All reversible dynamics in maximally non-local theories are trivial

David Gross,[1] Markus Müller,[2,3] Roger Colbeck,[4,5] and Oscar C. O. Dahlsten[4]

[1]Institute for Theoretical Physics, Leibniz University Hannover
[2]Institute of Mathematics, Technical University of Berlin
[3]Institute of Physics and Astronomy, University of Potsdam
[4]Institute of Theoretical Physics, ETH Zurich
[5]Institute of Theoretical Computer Science, ETH Zurich

Outline

1. Beyond quantum: CHSH and PR-boxes
   - The CHSH inequality
   - Two questions about physics

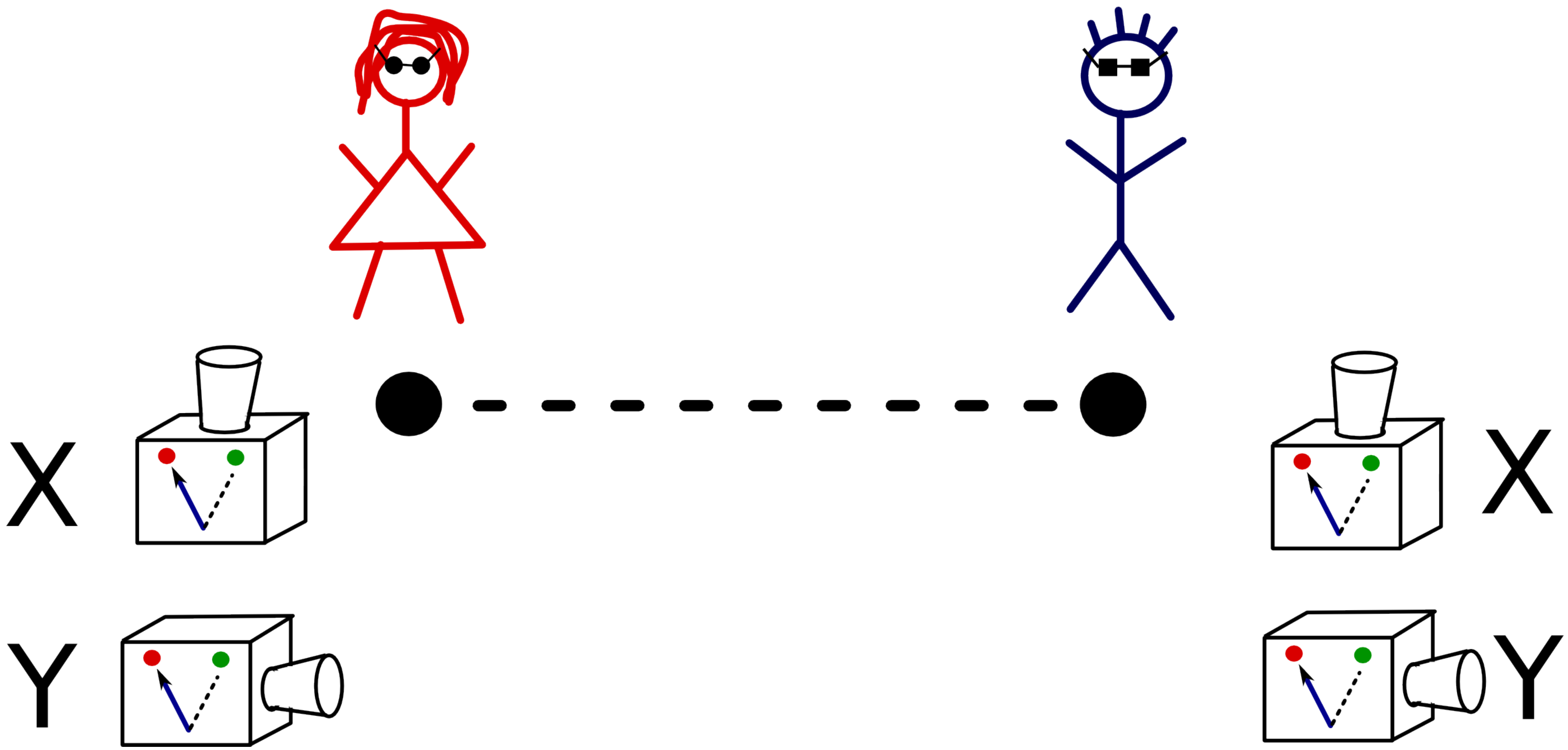
2. All reversible transformations in boxworld
   - State spaces and their symmetries
   - Main Results

3. Conclusions

1. Beyond quantum: CHSH and PR-boxes

The CHSH inequality

Quantum theory allows for **stronger correlations** than classical theories. Particular example: the **CHSH inequality**.



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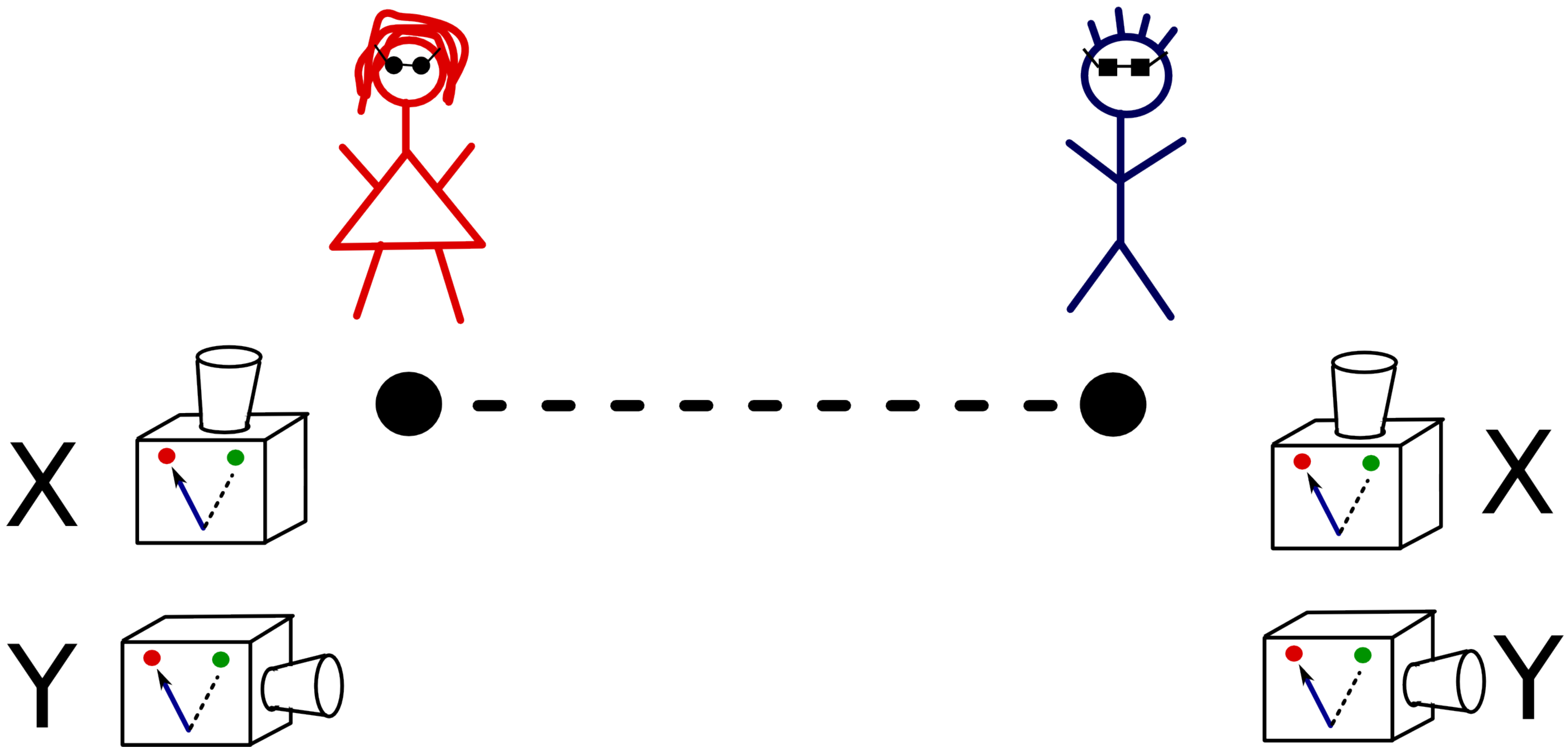

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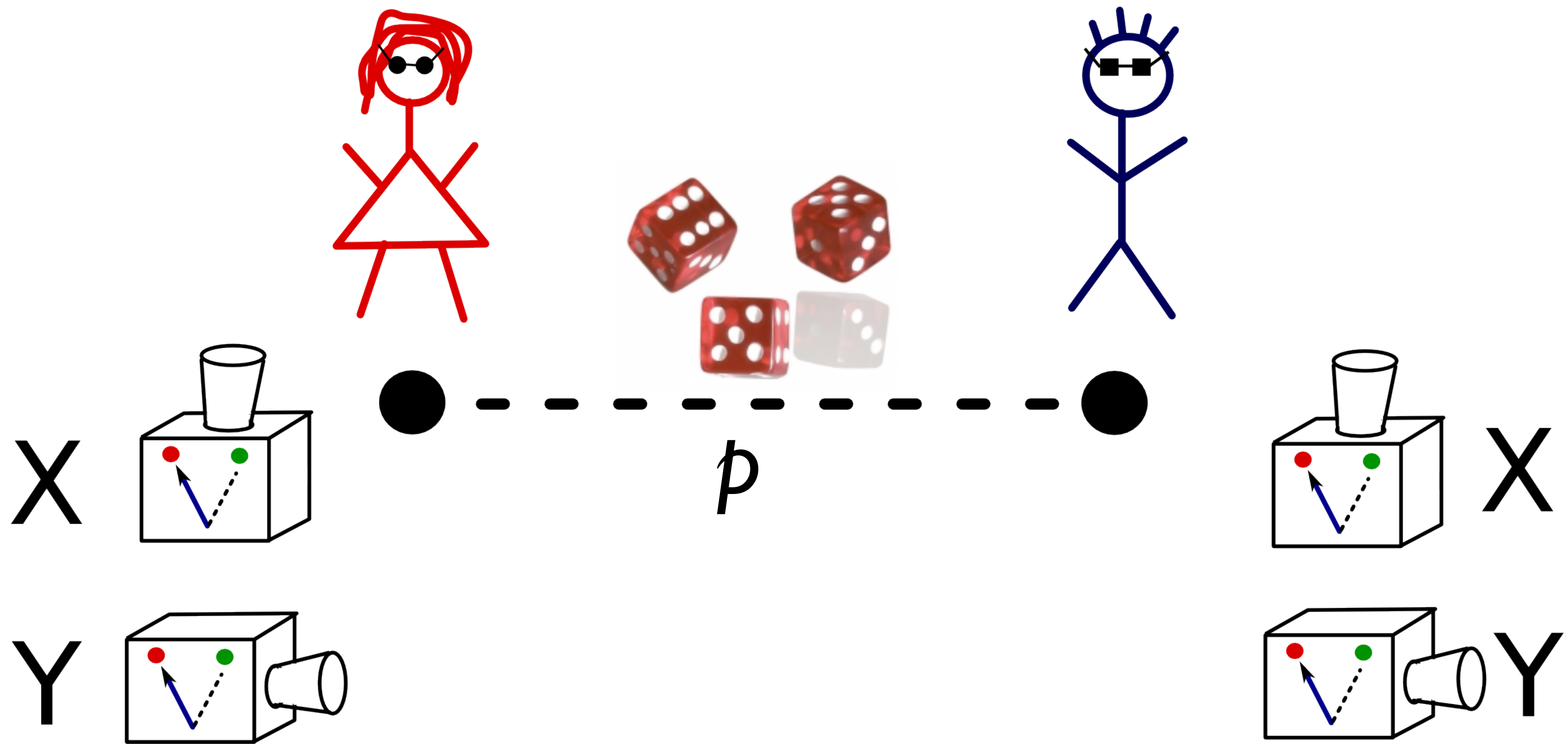

Measurement outcomes: -1 and +1. $\mathbf{C} := |\langle \mathbf{XX} \rangle - \langle \mathbf{XY} \rangle - \langle \mathbf{YX} \rangle - \langle \mathbf{YY} \rangle|$

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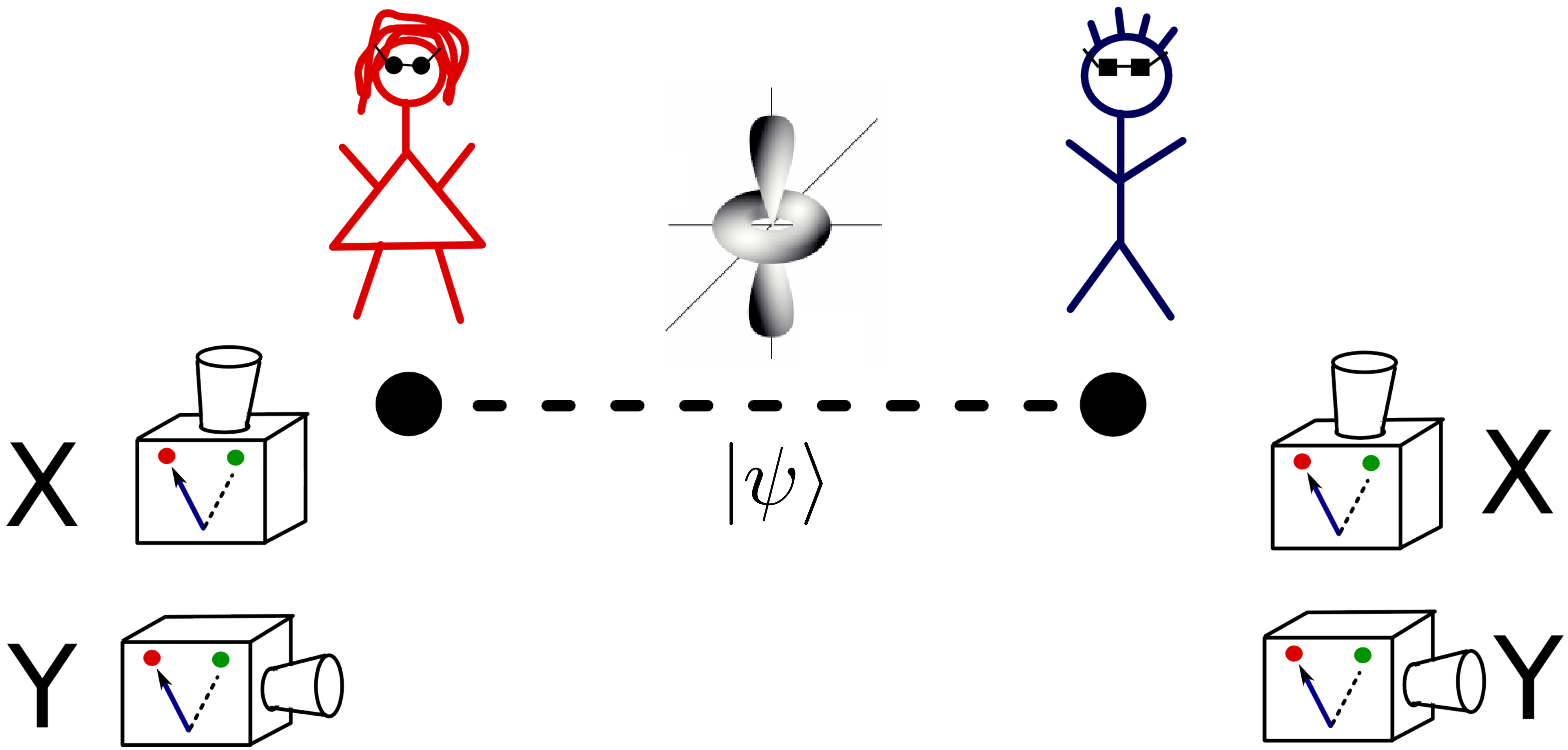

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- If $|\psi\rangle$ is a quantum state, then $\mathbf{C} > \mathbf{2}$ is possible, but still $\mathbf{C} \leq \mathbf{2\sqrt{2}}.$

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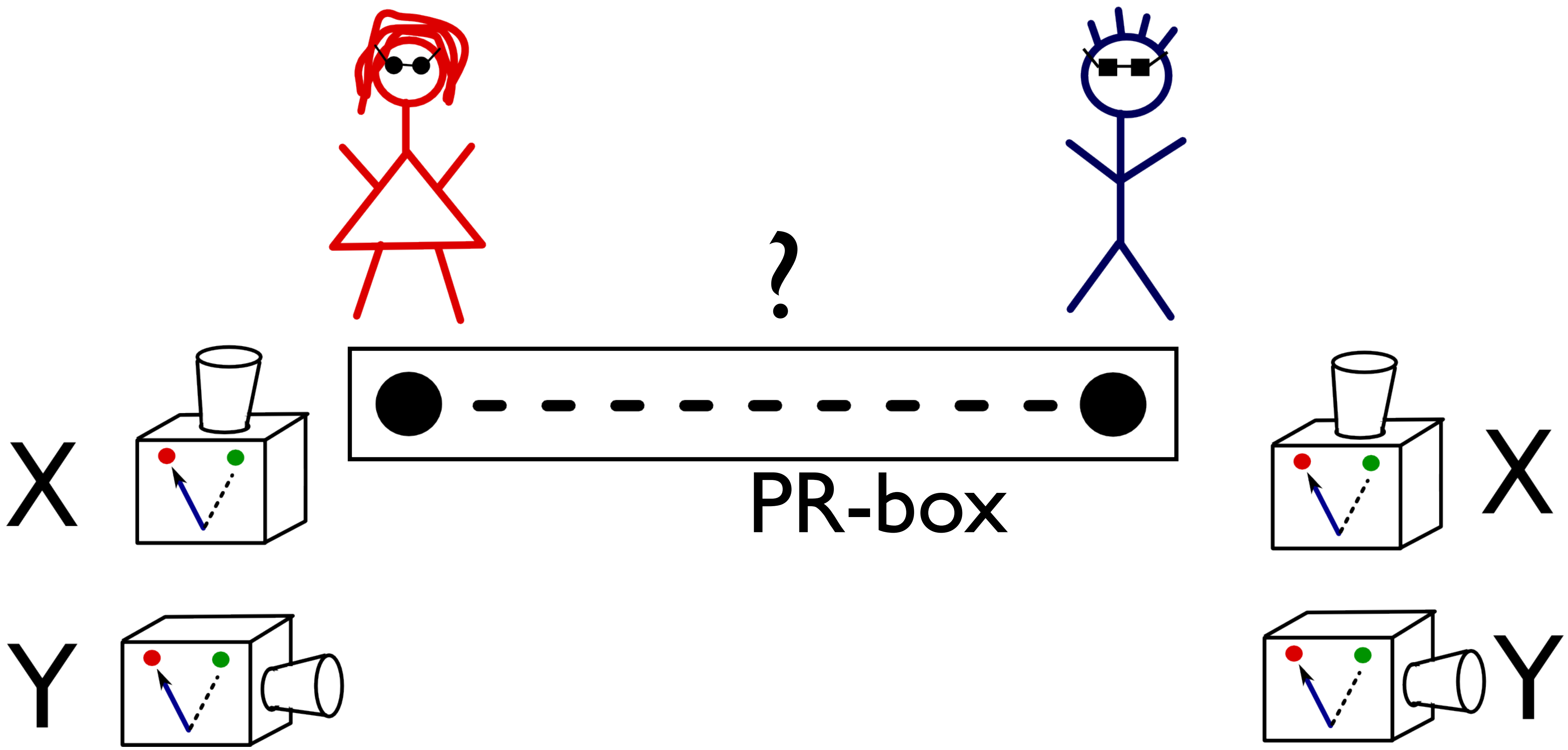

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- Hypothetical non-local boxes (e.g. PR-box) can have $\mathbf{C} = \mathbf{4}$.

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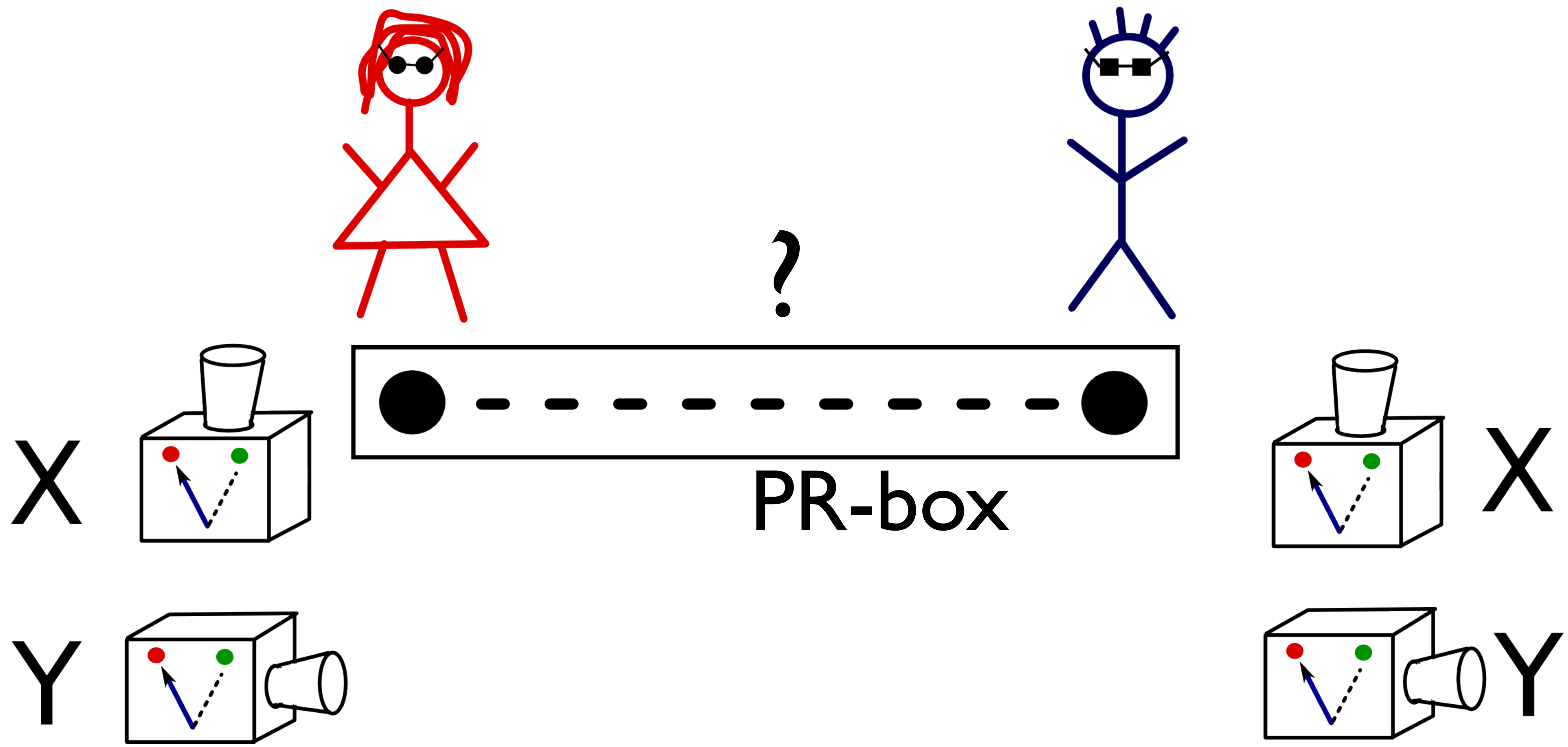

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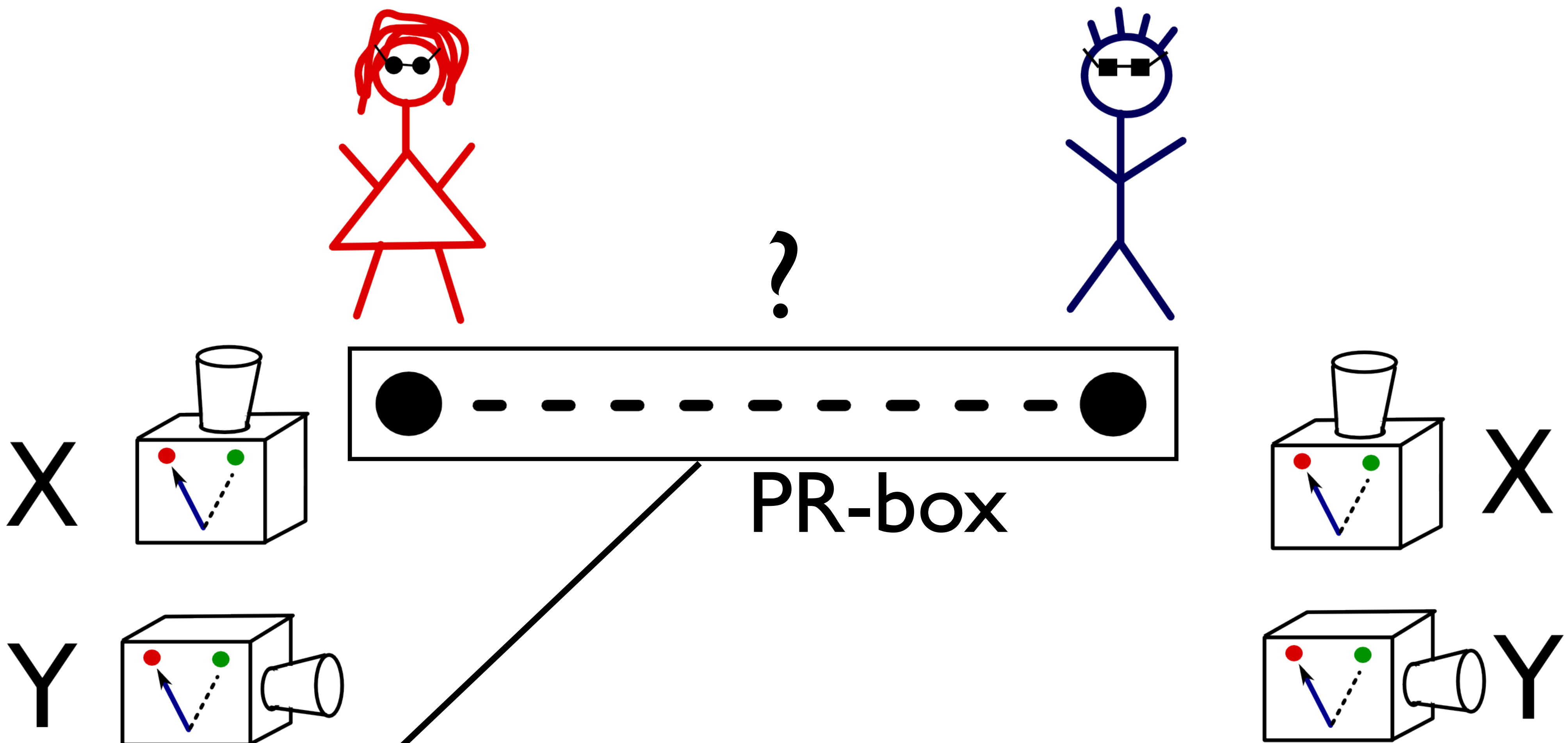

$$P(+1,-1|XX) = P(-1,+1|XX) = \frac{1}{2}$$

$$P\left(+1,+1 \,\middle|\, \begin{array}{c} XY \\ YX \\ YY \end{array}\right) = P\left(-1,-1 \,\middle|\, \begin{array}{c} XY \\ YX \\ YY \end{array}\right) = \frac{1}{2}$$

Satisfies **no-signalling-principle**: Alice's choice of measurement does not affect Bob's observed probabilities.

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- information causality would be violated (Pawlowski et al., Nature **461**, 1101 (2009))...

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What about **reversible computation** in probabilistic theories?

2. All reversible transformations in boxworld

State spaces and their symmetries

- N: number of parties (Alice, Bob, Charlie, ...)
- M: number of measurement devices (*X*, *Y*, ...)
- K: number of outcomes per device (-*1*, +*1*, ...)

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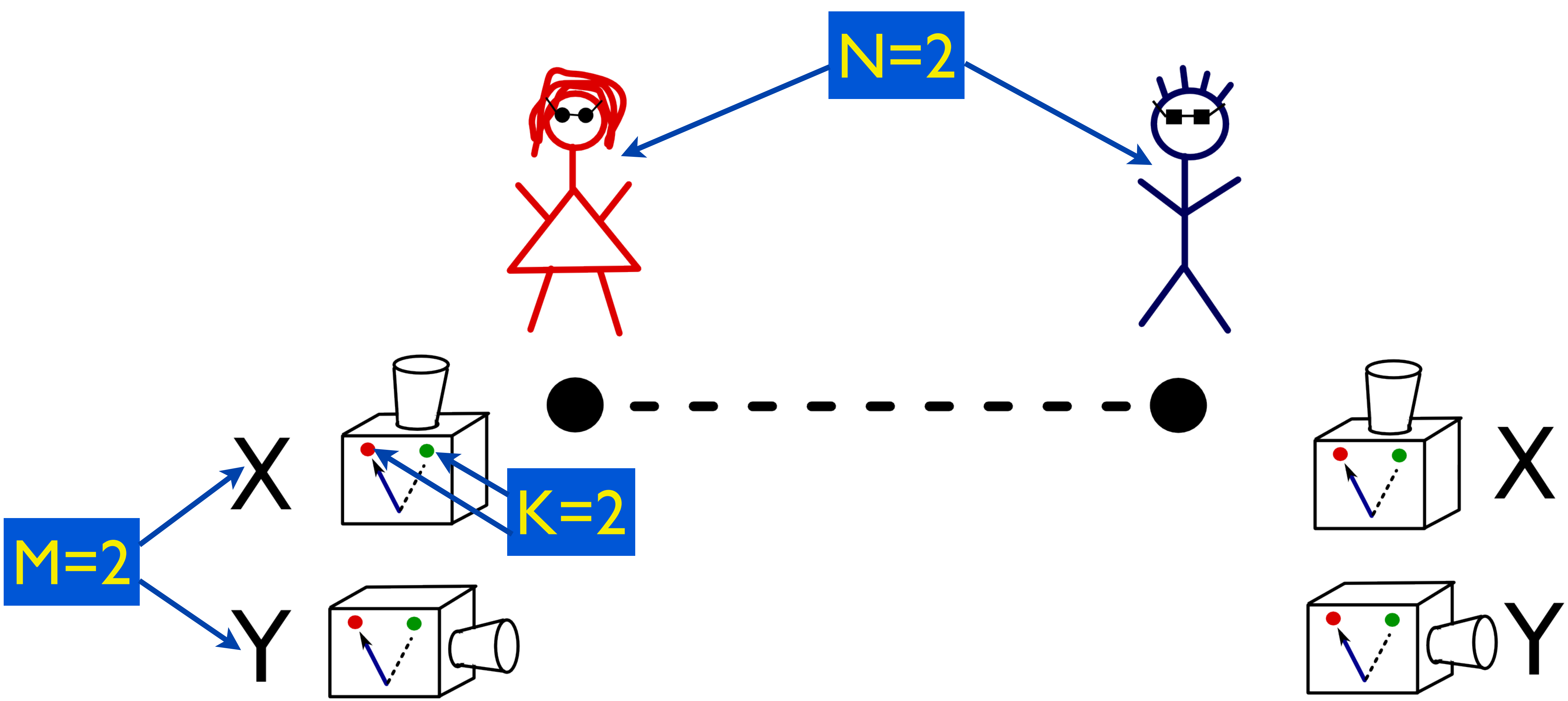

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$$P(a_1, \ldots, a_N | A_1, \ldots, A_N)$$

to the outcomes $a_1, \ldots, a_N$, given measurements $A_1, \ldots, A_N$.

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***(N,M,K)*-boxworld** consists of all states P that are

- non-negative,
- normalized in the obvious sense, and
- satisfy the no-signalling condition: $\sum_{a_i} P(a_1, \ldots, a_i, \ldots, a_N | A_1, \ldots, A_i, \ldots, A_N)$ this sum is independent of A_i.

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A **reversible transformation** is a linear map *T* such that *T* and T^{-1} map the state space to itself.

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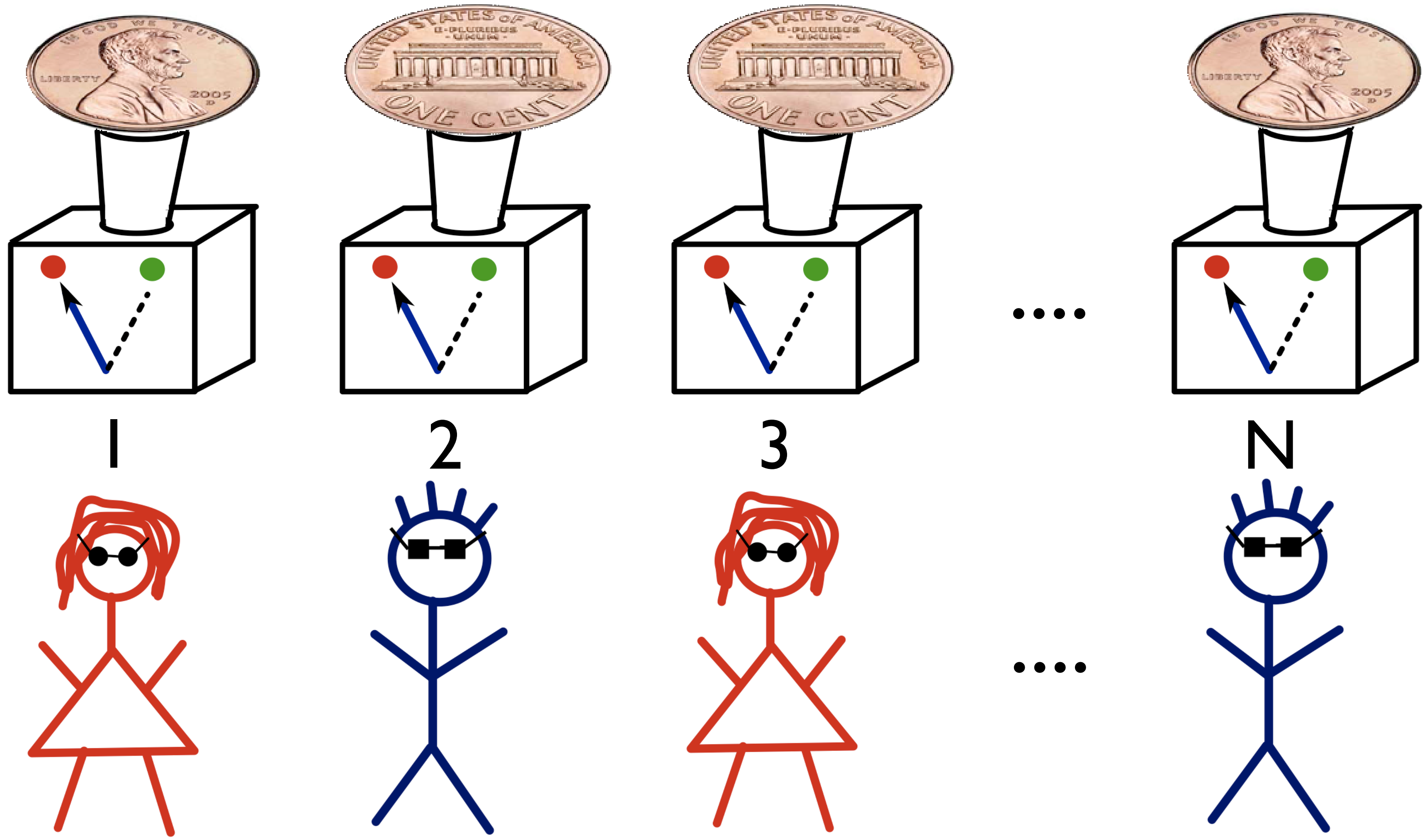
- M=1 (single device): classical probability theory

For simplicity, assume K=2 (classical bits, or coins).



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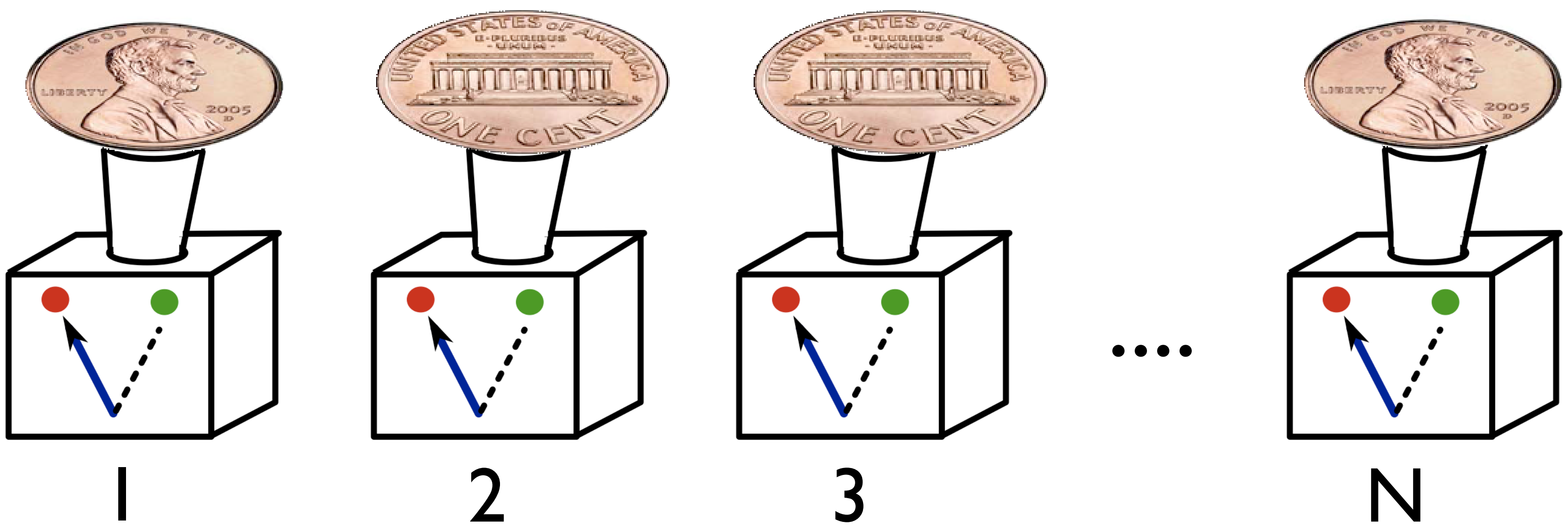

State space consists of all probability distributions on the 2^N bit strings.

All permutations are reversible transformations (these are many!).

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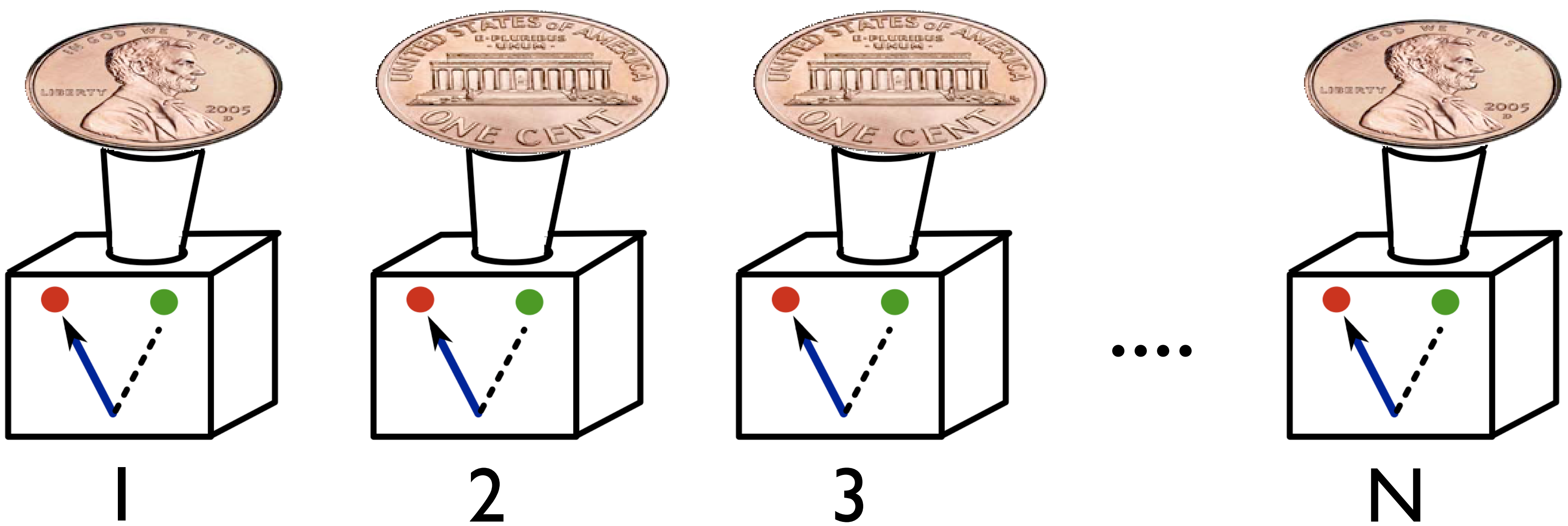

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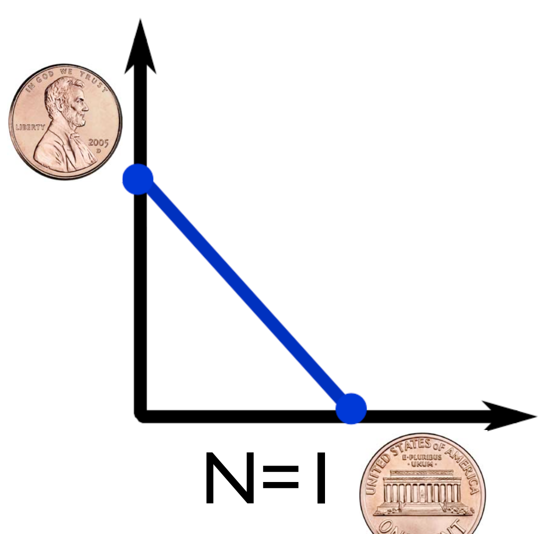

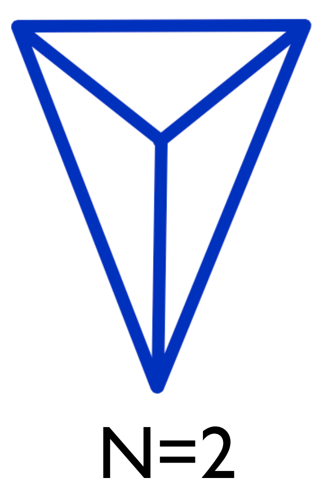

Geometrically, the state space is a simplex (highly symmetric).

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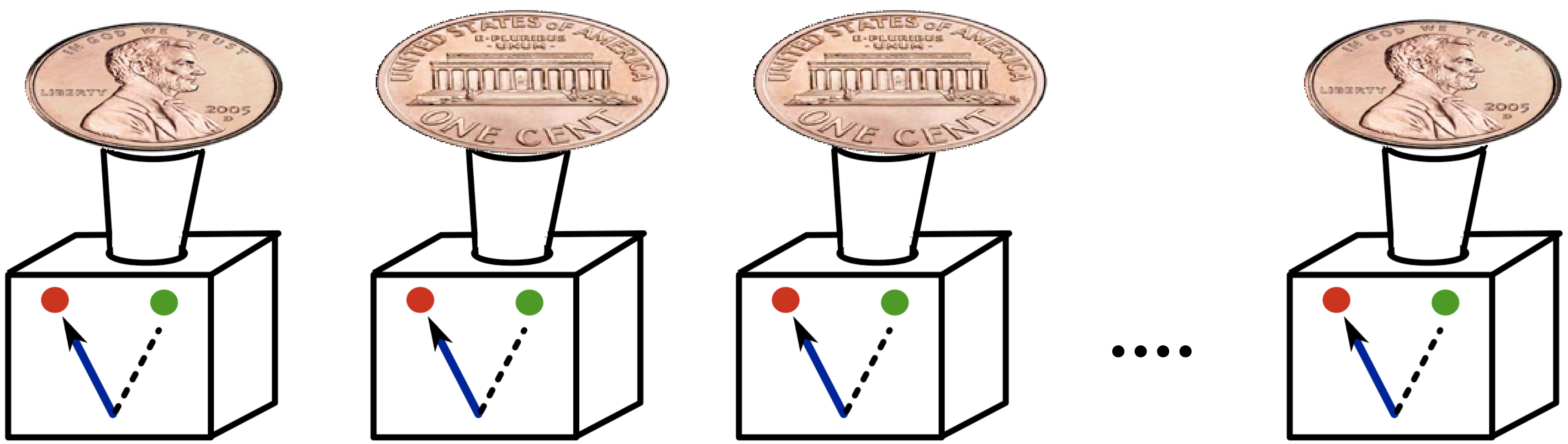

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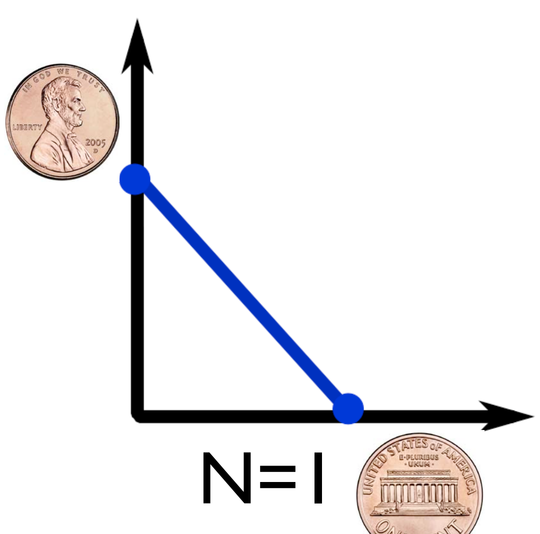
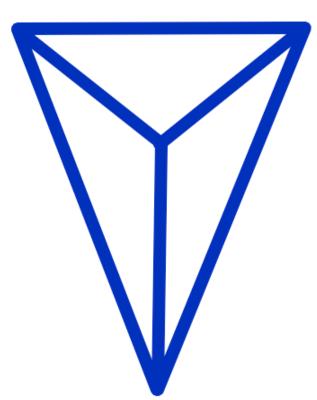


N=1



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Example: CNOT

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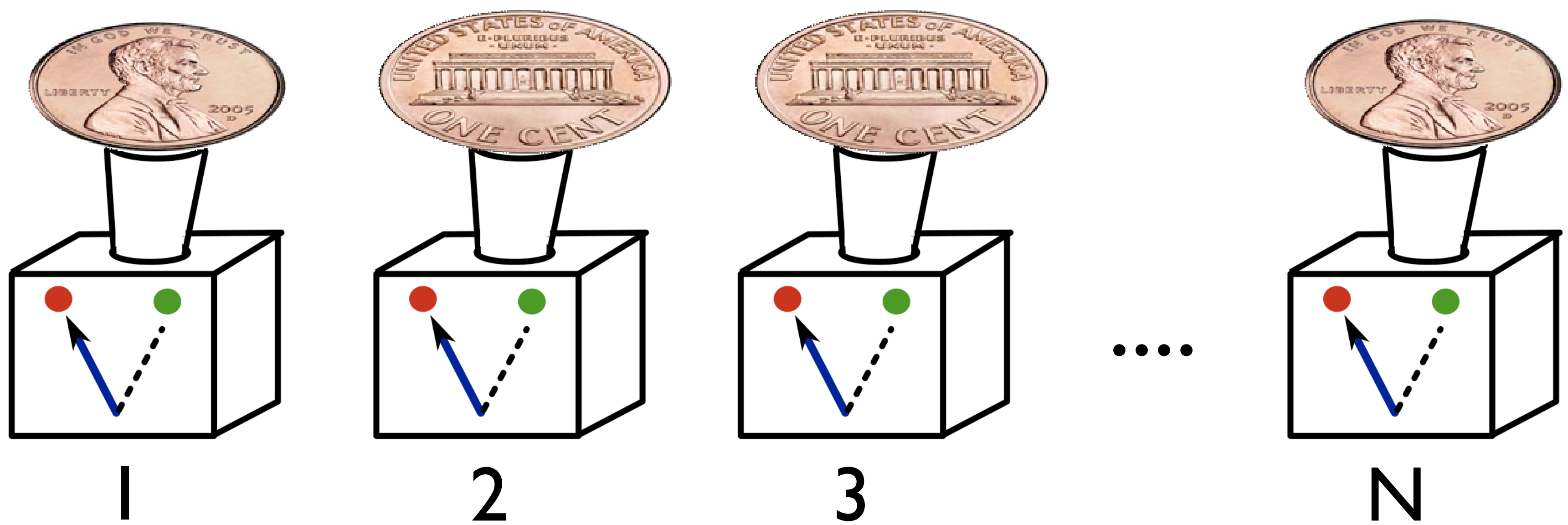

"Many" trans-formations: Classical reversible computation is possible.

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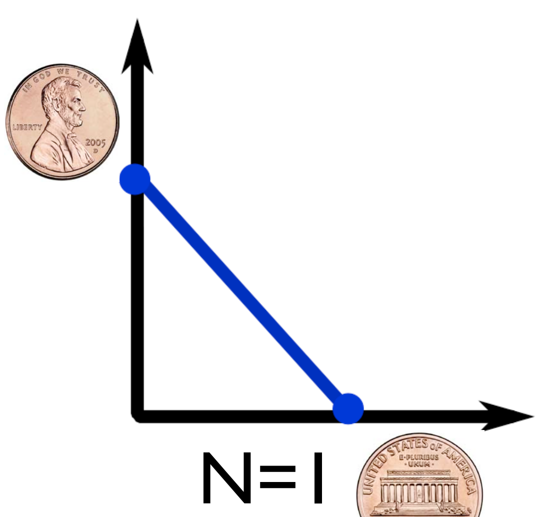

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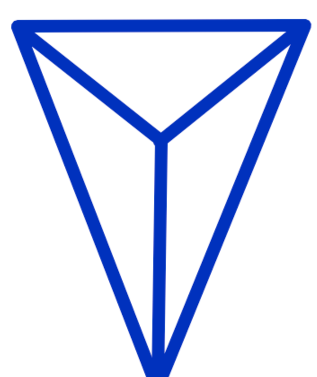

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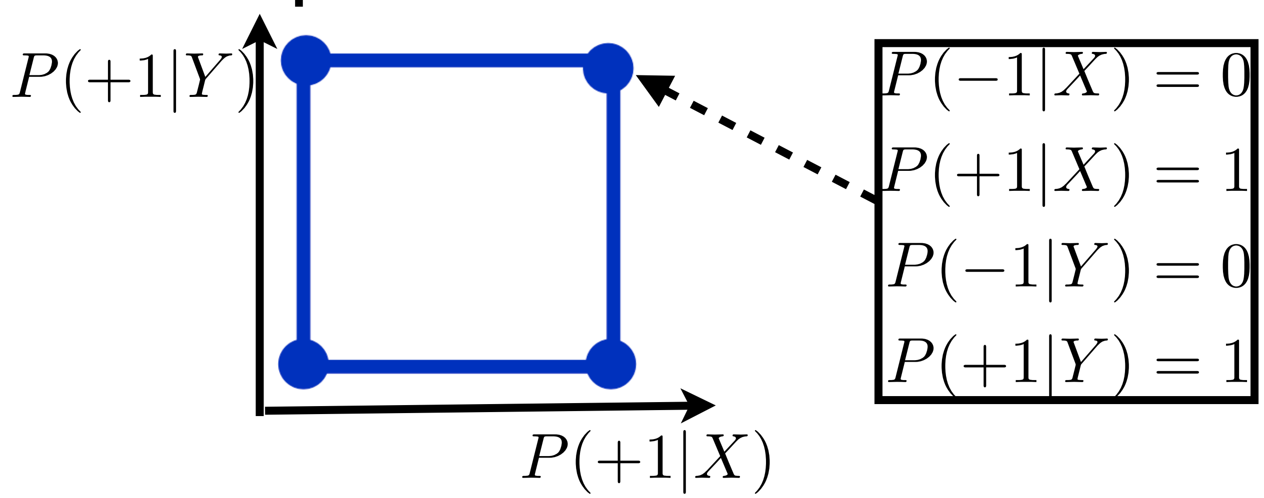
- M=2 (two devices X and Y): "generalized bits" (**gbits**) if K=2.
  - N=1: prob. for X- and Y-outcomes vary independently



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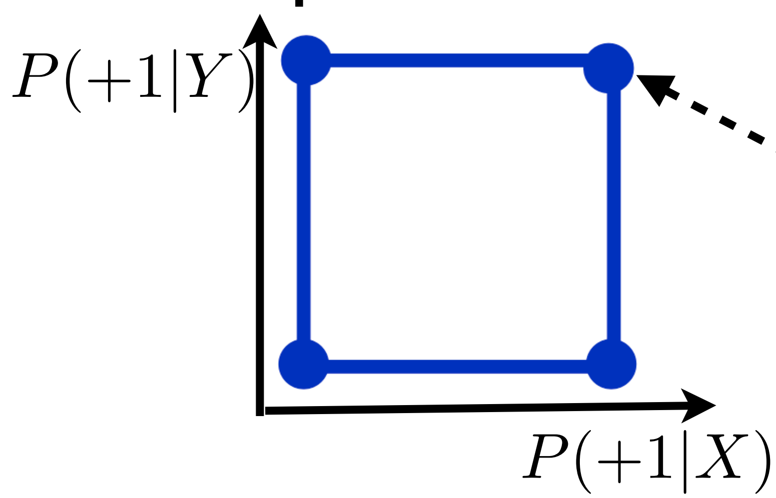

$$P(-1|X) = 0$$
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Reversible transformations:
Dihedral group D_4

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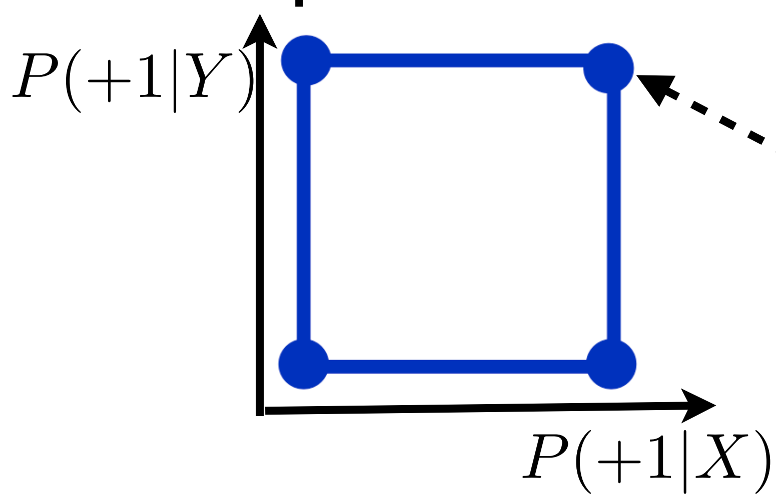

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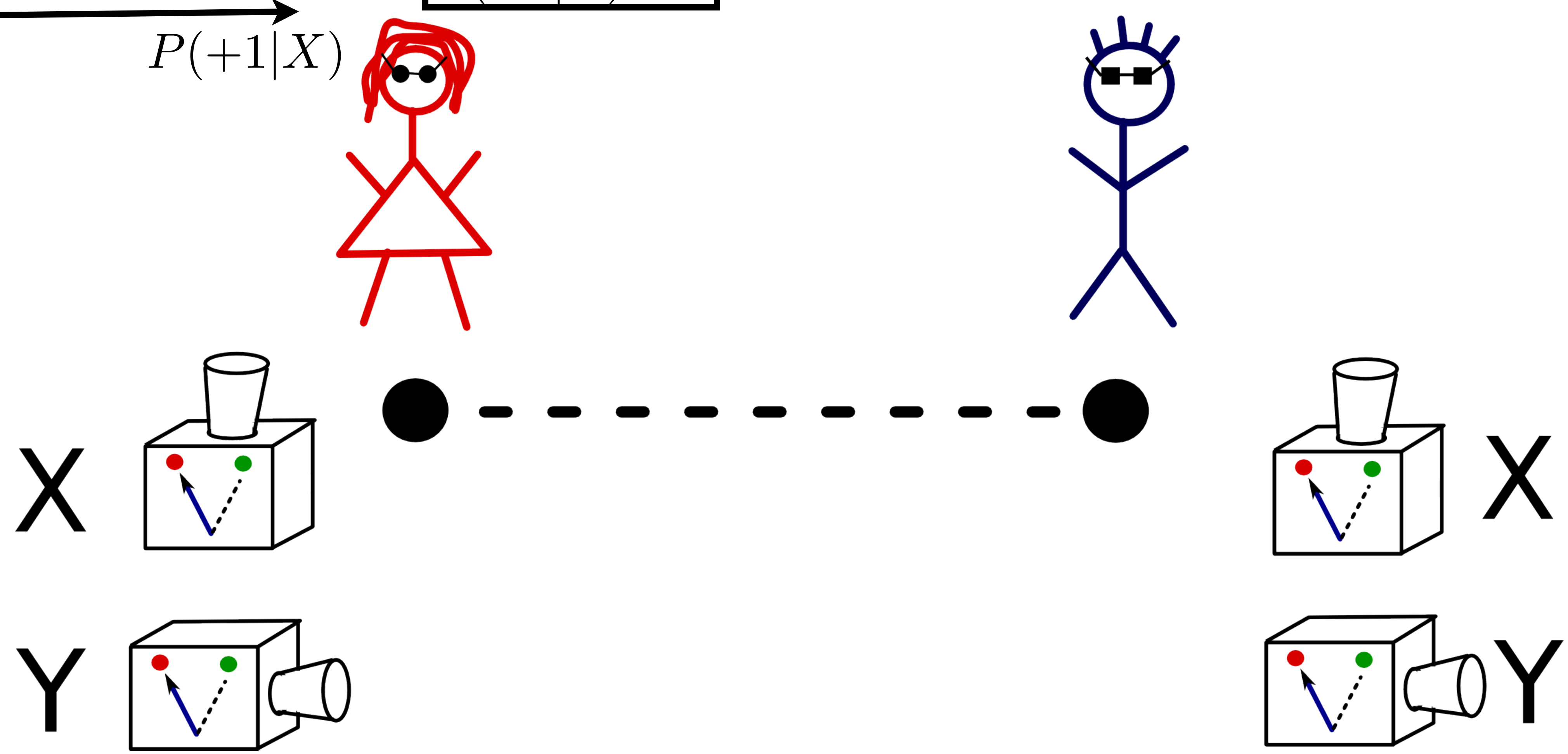

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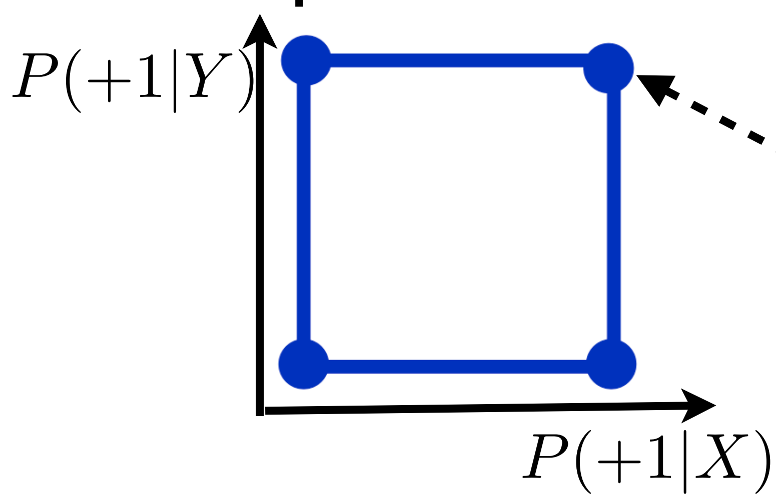

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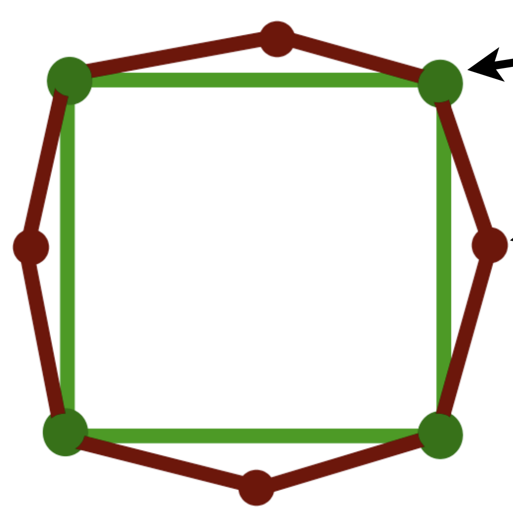
    24 vertices = 16 (4x4) product states + 8 PR-boxes



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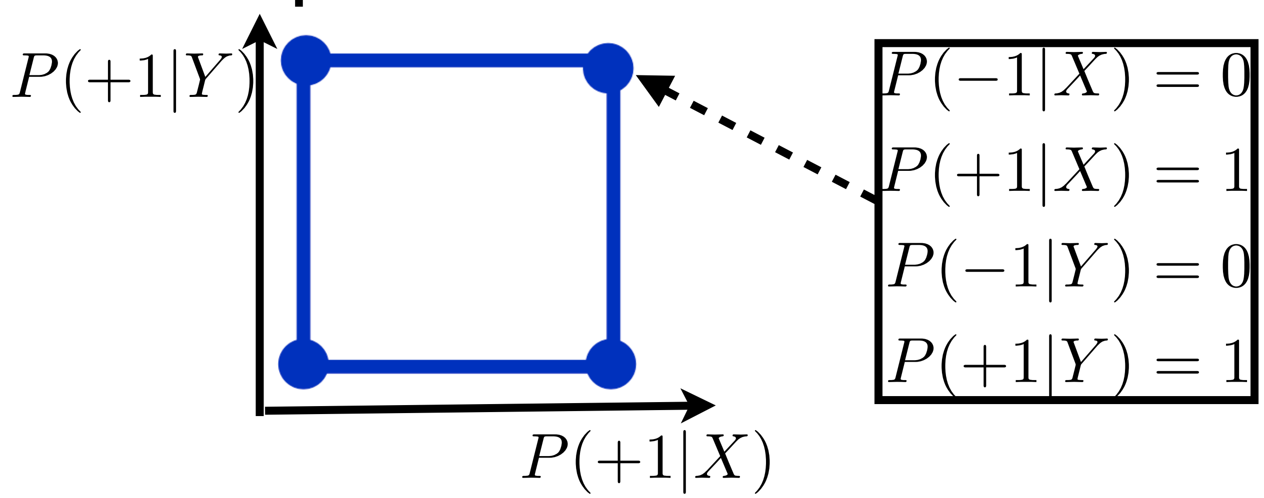

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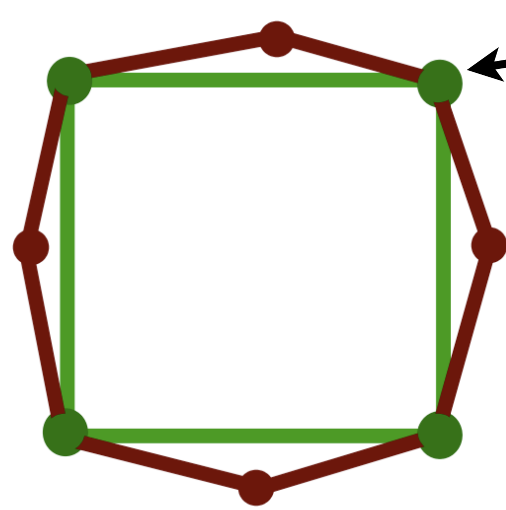

No reversible transformation maps product states to PR boxes.

The only reversible transformations are SWAP and local transformations.

2. All reversible transformations in boxworld

Main Results

Theorem 1: If $M \geq 2$ (at least two devices), then all reversible transformations in (N,M,K)-boxworld are combinations of

- local relabellings of measurements,
- local relabellings of outcomes,
- and permutations of subsystems.

No other operations (e.g. CNOT) are possible.

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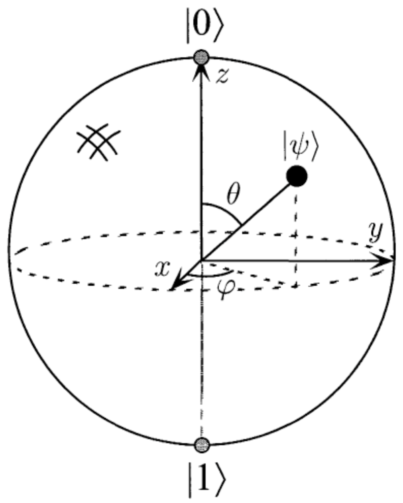
- More non-locality does not necessarily imply more powerful computation.
- There must be lots of symmetry in the state space of a theory for reversible computation.



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Hybrid systems: # of devices and outcomes varies among the subsystems.

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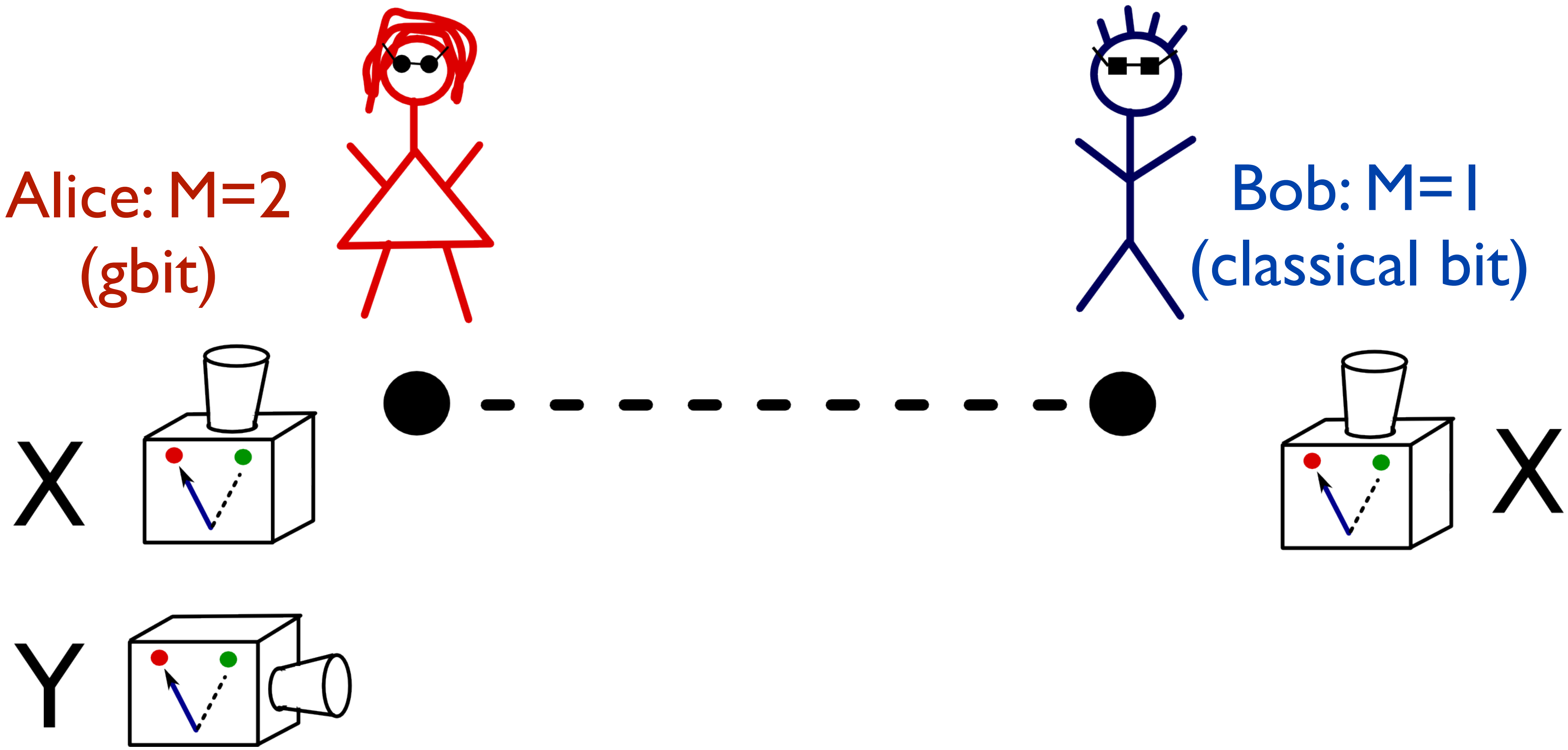
Theorem 1 remains valid in some cases, but not in all. Counterex.:



There is a CNOT operation: Bob's bit can control Alice's gbit, but **not vice versa**.

2. All reversible transformations in boxworld

Main Results

Hybrid systems: # of devices and outcomes varies among the subsystems.

Theorem 2: In every hybrid boxworld system, all reversible transformations map pure product states to pure product states.

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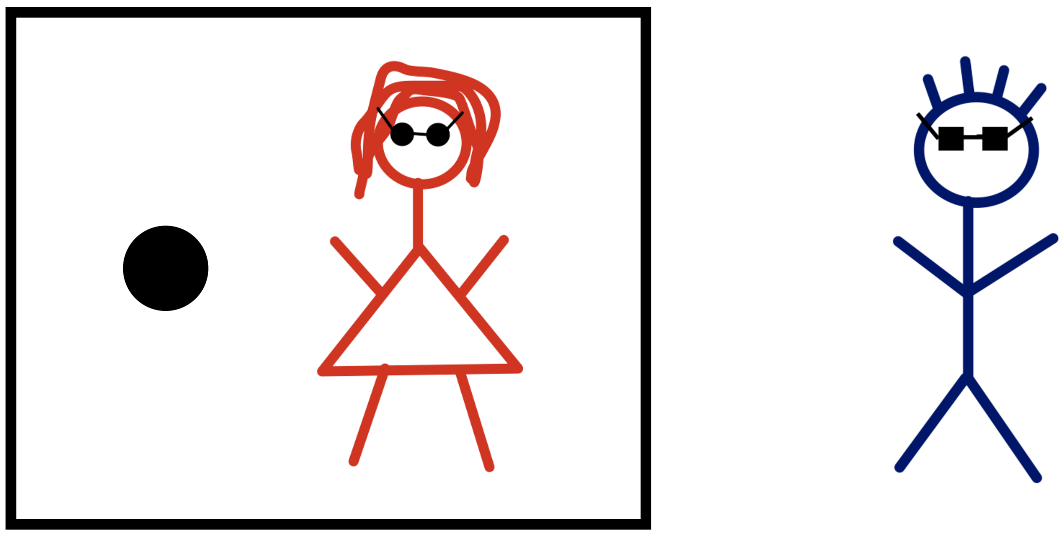
Hybrid systems: # of devices and outcomes varies among the subsystems.

Theorem 2: In every hybrid boxworld system, all reversible transformations map pure product states to pure product states.

- No non-locality can ever be reversibly created.
- Measurements done by third parties **must** be modelled as irreversible processes (in contrast to QM!)



2. All reversible transformations in boxworld
Proof Idea

- Switch from "Schrödinger" to "Heisenberg" picture.

  QM: states ρ, effects=projectors $\Pi \longrightarrow$ probabilities $\mathrm{tr}(\rho\Pi)$

  $$\mathcal{U}(\rho) := U\rho U^\dagger \longrightarrow \mathcal{U}^\dagger(\Pi) := U^\dagger \Pi U$$

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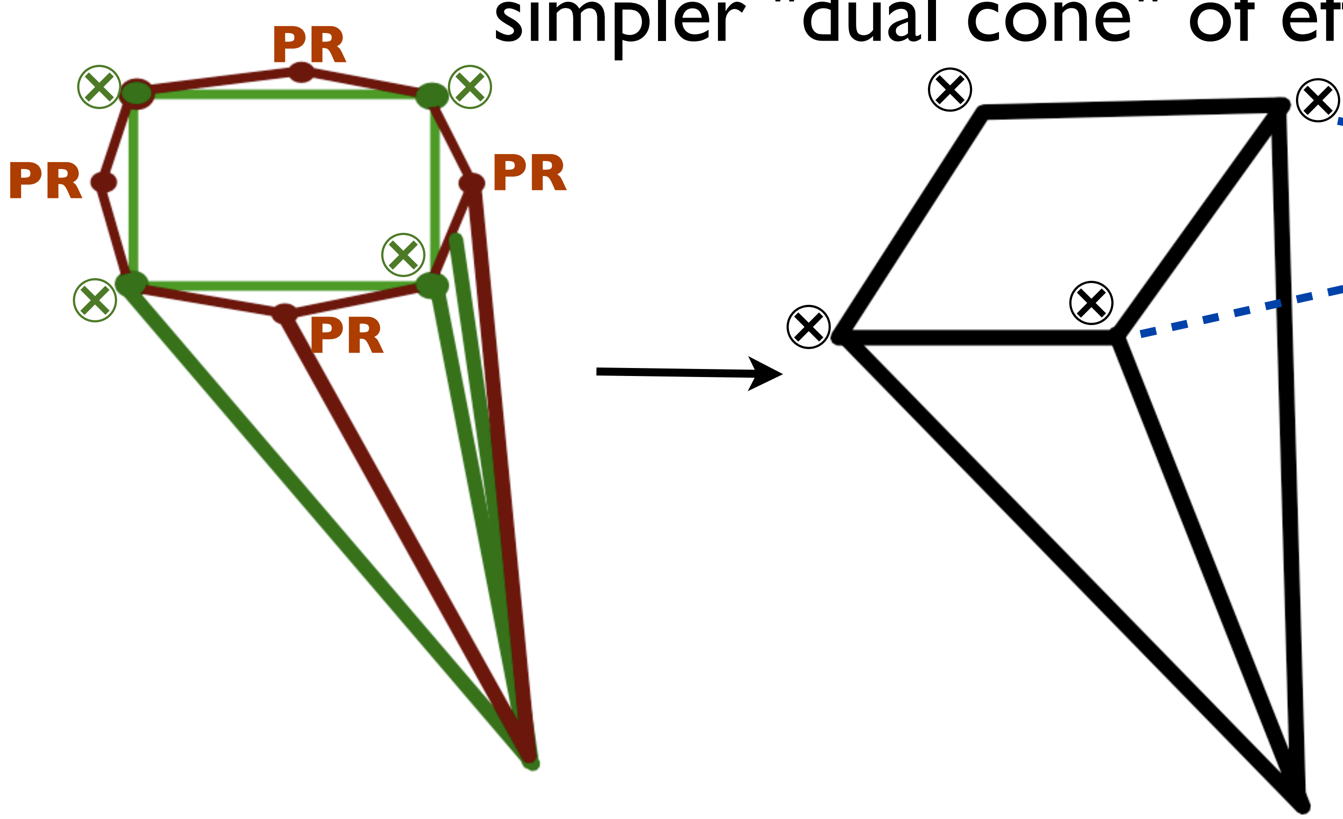
- Boxworld: difficult cone of unnormalized states $\longrightarrow$ simpler "dual cone" of effects



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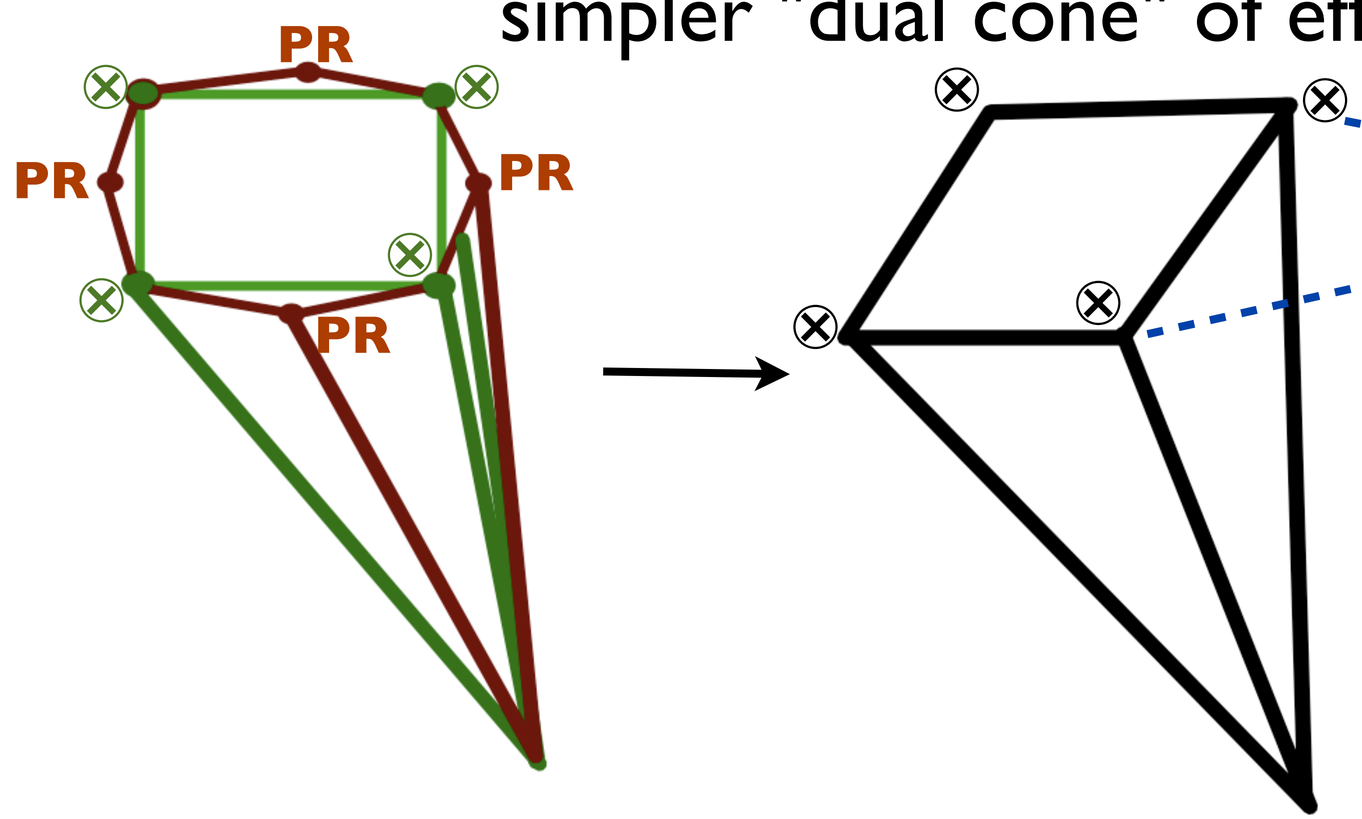

Extremal rays are all spanned by product effects

- Reversible transformations map product effects to product effects.
- Preservation of scalar products $\longrightarrow$ enough invariants.

Conclusions

- We have classified all reversible transformations in boxworld.
- Except for classical theory (M=1), all reversible transformations are local operations and permutations of subsystems.
- More generally: for hybrid boxworld systems, no entangled states can ever be reversibly prepared from product states.

arXiv:0910.1840, Phys. Rev. Lett. **104**, 080402 (2010)

Thank you!