Entropy and Algorithmic Complexity in Quantum Information Theory: a Quantum Brudno's Theorem (quant-ph/0506080)

Fabio Benatti[1] Tyll Krüger[2]
Markus Müller[2] Rainer Siegmund-Schultze[2]
Arleta Szkoła[2*]

[1]University of Trieste
Department of Theoretical Physics
Strada Costiera, 11, 34014 Trieste, Italy

[2]Technische Universität Berlin
Fakultät II - Mathematik und Naturwissenschaften
Institut für Mathematik MA 7-2
Straße des 17. Juni 136, 10623 Berlin, Germany

Overview

- Brudno's Theorem (classical),
- Quantum Algorithmic (=Kolmogorov) Complexity,
- Ergodic Quantum Sources,
- Quantum Brudno's Theorem.
- Outline of the Proof:
  - Upper Bound: Universal Compression,
  - Lower Bound: Quantum Counting Argument,
- Discussion.

Brudno's Theorem (classical)

compares the **entropy rate** and **complexity rate** of strings, emitted by ergodic information sources.

- **Kolmogorov (=algorithmic) complexity** K:
  Let $x \in \{0,1\}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$ be a binary string, and $U : \{0,1\}^* \to \{0,1\}^*$ a universal Turing Machine. Then,

  $$K(x) := \min\{\ell(s) : U(s) = x\}.$$

  Up to an additive constant *independent* of U.

- **Complexity rate** k:
  Let $x \in \{0,1\}^{\mathbb{N}}$, and $x^{(n)}$:=first n bits of x.

  $$k(x) := \lim_{n\to\infty} \frac{1}{n} K(x^{(n)}).$$

Brudno's Theorem (classical)

$$\text{complexity rate } k(x) := \lim_{n\to\infty} \frac{1}{n} K(x^{(n)})$$

information source p: emits strings $x^{(n)}$ of length n, according to probability distributions $p^{(n)}$.

- (Shannon) Entropy rate h:

$$\begin{aligned} h(p) &:= \lim_{n\to\infty} \frac{1}{n} H(p^{(n)}) \\ &= -\lim_{n\to\infty} \frac{1}{n} \sum_{s\in\{0,1\}^n} p^{(n)}(s) \log p^{(n)}(s) \end{aligned}$$

Theorem 1. [Brudno, 1983] *If p is a (classical) ergodic information source, then*

$$k(x) = h(p) \qquad \textit{for } p\textit{-almost all } x.$$

Quantum Algorithmic Complexity

Goal: Find quantum version of Brudno's Theorem.

$\longrightarrow$needed: quantum versions of binary strings, Kolmogorov complexity, and entropy rate.

- Qubit string σ = any density operator on $\mathcal{H}_{Fock} := \bigoplus_{k=0}^{\infty} \mathcal{H}_k$, where $\mathcal{H}_k := \left(\quad ^{\{0,1\}} \right)^{\otimes k}$. Let $\mathcal{H}_{\leq n} := \bigoplus_{k=0}^{n} \mathcal{H}_k$, and define the length

$$\ell(\sigma) := \min\{n \in \quad _0 \mid \sigma \in \mathcal{T}_1^+(\mathcal{H}_{\leq n})\},$$

where $\mathcal{T}_1^+(\mathcal{H})$:=set of density operators on $\mathcal{H}$.

Example: $\sigma := \frac{1}{2}\left(|0\rangle + |11\rangle\right)\left(\langle 0| + \langle 11|\right)$ is a qubit string of length $\ell(\sigma) = 2$.

- Trace distance $\|\rho - \sigma\|_{tr} := \frac{1}{2}\mathrm{Tr}|\rho - \sigma|$.

Quantum Algorithmic Complexity

- **Quantum Turing Machine (*QTM*)**: E. Bernstein, U. Vazirani, "Quantum Complexity Theory", *SIAM Journal on Computing* **26** 1411-1473 (1997)

  A Quantum Turing Machine M

  – is defined by a local (computable) transition amplitude,
  – an input qubit string σ is mapped to an output qubit string $\rho := M(\sigma)$,
  – may halt or may not halt on input σ,
  – is a partial mapping

$$\begin{aligned} M : \mathcal{T}_1^+(\mathcal{H}_{Fock}) &\longrightarrow \mathcal{T}_1^+(\mathcal{H}_{Fock}) \\ \sigma &\longmapsto \rho, \end{aligned}$$

  undefined if M does not halt.

Quantum Algorithmic Complexity

Definition 2. [Quantum Algorithmic Complexity] *Let* M *be a QTM and* $\rho \in \mathcal{T}_1^+(\mathcal{H}_{Fock})$ *a qubit string. For every* $\delta \geq 0$*, let*

$$QC_M{}^{\delta}(\rho) := \min \{\ell(\sigma) \mid \|\rho - M(\sigma)\|_{\mathrm{tr}} \leq \delta\},$$

$$QC_M^{\searrow 0}(\rho) := \min \left\{\ell(\sigma) \middle| \|\rho - M(k,\sigma)\|_{\mathrm{tr}} \leq \frac{1}{k} \forall k \in \quad \right\}$$

Fix a univ. QTM U*, and set* $QC^{\bullet}(\rho) := QC_U^{\bullet}(\rho)$.

A. Berthiaume, W. Van Dam and S. Laplante, "Quantum Kolmogorov complexity", *J. Comput, System Sci.* **63** 201-221 (2001)

Ergodic Quantum Sources

- C^*-Algebra $\mathcal{A} := \mathcal{B}(\quad^2) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| a, b, c, d \in \qquad \right\}$

- assign to each $x \in \quad$ an algebra $\mathcal{A}_x \simeq \mathcal{A}$.

- $\mathcal{A}^{(n)} := \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \ldots \otimes \mathcal{A}_n \qquad (n \in \quad)$,
  $\mathcal{A}_\Lambda := \bigotimes_{x \in \Lambda} \mathcal{A}_x \qquad (\Lambda \subset \quad$ finite$)$

- quasilocal algebra $\mathcal{A}^\infty$:=operator norm completion of $\bigcup_{\Lambda \subset \mathbb{Z}} \mathcal{A}_\Lambda \approx \ldots \otimes \mathcal{A}_{-1} \otimes \mathcal{A}_0 \otimes \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \ldots$

- state Ψ:=normalized positive linear functional on $\mathcal{A}^\infty$. Restriction to interval $[1, n]$: $\Psi^{(n)} := \Psi|_{\mathcal{A}^{(n)}}$.

Ergodic Quantum Sources

- $\mathcal{A}^\infty \approx \ldots \otimes \mathcal{A}_{-1} \otimes \mathcal{A}_0 \otimes \underbrace{\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3}_{\mathcal{A}^{(3)}} \otimes \ldots$

- Correspondence: state $\Psi^{(n)}$ $\leftrightarrow$ density matrix $\rho^{(n)}$
  $(a \in \mathcal{A}^{(n)})$ $\quad \Psi^{(n)}(a) = \mathrm{Tr}(\rho^{(n)} a)$

- state Ψ is shift-invariant iff $\Psi(a \otimes \mathbf{1}) = \Psi(\mathbf{1} \otimes a)$.

- The set of shift-invariant states is a convex set. Its extremal points are called ergodic states.

- (von Neumann) entropy rate $s := \lim_{n\to\infty} \frac{1}{n} S(\rho^{(n)})$, where $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$.

Quantum Brudno's Theorem

Theorem 3. [Quantum Brudno's Theorem] *Let* $(\mathcal{A}^\infty, \Psi)$ *be an ergodic quantum information source with entropy rate* s*. For every* $\delta > 0$*, there exists a sequence of* Ψ*-typical projectors* $q_n(\delta) \in \mathcal{A}^{(n)}$*,* $n \in$ *, i.e.* $\lim_{n\to\infty} \Psi^{(n)}(q_n(\delta)) = 1$*, such that for every pure qubit string* $q \leq q_n(\delta)$ *and* n *large enough*

- $\frac{1}{n} QC^{\searrow 0}(q) \in (s - \delta, s + \delta)$,
- $\frac{1}{n} QC^{\delta}(q) \in (s - \delta(2 + \delta)s, s + \delta)$.

Outline of the Proof

$\frac{1}{n}QC(q) < s + \delta$: Universal compression

A. Kaltchenko and E. H. Yang, "Universal compression of ergodic quantum sources", *Quantum Information and Computation* **3, No. 4** 359-375 (2003)

- Generalization of Schumacher's quantum data compression to general ergodic sources,
- uses universal codebook *independent* of state Ψ.
- $\underset{\ell(q) = n}{\text{qubit string } q} \quad \xrightarrow{\text{compression}} \quad \underbrace{\underset{\ell(\tilde{q}) \approx n(s+\delta)}{\text{qubit string } \tilde{q}}}_{\text{valid QTM program}}$

Outline of the Proof

$\frac{1}{n} QC(q) > s - \delta$: Quantum Counting Argument

Quantum operations $\mathcal{E}$ cannot increase number of $\perp$ vectors. More in detail:

Lemma 4. [Counting Argument]
Let $\mathcal{E}$ *be any quantum operation and* ${N_c}^{\delta}$ *:=max. number of* $\perp$ *pure qubit strings* ρ *with* $QC_{\mathcal{E}}^{\delta}(\rho) \leq c$.
Then, $\log {N_c}^{\delta} < c + 1 + \frac{2+\delta}{1-2\delta}\delta c$.

- There are $2^{c+1} - 1$ $\perp$ input strings σ with $\ell(\sigma) \leq c$.

- Proves the lower bound together with: I. Bjelaković, T. Krüger, Ra. Siegmund-Schultze, A. Szkoła, "The Shannon-McMillan theorem for ergodic quantum lattice systems", *Invent. Math.* **155** 203-222 (2004)

Discussion

- Why ergodic sources (not only i.i.d.)? Ergodic sources naturally appear in statistical mechanics.

- Result: Correspondence between

  $$\text{Complexity rate } \frac{1}{n}QC \qquad \longleftrightarrow \qquad \text{Entropy rate } s$$

  Interesting, because

  – there are different (inequivalent) definitions of quantum Kolmogorov complexity (Gács 2001, Vitányi 2001, Rogers, Vedral 2005 etc.);
  – what roles do they play?
  – different complexities $\longleftrightarrow$ different entropies?

$\longrightarrow$ Quantum Algorithmic Information Theory?