rangle\langle\varphi_i|\right)\right) - \sum_i p_i S\left(\mathcal{M}(|\varphi_i\rangle\langle\varphi_i|)\right)\right]$

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The quest for a single-letter formula:

- <2008: maybe $C(\mathcal{N}) = \chi(\mathcal{N})$ **?**
- Hastings 2008: no!

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But: maybe $C(\mathcal{N})$ is not computable in general?

This would prove - once and for all - that there cannot be any single-letter formula.

# Conclusions

- Undecidability in quantum measurements:

| Quantum MOP | Classical MOP |
| --- | --- |
| ↕ | ↕ |
| MMP | $MMP_{\geq 0}$ |
| ↕ | ↕ |
| Destructive interference | Only constructive interference |
| undecidable | decidable |

- Speculation: are quantum channel capacities noncomputable?

# Thank you!

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arXiv:1111.3965