Convex Trace Functions on Quantum Channels and the Additivity Conjecture

Markus Müller

Max Planck Institute for Mathematics in the Sciences, Leipzig
& Technische Universität Berlin, Institut für Mathematik

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that is, if entanglement does not help to get purer output states.

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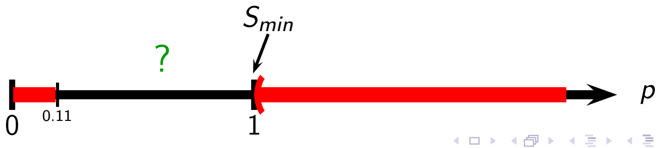

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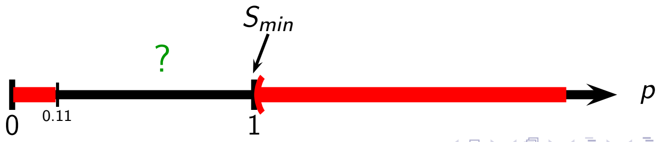

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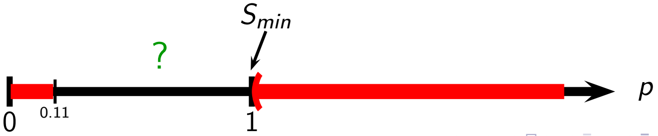

General Definition

A convex function $f : [0,1] \to \mathbb{R}$ is *additive* for a pair of channels (Φ, Ω) if there exists some unentangled input state ρ_u such that

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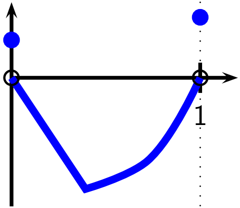

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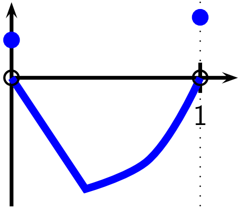

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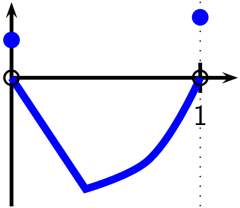

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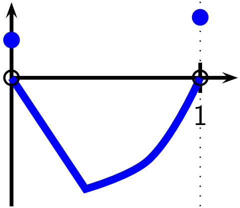

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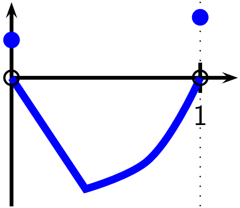

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Ultimate Goal: Classify the set of additive functions!

Structural Properties (an Example)

Theorem

If a convex function $f : [0,1] \to \mathbb{R}$ *is additive for all channels, and* $(\lambda_1, \ldots, \lambda_n)$ *is any probability vector, then the function*

$$\tilde{f}(x) := \sum_{i=1}^{n} f(\lambda_i x)$$

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Proof. Let Φ, Ω arbitrary and $\sigma := \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$; set $\tilde{\Phi} := \Phi \otimes \sigma$. The function f is additive for $\tilde{\Phi} \otimes \Omega$, so $\mathrm{Tr}\, f(\tilde{\Phi} \otimes \Omega(\rho)) = \mathrm{Tr}\, \tilde{f}(\Phi \otimes \Omega(\rho))$ has an unentangled global maximizer.

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- The set of additive functions for $\Phi \otimes \Omega$ is a closed cone. It is convex if both channels are "unitarily covariant".

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More Structural Properties

(Reminder: "Additive" means "additive for all channels".)

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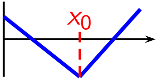
- 

Functions f with two affine pieces and kink at x_0 are additive iff $x_0 \geq \gamma$, where $\frac{1}{3} \leq \gamma \leq 1$ is a constant.

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If there exists any additive convex function on $[0,1]$ of the form

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