Effective Complexity

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- 4. Effective Complexity and Logical Depth

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  then $K(x) \approx n$. $\Rightarrow$ K **is not really a complexity measure!**

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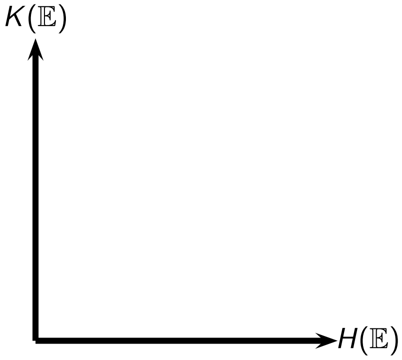

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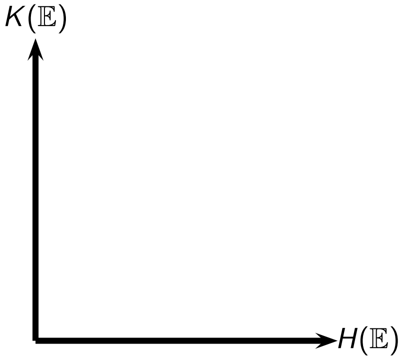

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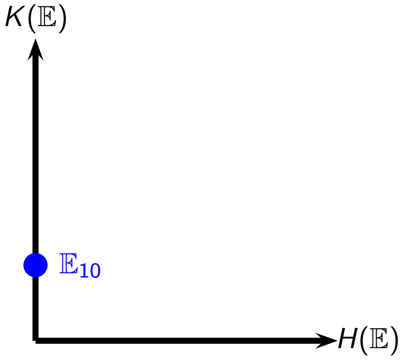

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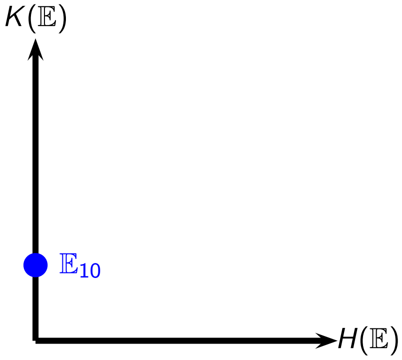

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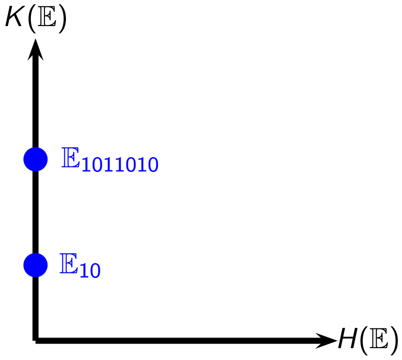

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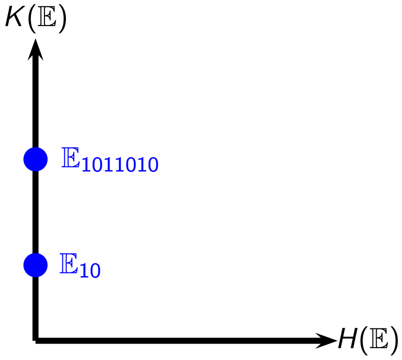

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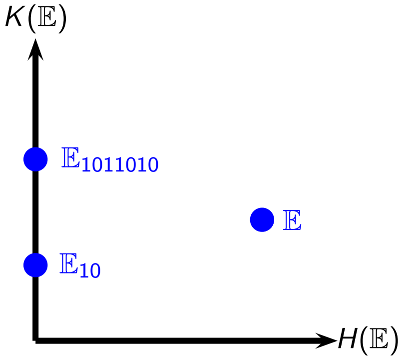

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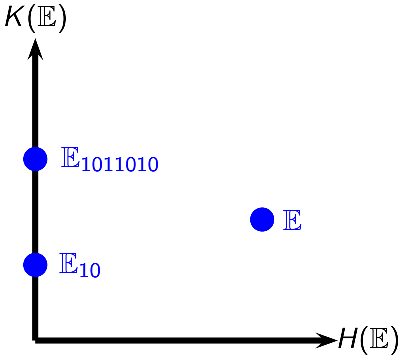

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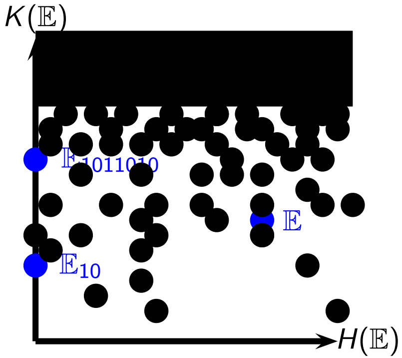

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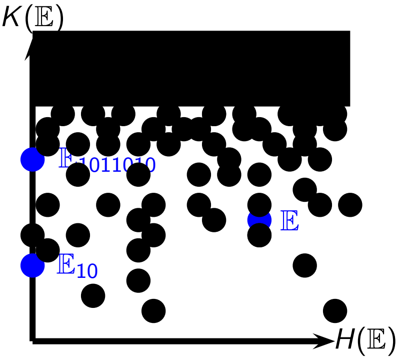

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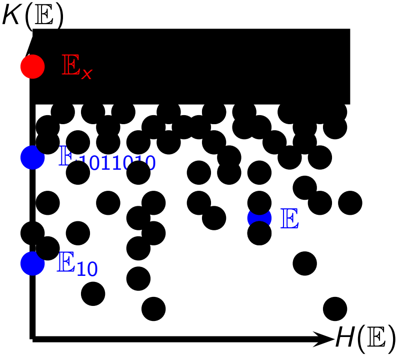

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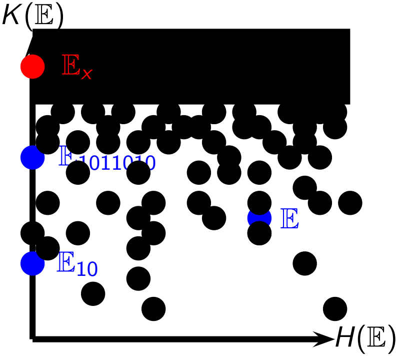

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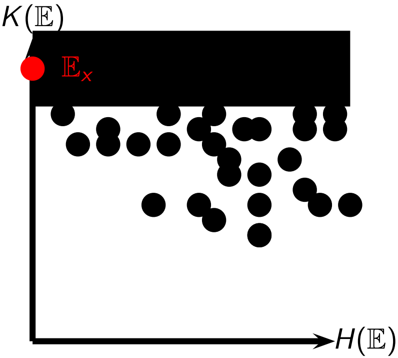

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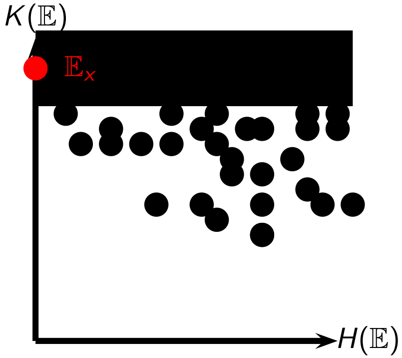
Observation: The remaining ensembles all have total information $\Sigma(\mathbb{E}) := K(\mathbb{E}) + H(\mathbb{E}) \geq K(\mathbb{E}_x) - \mathcal{O}(1) = K(x) - \mathcal{O}(1)$.



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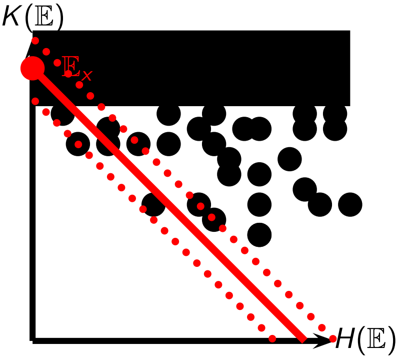

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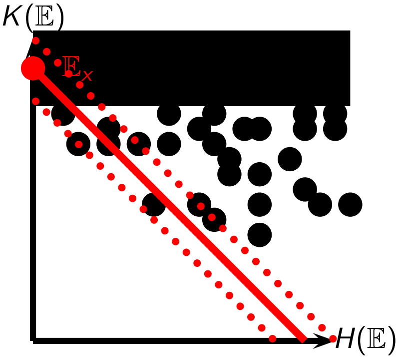

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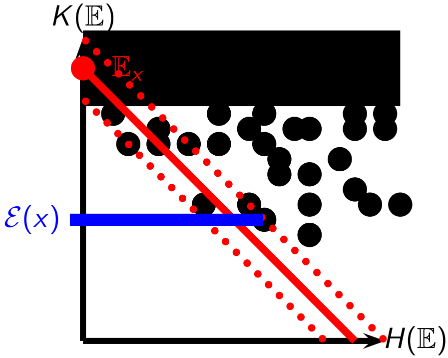

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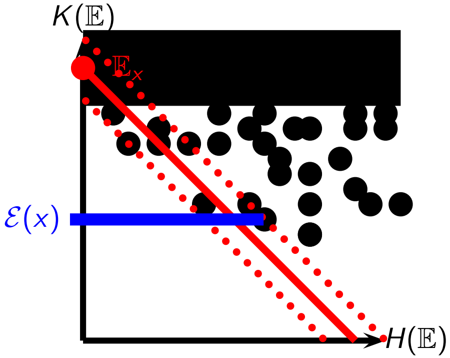

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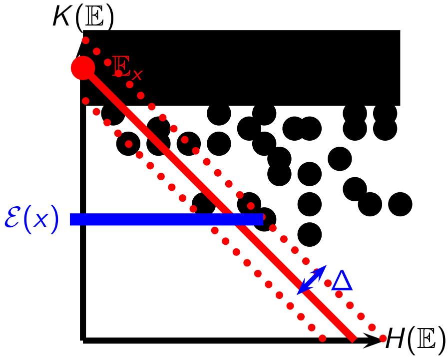

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There is a constant $c > 0$ such that

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- $\mathbb{E}$:=uniform distribution on $\{0,1\}^n$, then
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Random Strings are Effectively Simple:

Theorem (after Gell-Mann and Lloyd 1996)

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Open Problem: Find more intuitive examples, or give a better interpretation! ($\rightarrow$ Gell-Mann, Lloyd: DNA?)

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- The Quark and the Jaguar: Gell-Mann discusses interrelations between algorithmic information content and effective complexity as well as logical depth.
  $\longrightarrow$ relation between depth and complexity?

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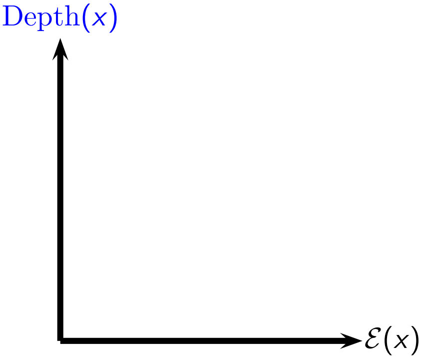

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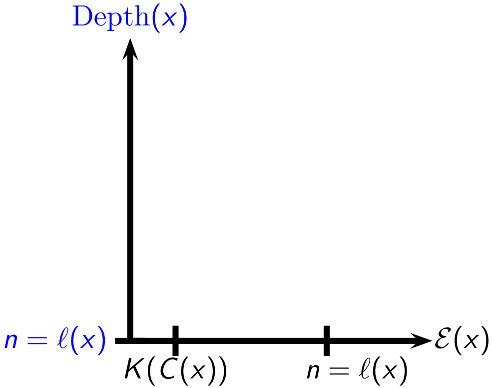

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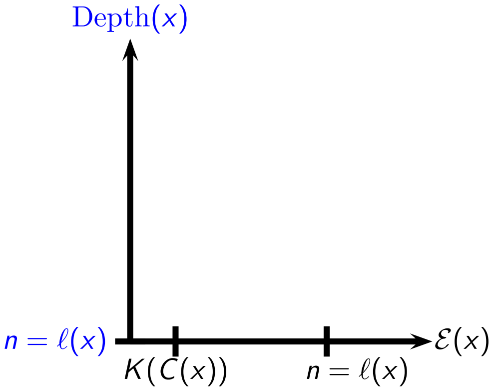

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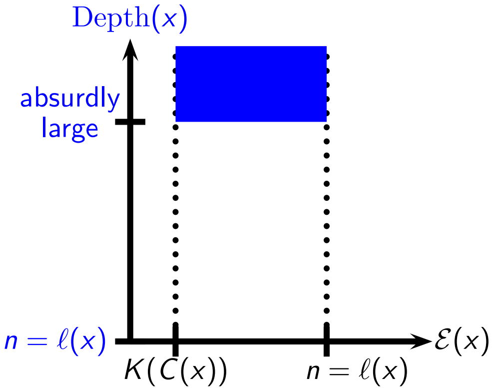

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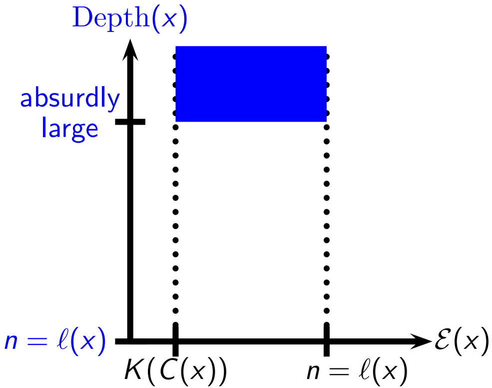

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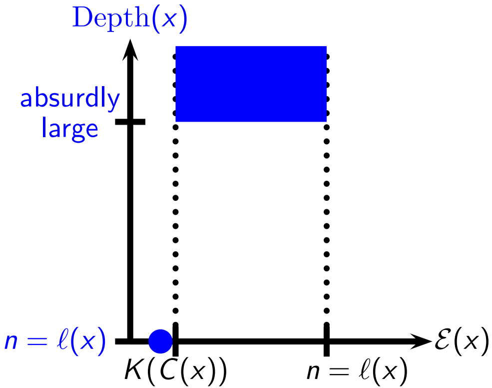

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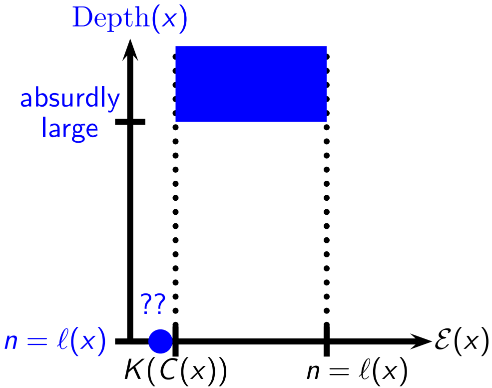

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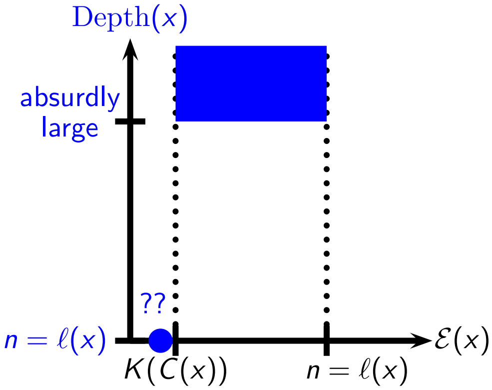

- The "edge of depth" at $K(C(x))$. Happens also for $\mathcal{E}$ with constraints.

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  - shown that there exist effectively complex strings x with $\mathcal{E}(x) \geq n - \mathcal{O}(\log n)$,
  - found a relation between effective complexity and Bennett's logical depth: If $\mathcal{E}(x) > K(C(x)) + \mathcal{O}(1)$, then $\mathrm{Depth}(x)$ is astronomically large. Otherwise, it can be arbitrarily small.