Quantum Bit Strings and Prefix-Free Subspaces

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Las Vegas, July 16, 2008

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Not prefix-free sets like $\{0, 00\}$: $0 \circ 00 \xleftarrow{?} 000 \xrightarrow{?} 00 \circ 0$

Theorem (Kraft Inequality)

There exists a prefix code $\{c_1, \ldots, c_n\}$ *with code word lengths* $\{l_1, \ldots, l_n\} = \{\ell(c_1), \ldots, \ell(c_n)\} \subset \mathbb{N}_0$, *if and only if*

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- Restrictions or prefixes ψ_1^2 (:=the first two qubits of $|\psi\rangle$) should be given by partial trace:

$$\psi_1^2 = \mathrm{Tr}_{[3,\infty)}|\psi\rangle\langle\psi| \qquad \text{Tensor product structure??}$$

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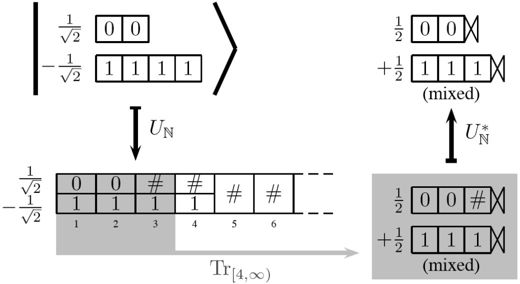

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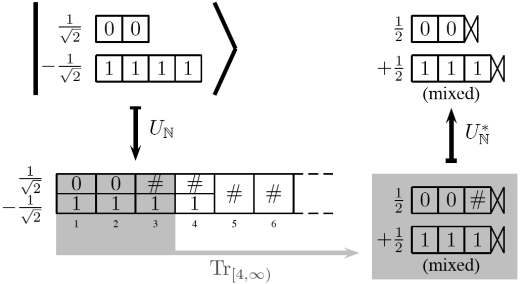

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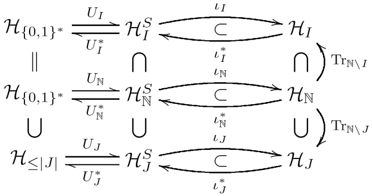

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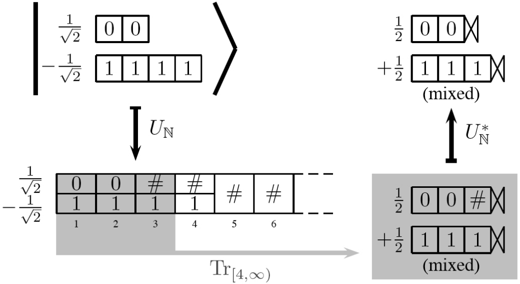

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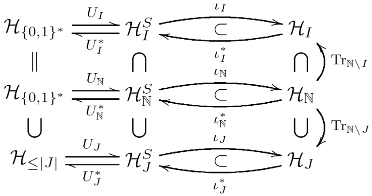

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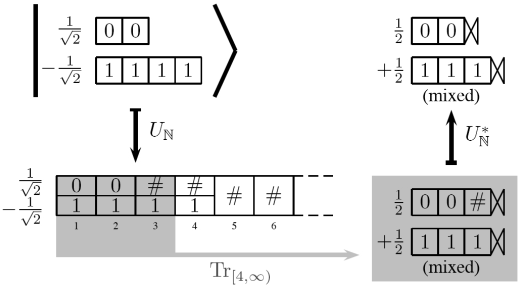

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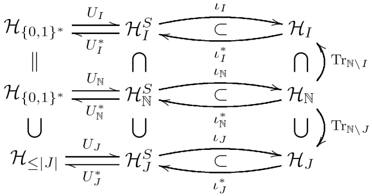

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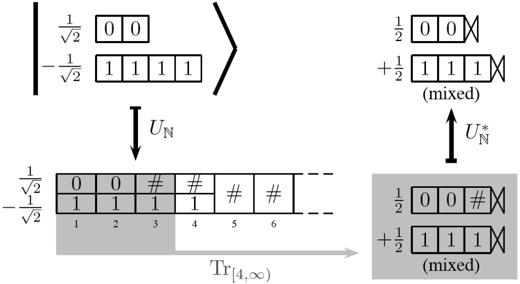

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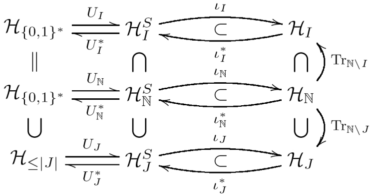

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- Prefix $\rho_1^n := \rho_{[1,n]}$ is in general a mixed state!

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This definition of restriction is consistent with a natural tensor product in $\mathcal{H}_{\{0,1\}^*}$.

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- If $|\psi\rangle$ is a length eigenvector, then $|\psi \circ \varphi\rangle = |\psi\rangle \otimes |\varphi\rangle$.
- The concatenation is in general not isometric: If $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |00\rangle\right)$, then

$$|\psi \circ \psi\rangle = \frac{1}{2}|00\rangle + |000\rangle + \frac{1}{2}|0000\rangle.$$

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Theorem: A closed subspace $\mathcal{H} \subset \mathcal{H}_{\{0,1\}^*}$ is prefix-free if and only if one (and thus every) orthonormal basis of $\mathcal{H}$ is prefix-free.

- In contrast to classical strings, qubit strings can be proper prefixes of themselves: E.g. for $|\varphi\rangle := \frac{3}{5}|1\rangle + \frac{4}{5}|10\rangle$ it holds

$$\langle\varphi|\varphi\circ 0\rangle = \frac{3\cdot 3}{5\cdot 5}\underbrace{\langle 1|10\rangle}_{0} + \frac{3\cdot 4}{5\cdot 5}\underbrace{\langle 1|100\rangle}_{0} + \frac{4\cdot 3}{5\cdot 5}\underbrace{\langle 10|10\rangle}_{1} + \frac{4\cdot 4}{5\cdot 5}\underbrace{\langle 10|100\rangle}_{0} \neq 0,$$

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Theorem (Perfect Distinguishability by means of Prefixes)

An orthonormal system $M \subset \mathcal{H}_{\{0,1\}^*}$ of length eigenvectors is prefix-free if and only if for all $|\varphi\rangle \neq |\psi\rangle \in M$ it holds

$$\langle\psi|\varphi_1^{\ell(\psi)}|\psi\rangle = 0. \qquad (1)$$

- In contrast to classical strings, qubit strings can be proper prefixes of themselves: E.g. for $|\varphi\rangle := \frac{3}{5}|1\rangle + \frac{4}{5}|10\rangle$ it holds

$$\langle\varphi|\varphi\circ 0\rangle = \frac{3\cdot 3}{5\cdot 5}\underbrace{\langle 1|10\rangle}_{0} + \frac{3\cdot 4}{5\cdot 5}\underbrace{\langle 1|100\rangle}_{0} + \frac{4\cdot 3}{5\cdot 5}\underbrace{\langle 10|10\rangle}_{1} + \frac{4\cdot 4}{5\cdot 5}\underbrace{\langle 10|100\rangle}_{0} \neq 0,$$

  so the singleton $\{|\varphi\rangle\}$ is not prefix-free.
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Schumacher+Westmoreland (2001) use (1) as a definition of prefix freedom $\longrightarrow$ restriction to M and $\mathcal{H} \subset \mathcal{H}_{\{0,1\}^*}$!

Example for a "skew" prefix-free Hilbert space: $\mathcal{H} := \mathrm{span}\, M$ with

$$M := \left\{ \underbrace{\frac{1}{\sqrt{2}}\left(|1\rangle + |01\rangle\right)}_{=:|\psi\rangle}, \underbrace{\frac{1}{\sqrt{2}}\left(|10\rangle - |010\rangle\right)}_{=:|\varphi\rangle} \right\}.$$

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- $\mathcal{H}$ has no basis of length eigenvectors.
- But $|\varphi\rangle$ and $|\psi\rangle$ can not be distinguished by means of the first two qubits: $\langle\psi|\varphi_1^2|\psi\rangle = \frac{1}{4} \neq 0$.

Theorem (Quantum Kraft Inequality, MM & CR 2008)

Let $\mathcal{H} \subset \mathcal{H}_{\{0,1\}^*}$ be a prefix-free Hilbert space, spanned by an orthonormal system $\{|e_i\rangle\}_{i \in I} \subset \mathcal{H}_{\{0,1\}^*}$. Then it holds

$$\sum_{i \in I} 2^{-\ell(e_i)} \leq \sum_{i \in I} 2^{-\bar{\ell}(e_i)} \leq \mathrm{Tr}\left(2^{-\Lambda}\mathbb{P}(\mathcal{H})\right) \leq 1.$$

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Conjecture (Quantum Kraft Inequality and its Converse)

For a closed subspace $\mathcal{H} \subset \mathcal{H}_{\{0,1\}^*}$, there exists a length-preserving unitary $U : \mathcal{H}_{\{0,1\}^*} \to \mathcal{H}_{\{0,1\}^*}$ such that $U\mathcal{H}$ is prefix-free if and only if $\mathrm{Tr}\left(2^{-\Lambda}\mathbb{P}(\mathcal{H})\right) \leq 1$.

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Let $\mathcal{H} \subset \mathcal{H}_{\{0,1\}^*}$ be a prefix-free subspace, and let $|\varphi_1\rangle, |\varphi_2\rangle \in \mathcal{H}$. Then

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Hence there is a unique isometry $U : \mathcal{H} \otimes \mathcal{H}_{\{0,1\}^*} \to \mathcal{H}_{\{0,1\}^*}$ with $U|\varphi\rangle \otimes |\psi\rangle = |\varphi \circ \psi\rangle$ for all $|\varphi\rangle \in \mathcal{H}$ and $|\psi\rangle \in \mathcal{H}_{\{0,1\}^*}$.

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- Asymptotic compression of a source ρ with rate $S(\rho)$.

- If ρ is a density operator on some $\mathcal{H}$, then

$$S(\rho) \leq \min_U \bar{\ell}(U\rho U^*) \leq S(\rho) + 1. \tag{2}$$

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- Lossless Quantum Block Coding: ψ q-information source, ψ_I:=restriction of ψ to local algebra at $I \subset \mathbb{Z}$.

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- This way, $\bar{\ell}$ is minimized losslessly. (But the base length ℓ remains large, and yields some loss if the output is "cut down" to some fixed length e.g. before transmission.)

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- **ArXiv:0804.0022**