Concentration of measure, typical quantum states with fixed mean energy, and emergence of Gibbs states

Markus Müller[1,3], David Gross[2], Jens Eisert[1]

[1] Physics Department, University of Potsdam
[2] Institute for Mathematical Physics, TU Braunschweig
[3] Institute of Mathematics, TU Berlin

Outline of the talk

1. Introduction to "concentration of measure"
   - high-dimensional spheres: *Lévy's Lemma*
   - consequences for *quantum information*
   - applications in *statistical physics*

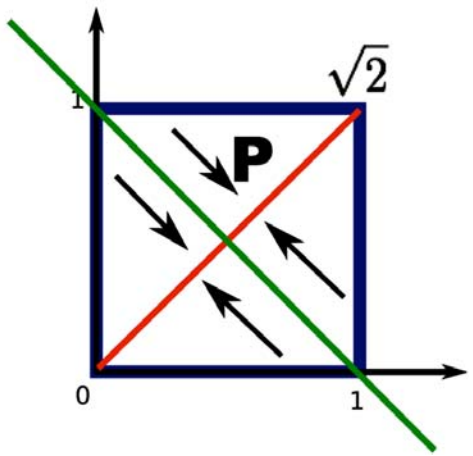
2. Random states with fixed energy
   - concentration on energy submanifolds
   - *proof Idea* + tools
   - how *Gibbs* states emerge

I. Introduction to "Concentration of Measure"

Invitation: the n-Cube

Choose n real numbers $x_1, \ldots, x_n$ uniformly i.i.d. from $[0,1]^n$.

$\longrightarrow$ the mean $\frac{1}{n}\sum_{i=1}^{n} x_i$ concentrates strongly around $\frac{1}{2}$



$$P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x_1+x_2) \\ \frac{1}{2}(x_1+x_2) \end{pmatrix}$$

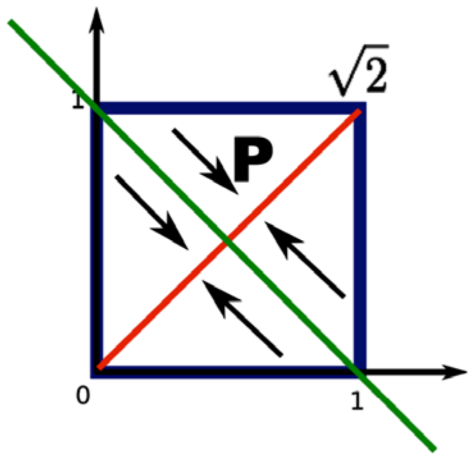
$$\longrightarrow \frac{x_1+x_2}{2} \cdot \sqrt{n=2}$$

I. Introduction to "Concentration of Measure"

Invitation: the n-Cube

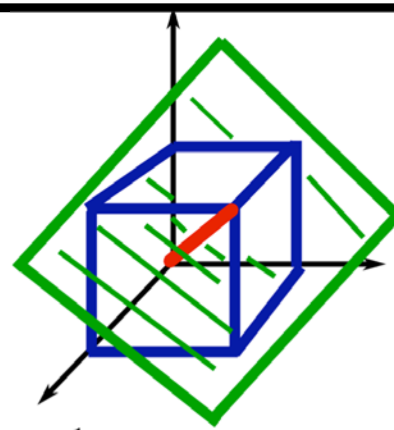
Choose n real numbers $x_1, \dots, x_n$ uniformly i.i.d. from $[0,1]^n$.

$\longrightarrow$ the mean $\frac{1}{n}\sum_{i=1}^{n} x_i$ concentrates strongly around $\frac{1}{2}$



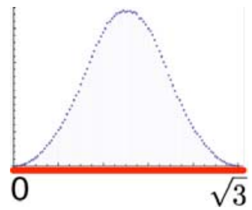
$$P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x_1+x_2) \\ \frac{1}{2}(x_1+x_2) \end{pmatrix}$$

$$\longrightarrow \frac{x_1+x_2}{2} \cdot \sqrt{n=2}$$



$$P(\vec{x}) = \frac{1}{3}(x_1+x_2+x_3)(1,1,1)^T$$

$$\longrightarrow \frac{x_1+x_2+x_3}{3} \cdot \sqrt{n=3}$$

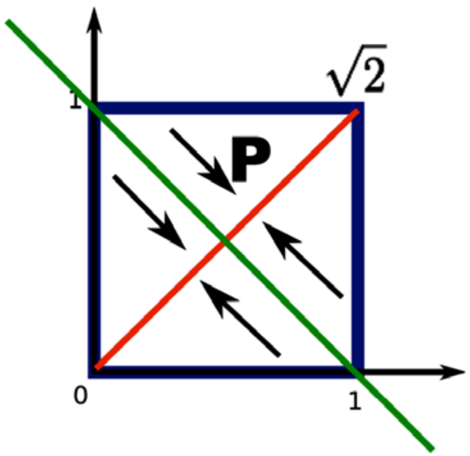


I. Introduction to "Concentration of Measure"

Invitation: the n-Cube

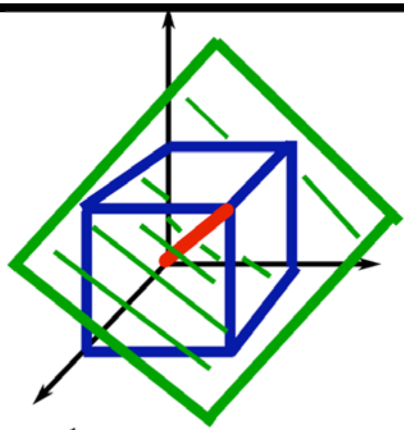
Choose n real numbers $x_1, \ldots, x_n$ uniformly i.i.d. from $[0,1]^n$.

$\longrightarrow$ the mean $\frac{1}{n}\sum_{i=1}^{n} x_i$ concentrates strongly around $\frac{1}{2}$



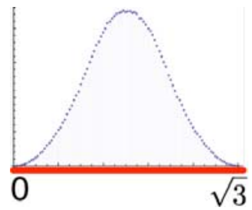
$$P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x_1+x_2) \\ \frac{1}{2}(x_1+x_2) \end{pmatrix}$$

$$\longrightarrow \frac{x_1+x_2}{2} \cdot \sqrt{n=2}$$



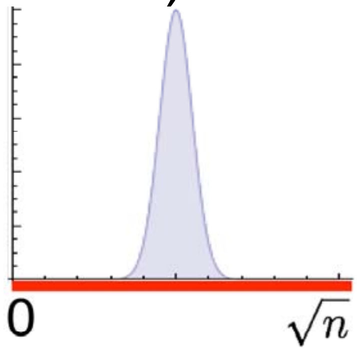
$$P(\vec{x}) = \frac{1}{3}(x_1+x_2+x_3)(1,1,1)^T$$

$$\longrightarrow \frac{x_1+x_2+x_3}{3} \cdot \sqrt{n=3}$$



n-cube for $n \to \infty$:

$\varepsilon\sqrt{n}$ -neighbourhood of hyperplane perp. to diagonal in $[0,1]^n$ contains almost all (uniform) measure

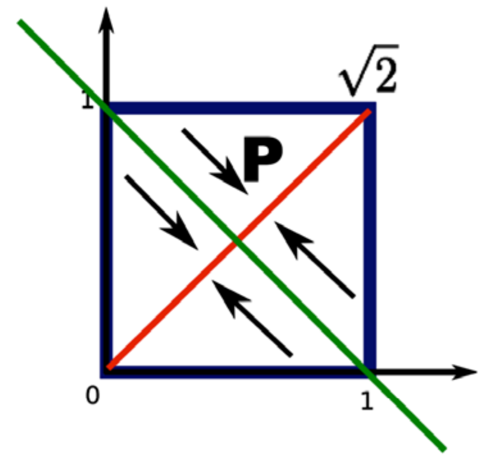


1. Introduction to "Concentration of Measure"

Invitation: the n-Cube

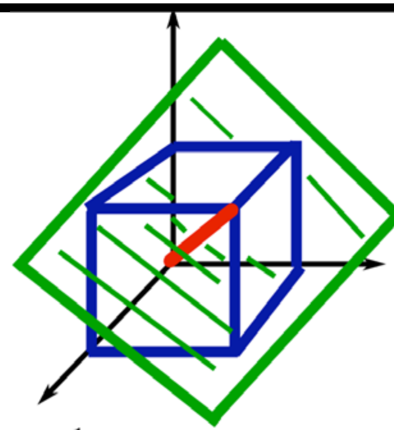
Q: How much is "almost all"?

A: A lot! Hoeffding bound: $\text{Prob}\left(\left|\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{2}\right| \geq \varepsilon\right) \leq 2\exp(-2n^2\varepsilon^2)$



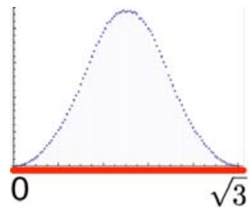
$$P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x_1 + x_2) \\ \frac{1}{2}(x_1 + x_2) \end{pmatrix}$$

$$\longrightarrow \frac{x_1 + x_2}{2} \cdot \sqrt{n = 2}$$



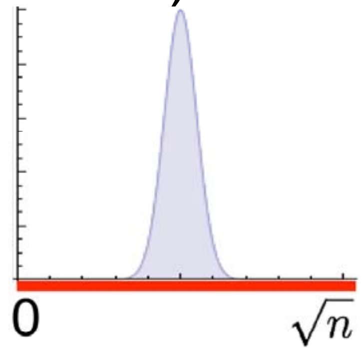
$$P(\vec{x}) = \frac{1}{3}(x_1 + x_2 + x_3)(1,1,1)^T$$

$$\longrightarrow \frac{x_1 + x_2 + x_3}{3} \cdot \sqrt{n = 3}$$



n-cube for $n \to \infty$:

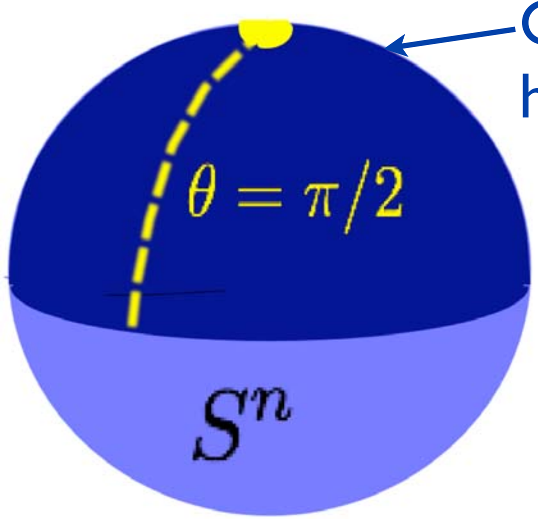
$\varepsilon\sqrt{n}$ -neighbourhood of hyperplane perp. to diagonal in $[0,1]^n$ contains almost all (uniform) measure



1. Introduction to "Concentration of Measure"

Let's get quantum: the n-Sphere

Let μ_n be the normalized (volume) measure on S^n.

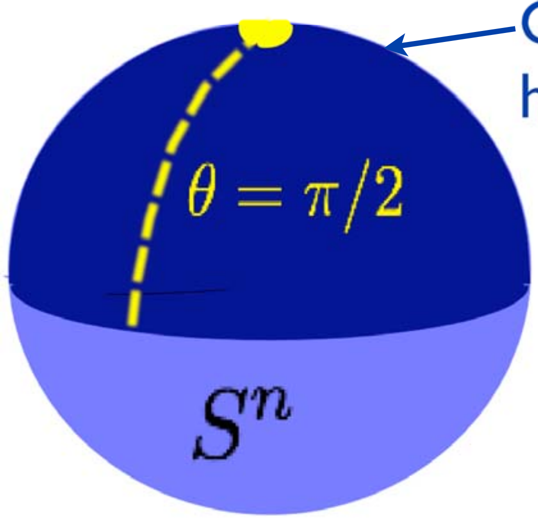


Clearly, a polar cap with $\theta = \pi/2$ has measure $\mu_n = 1/2$.

1. Introduction to "Concentration of Measure"

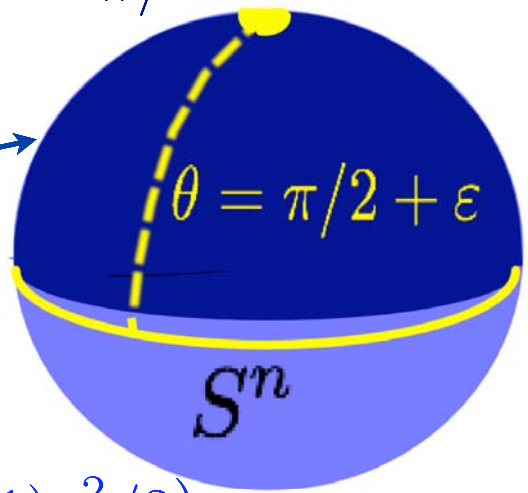
Let's get quantum: the n-Sphere

Let μ_n be the normalized (volume) measure on S^n.



Clearly, a polar cap with $\theta = \pi/2$ has measure $\mu_n = 1/2$.

As soon as the angle exceeds $\pi/2 + \varepsilon$, the cap has almost full measure 1.

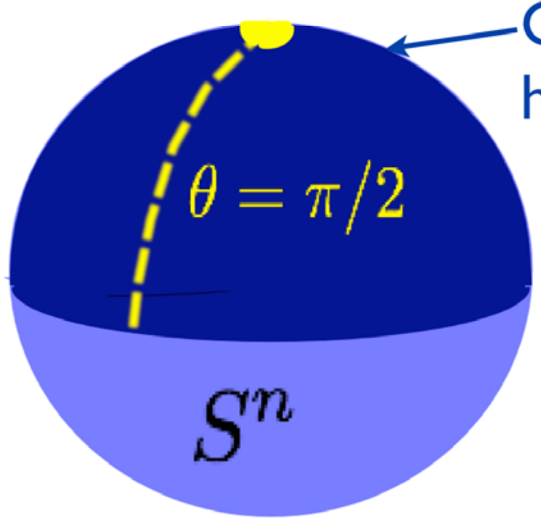


$$\mu_n \geq 1 - \exp\big(- (n-1)\varepsilon^2/2 \big)$$

1. Introduction to "Concentration of Measure"

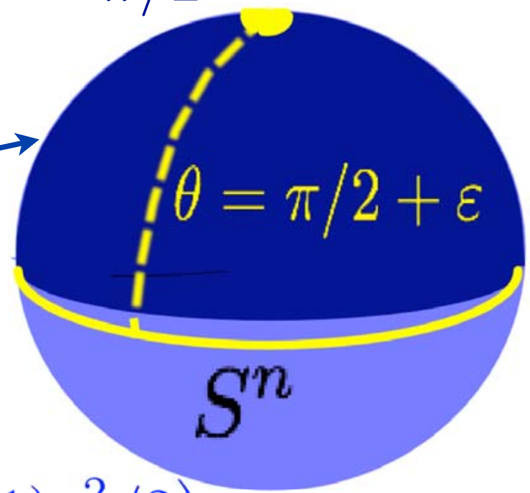
Let's get quantum: the n-Sphere

Let μ_n be the normalized (volume) measure on S^n.

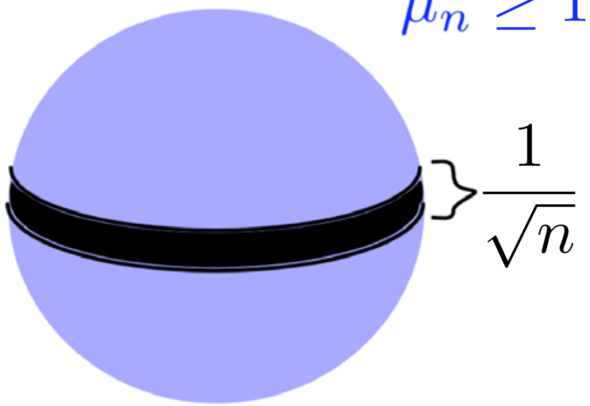


Clearly, a polar cap with $\theta = \pi/2$ has measure $\mu_n = 1/2$.

As soon as the angle exceeds $\pi/2 + \varepsilon$, the cap has almost full measure 1.



$$\mu_n \geq 1 - \exp\big(-(n-1)\varepsilon^2/2\big)$$



Measure is exponentially concentrated around any equator.

1. Introduction to "Concentration of Measure"

Let's get quantum: the n-Sphere

Some consequences:

A. Draw two vectors at random, then they are almost surely almost perpendicular. (also quantum!)

1. Introduction to "Concentration of Measure"

Let's get quantum: the n-Sphere

Some consequences:

A. Draw two vectors at random, then they are almost surely almost perpendicular. (also quantum!)

B. Lévy's Lemma:

Let $f : S^n \to \mathbb{R}$ be Lipschitz with constant η. Then,

$$\mu_n \{|f(x) - \mathbb{E}f| > \varepsilon\} \leq 2 \exp\left(c(n+1)\varepsilon^2/\eta^2\right),$$

where $c = (9\pi^3 \ln 2)^{-1}$.

1. Introduction to "Concentration of Measure"

Let's get quantum: the n-Sphere

Some consequences:

A. Draw two vectors at random, then they are almost surely almost perpendicular. (also quantum!)

B. Lévy's Lemma:

Let $f : S^n \to \mathbb{R}$ be Lipschitz with constant η. Then,

$$\mu_n \{|f(x) - \mathbb{E}f| > \varepsilon\} \leq 2\exp\left(c(n+1)\varepsilon^2/\eta^2\right),$$

where $c = (9\pi^3 \ln 2)^{-1}$.

C. By embedding $\mathbb{C}^n$ into $\mathbb{R}^{2n}$, measure on sphere is the unitarily invariant (Haar) measure on pure quantum states $\longrightarrow$ measure concentration in QInfo

1. Introduction to "Concentration of Measure"

Lévy's Lemma: Consequences for QInfo

Hayden, Leung, Winter, "Aspects of Generic Entanglement" (2004):

If $|\varphi\rangle$ is a random state on $A \otimes B$ with $|B| \geq |A| \geq 3$, then

$$\mathbb{P}\left\{S(\varphi_A) < \log|A| - \alpha - \frac{|A|}{|B|\ln 2}\right\} \leq \exp\left(-\frac{(|A|\cdot|B|-1)c\alpha^2}{(\log|A|)^2}\right),$$

where $c = (8\pi^2 \ln 2)^{-1}$.

I. Introduction to "Concentration of Measure"

Lévy's Lemma: Consequences for QInfo

Hayden, Leung, Winter, "Aspects of Generic Entanglement" (2004):

If $|\varphi\rangle$ is a random state on $A \otimes B$ with $|B| \geq |A| \geq 3$, then

$$\mathbb{P}\left\{ S(\varphi_A) < \log|A| - \alpha - \frac{|A|}{|B|\ln 2} \right\} \leq \exp\left(-\frac{(|A|\cdot|B|-1)c\alpha^2}{(\log|A|)^2} \right),$$

where $c = (8\pi^2 \ln 2)^{-1}$.

⟶ Most states are highly entangled (same for subspaces). Application: Additivity Conjecture

For quantum channels $\mathcal{M}$, let $S^{min}(\mathcal{M}) := \min_\psi S(\mathcal{M}(\psi))$.

Conjecture: it holds

$$S^{min}(\mathcal{M} \otimes \mathcal{N}) = S^{min}(\mathcal{M}) + S^{min}(\mathcal{N}) \quad \text{for all } \mathcal{M}, \mathcal{N}.$$

1. Introduction to "Concentration of Measure"

Lévy's Lemma: Consequences for QInfo

- Proven for several *special classes of channels* (identity, depolarizing,...)

Application: Additivity Conjecture (* 1997, † 2009)

For quantum channels $\mathcal{M}$, let $S^{min}(\mathcal{M}) := \min_\psi S(\mathcal{M}(\psi))$.

Conjecture: it holds

$$S^{min}(\mathcal{M} \otimes \mathcal{N}) = S^{min}(\mathcal{M}) + S^{min}(\mathcal{N}) \quad \text{for all } \mathcal{M}, \mathcal{N}.$$

1. Introduction to "Concentration of Measure"

Lévy's Lemma: Consequences for QInfo

- Proven for several *special classes of channels* (identity, depolarizing,...)
- Since '07: *counterexamples to generalizations* (p-Rényi entropy) via *measure concentration*

Application: Additivity Conjecture (* 1997, † 2009)

For quantum channels $\mathcal{M}$, let $S^{min}(\mathcal{M}) := \min_\psi S(\mathcal{M}(\psi))$.
Conjecture: it holds

$$S^{min}(\mathcal{M} \otimes \mathcal{N}) = S^{min}(\mathcal{M}) + S^{min}(\mathcal{N}) \quad \text{for all } \mathcal{M}, \mathcal{N}.$$

1. Introduction to "Concentration of Measure"

Lévy's Lemma: Consequences for QInfo

- Proven for several *special classes of channels* (identity, depolarizing,...)
- Since '07: *counterexamples to generalizations* (p-Rényi entropy) via *measure concentration*
- Matt Hastings arXiv:0809.3972: **counter-example to additivity conjecture**! Main tool: Concentration of measure.

Application: Additivity Conjecture (* 1997, † 2009)

For quantum channels $\mathcal{M}$, let $S^{min}(\mathcal{M}) := \min_\psi S(\mathcal{M}(\psi))$.
Conjecture: it holds

$$S^{min}(\mathcal{M} \otimes \mathcal{N}) = S^{min}(\mathcal{M}) + S^{min}(\mathcal{N}) \quad \text{for all } \mathcal{M}, \mathcal{N}.$$

1. Introduction to "Concentration of Measure"

Lévy's Lemma: Consequences for QInfo

- Proven for several *special classes of channels* (identity, depolarizing,...)
- Since '07: *counterexamples to generalizations* (p-Rényi entropy) via *measure concentration*
- Matt Hastings arXiv:0809.3972: **counter-example to additivity conjecture**! Main tool: Concentration of measure.

Application: Additivity Conjecture (* 1997, † 2009)

For quantum channels $\mathcal{M}$, let $S^{min}(\mathcal{M}) := \min_\psi S(\mathcal{M}(\psi))$.

Conjecture: it holds

$$S^{min}(\mathcal{M} \otimes \mathcal{N}) = S^{min}(\mathcal{M}) + S^{min}(\mathcal{N}) \quad \text{for all } \mathcal{M}, \mathcal{N}.$$

1. Introduction to "Concentration of Measure"

Lévy's Lemma: Consequences for QInfo

- Proven for several *special classes of channels* (identity, depolarizing,...)
- Since '07: *counterexamples to generalizations* (p-Rényi entropy) via *measure concentration*
- Matt Hastings arXiv:0809.3972: **counter-example to additivity conjecture**! Main tool: Concentration of measure.

Application: Additivity Conjecture (* 1997, † 2009)

For quantum channels $\mathcal{M}$, let $S^{min}(\mathcal{M}) := \min_\psi S(\mathcal{M}(\psi))$.
Conjecture: it holds

$$S^{min}(\mathcal{M} \otimes \mathcal{N}) = S^{min}(\mathcal{M}) + S^{min}(\mathcal{N}) \quad \text{for all } \mathcal{M}, \mathcal{N}.$$

I. Introduction to "Concentration of Measure"

Lévy's Lemma: Consequences for QInfo

- Proven for several *special classes of channels* (identity, depolarizing,...)
- Since '07: *counterexamples to generalizations* (p-Rényi entropy) via *measure concentration*
- Matt Hastings arXiv:0809.3972: **counter-example to additivity conjecture**! Main tool: Concentration of measure.

Main Idea:

- Stinespring dilation: $\mathcal{N}(\rho) = \mathrm{Tr}_{env}\left(\underbrace{U_{\mathcal{N}}\rho U_{\mathcal{N}}^{\dagger}}_{R\subset\mathcal{H}\otimes env}\right)$
- $S^{min}(\mathcal{N})$ large $\Leftrightarrow$ all states in R highly entangled
- Measure concentration: $S^{min}(\mathcal{N})$ generically *very* large $\Rightarrow$ entangled ψ can easily have $S(\mathcal{M}\otimes\mathcal{N}(\psi)) < S^{min}(\mathcal{M}) + S^{min}(\mathcal{N})$.

1. Introduction to "Concentration of Measure"

Applications in Statistical Physics

Motivation:

- Composite system (system+bath $S \otimes B$), initially in state $|\psi\rangle$, with time evolution $\rho_{SB}(t) = e^{-itH}|\psi\rangle\langle\psi|e^{itH}$

1. Introduction to "Concentration of Measure"

Applications in Statistical Physics

Motivation:

- Composite system (system+bath $S \otimes B$), initially in state $|\psi\rangle$, with time evolution $\rho_{SB}(t) = e^{-itH}|\psi\rangle\langle\psi|e^{itH}$
- Key problem: show that system relaxes locally, i.e. that $\rho_S(t) := \mathrm{Tr}_B \rho_{SB}(t)$ is equal to some standard ensemble ρ_S for "most" times *t*, and...

1. Introduction to "Concentration of Measure"

Applications in Statistical Physics

Motivation:

- Composite system (system+bath $S \otimes B$), initially in state $|\psi\rangle$, with time evolution $\rho_{SB}(t) = e^{-itH}|\psi\rangle\langle\psi|e^{itH}$
- Key problem: show that system relaxes locally, i.e. that $\rho_S(t) := \mathrm{Tr}_B \rho_{SB}(t)$ is equal to some standard ensemble ρ_S for "most" times *t*, and...
- ... the standard ensemble should only depend on a few "macroscopic observables": if *M* is the set of states compatible with macroscopic constraints, then

$$\rho_s \simeq \int_{|\psi\rangle \in M} \mathrm{Tr}_B |\psi\rangle\langle\psi| \, d\psi$$

1. Introduction to "Concentration of Measure"

Applications in Statistical Physics

Motivation:

- Composite system (system+bath $S \otimes B$), initially in state $|\psi\rangle$, with time evolution $\rho_{SB}(t) = e^{-itH}|\psi\rangle\langle\psi|e^{itH}$
- Key problem: show that system relaxes locally, i.e. that $\rho_S(t) := \mathrm{Tr}_B \rho_{SB}(t)$ is equal to some standard ensemble ρ_S for "most" times *t*, and...
- ... the standard ensemble should only depend on a few "macroscopic observables": if *M* is the set of states compatible with macroscopic constraints, then

$$\rho_s \simeq \int_{|\psi\rangle \in M} \mathrm{Tr}_B |\psi\rangle\langle\psi| \, d\psi$$

Idea: Show that most states $|\psi\rangle \in M$ are very close to ρ_S via measure concentration ⟶ also plausible for $\rho_S(t)$

I. Introduction to "Concentration of Measure"

Applications in Statistical Physics

Papers with this (or similar) approach:

- S. Popescu, A. J. Short, A. Winter, *Nature Physics* (2006): State of the universe $S \otimes E$ restricted to subspace $R \subset S \otimes E$ then almost all states of S are still very close to maximal mixture $\rho_S = \mathbf{1}/|S|$.

- S. Goldstein, J. Lebowitz, R. Tumulka, N. Zanghi, *PRL* (2006)

- J. Gemmer, G. Mahler, *Phys. Rev. E* (2002)

- N. Linden, S. Popescu, A. J. Short A. Winter, *arXiv:0812.2385*

2. Random States with Fixed Energy

Measure Concentration on Energy Submanifolds

Warning: Work in Progress. 

Setting: We fix some Hamiltonian *H* on $\mathbb{C}^n$, and we draw vector states $|\psi\rangle \in \mathbb{C}^n$ with $\|\psi\| = 1$ randomly under the constraint $\langle\psi|H|\psi\rangle = E.$

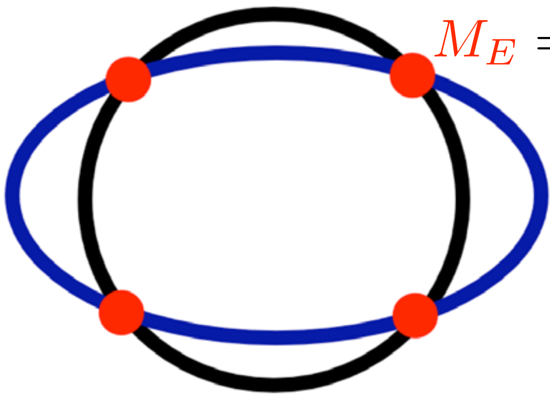
2. Random States with Fixed Energy

Measure Concentration on Energy Submanifolds

Warning: Work in Progress. 

Setting: We fix some Hamiltonian *H* on $\mathbb{C}^n$, and we draw vector states $|\psi\rangle \in \mathbb{C}^n$ with $\|\psi\| = 1$ randomly under the constraint $\langle\psi|H|\psi\rangle = E$.

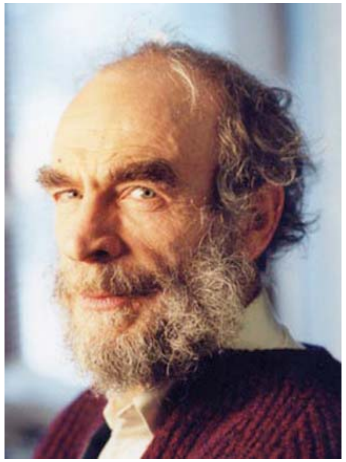
$$M_E = \{|\psi\rangle \in \mathbb{C}^n \mid \|\psi\|^2 = 1, \langle\psi|H|\psi\rangle = E\}$$



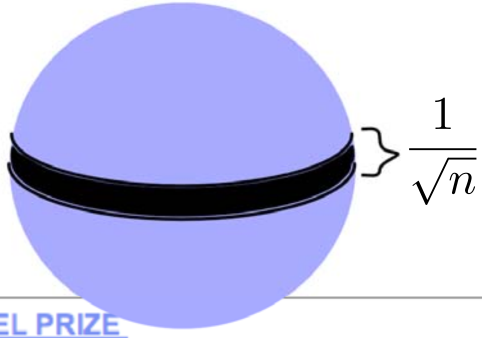
- Can we prove concentration of measure on M_E?
- Do typical states behave like Gibbs states?

2. Random States with Fixed Energy

Measure Concentration on Energy Submanifolds



M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces* (Birkhäuser '01):

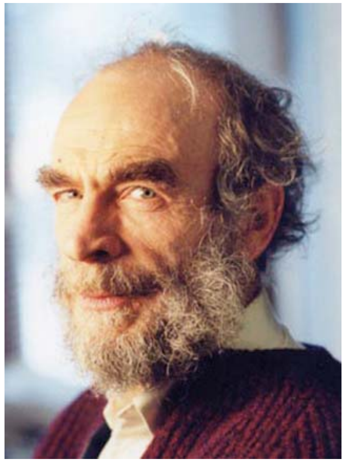


"Observable diameter" of the n-Sphere is about $1/\sqrt{n}$.

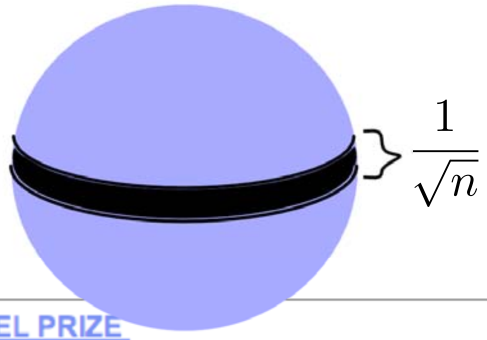
GROMOV AWARDED 2009 ABEL PRIZE
The 2009 Abel Prize is awarded to **Mikhail Leonidovich Gromov**, Permanent
France, "for his revolutionary contributions to geometry." The award is 6 million

2. Random States with Fixed Energy

Measure Concentration on Energy Submanifolds

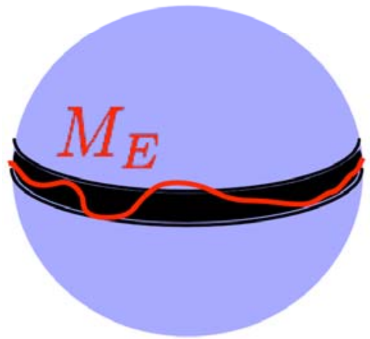


M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces* (Birkhäuser '01):



"Observable diameter" of the n-Sphere is about $1/\sqrt{n}$.

GROMOV AWARDED 2009 ABEL PRIZE
The 2009 Abel Prize is awarded to **Mikhail Leonidovich Gromov**, Permanent
France, "for his revolutionary contributions to geometry." The award is 6 million

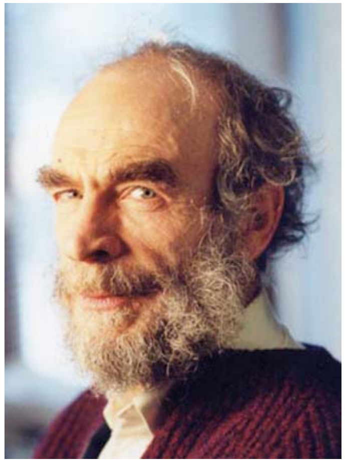


Concentration on (n-1)-dim. Submanifold M_E:

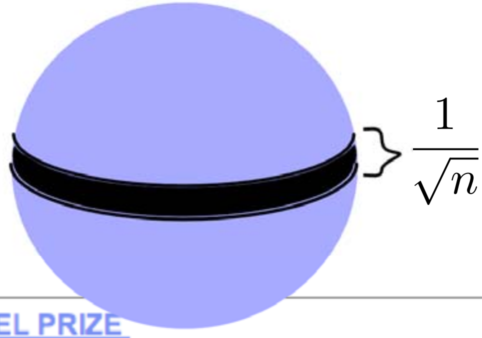
$$\mathrm{ObsDiam}(M_E) \lesssim \frac{1}{\sqrt{n}} \left(\mathrm{const.} + \log \frac{n}{c} \right)$$

2. Random States with Fixed Energy

Measure Concentration on Energy Submanifolds

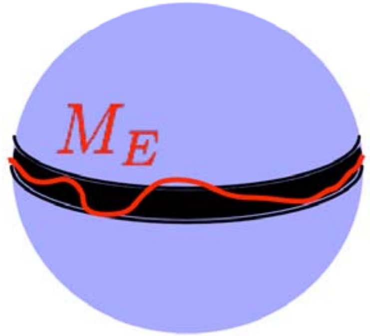


M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces* (Birkhäuser '01):



"Observable diameter" of the n-Sphere is about $1/\sqrt{n}$.

GROMOV AWARDED 2009 ABEL PRIZE
The 2009 Abel Prize is awarded to **Mikhail Leonidovich Gromov**, Permanent
France, "for his revolutionary contributions to geometry." The award is 6 million



Concentration on (n-1)-dim. Submanifold M_E:

$$\mathrm{ObsDiam}(M_E) \lesssim \frac{1}{\sqrt{n}} \left(\mathrm{const.} + \log \frac{n}{c}\right)$$

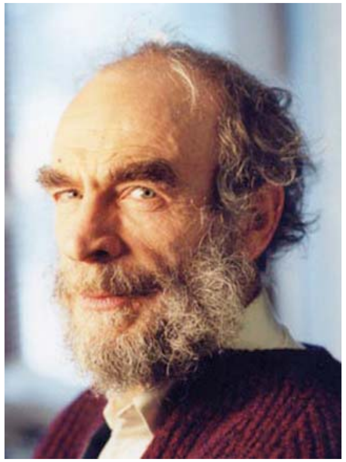
Need to lower-bound (n-1)-measure of M_E:

$$\mu_{n-1}(M_E) \geq c \cdot \mu_{n-1}(S^{n-1})$$

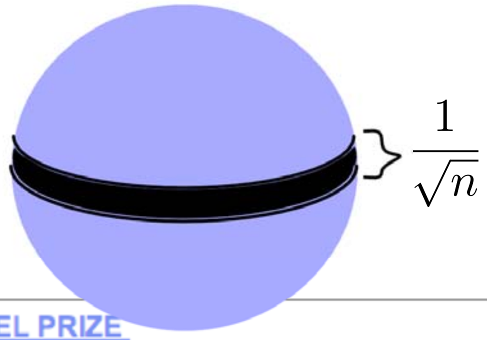
But c is tiny unless $E \approx \mathrm{Tr} H/n$. :-(

2. Random States with Fixed Energy

Measure Concentration on Energy Submanifolds

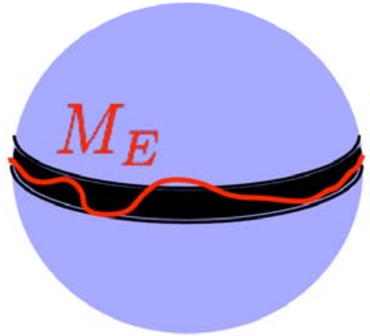


M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces* (Birkhäuser '01):



"Observable diameter" of the n-Sphere is about $1/\sqrt{n}$.

GROMOV AWARDED 2009 ABEL PRIZE
The 2009 Abel Prize is awarded to **Mikhail Leonidovich Gromov**, Permanent
France, "for his revolutionary contributions to geometry." The award is 6 million



Concentration on (n-1)-dim. Submanifold M_E:

$$\mathrm{ObsDiam}(M_E) \lesssim \frac{1}{\sqrt{n}}\left(\mathrm{const.} + \log\frac{n}{c}\right)$$

Need to lower-bound (n-1)-measure of M_E:

$$\mu_{n-1}(M_E) \geq c \cdot \mu_{n-1}(S^{n-1})$$

But c is tiny unless $E \approx \mathrm{Tr}H/n$. :-(→ Works (directly) only for infinite temperature.

2. Random States with Fixed Energy

Measure Concentration on Energy Submanifolds

Way out: look at an *norm-energy-ellipsoid* instead:

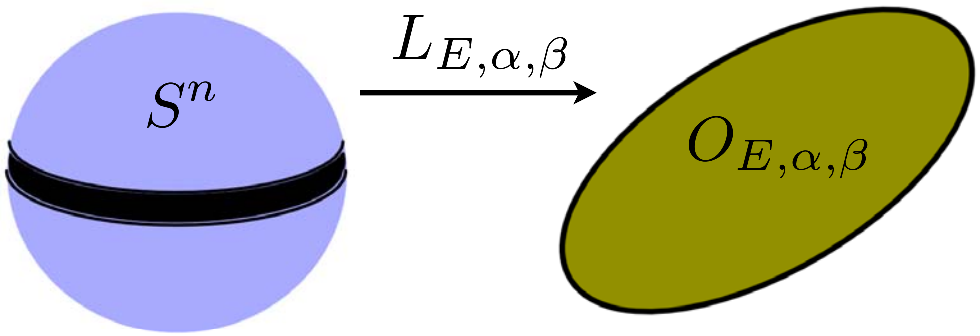
$$O_{E,\alpha,\beta} := \{\psi \in \mathbb{C}^n \mid \alpha\|\psi\|^2 + \beta\langle\psi|H|\psi\rangle = \alpha + \beta E\}$$

2. Random States with Fixed Energy

Measure Concentration on Energy Submanifolds

Way out: look at an *norm-energy-ellipsoid* instead:

$$O_{E,\alpha,\beta} := \{\psi \in \mathbb{C}^n \mid \alpha\|\psi\|^2 + \beta\langle\psi|H|\psi\rangle = \alpha + \beta E\}$$

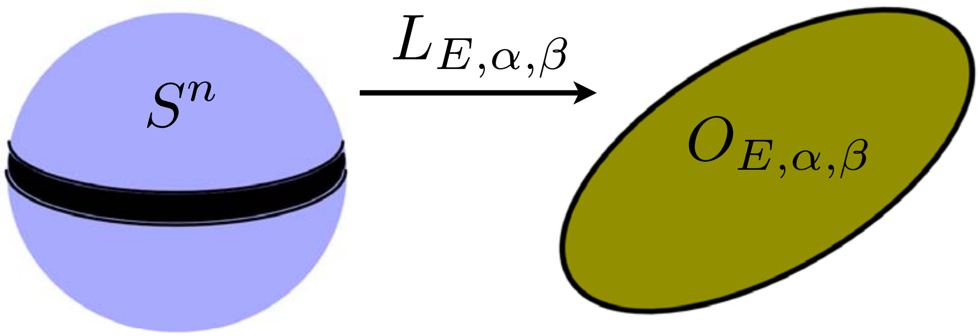


2. Random States with Fixed Energy

Measure Concentration on Energy Submanifolds

Way out: look at an *norm-energy-ellipsoid* instead:

$$O_{E,\alpha,\beta} := \{\psi \in \mathbb{C}^n \mid \alpha\|\psi\|^2 + \beta\langle\psi|H|\psi\rangle = \alpha + \beta E\}$$



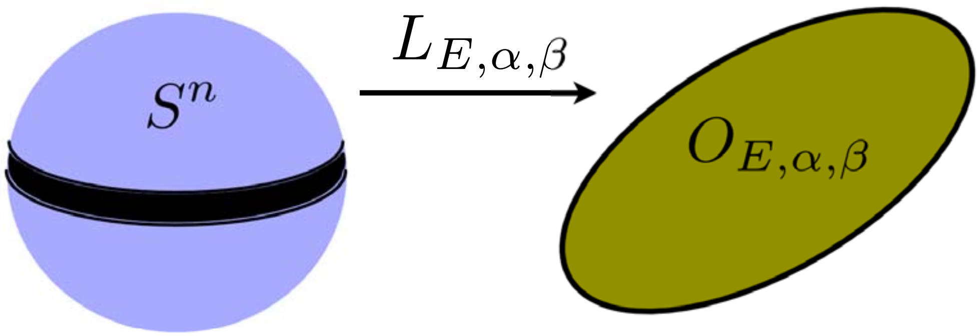
$L_{E,\alpha,\beta}$ pushes forward Haar measure on S^n to measure μ on $O_{E,\alpha,\beta}$. Then, choose α and β such that $\langle\|\psi\|^2\rangle_\mu = 1$ and $\langle\langle\psi|H|\psi\rangle\rangle_\mu = E$.

2. Random States with Fixed Energy

Measure Concentration on Energy Submanifolds

Way out: look at an *norm-energy-ellipsoid* instead:

$$O_{E,\alpha,\beta} := \{\psi \in \mathbb{C}^n \mid \alpha \|\psi\|^2 + \beta\langle\psi|H|\psi\rangle = \alpha + \beta E\}$$



$L_{E,\alpha,\beta}$ pushes forward Haar measure on S^n to measure μ on $O_{E,\alpha,\beta}$.
Then, choose α and β such that $\langle\|\psi\|^2\rangle_\mu = 1$ and $\langle\langle\psi|H|\psi\rangle\rangle_\mu = E$.

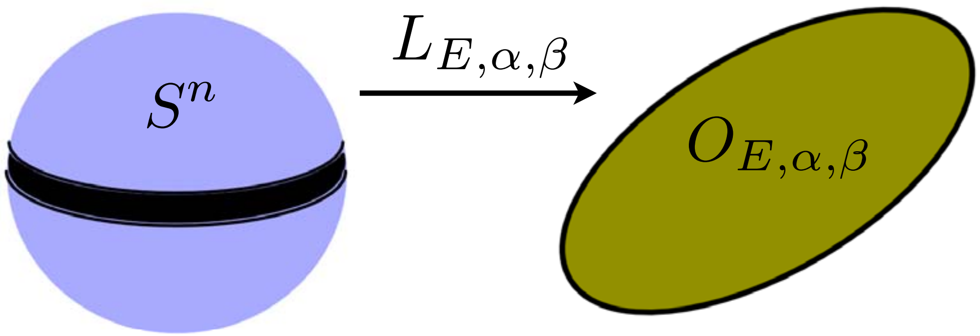
Transport $O_{E,\alpha,\beta} \cap S^n$ back via $L^{-1}_{E,\alpha,\beta}$: this gives "highly probable" submanifold in S^n.
$\rightarrow$ With Gromov, we get measure concentration.

2. Random States with Fixed Energy

Emergence of Gibbs States

Way out: look at an *norm-energy-ellipsoid* instead:

$$O_{E,\alpha,\beta} := \{\psi \in \mathbb{C}^n \mid \alpha\|\psi\|^2 + \beta\langle\psi|H|\psi\rangle = \alpha + \beta E\}$$



$\psi \in \mathbb{C}^{(n+1)/2} \leftrightarrow x \in \mathbb{R}^{n+1}$

$\langle\psi|H|\psi\rangle =: E(x)$

Sampling the n-Sphere:

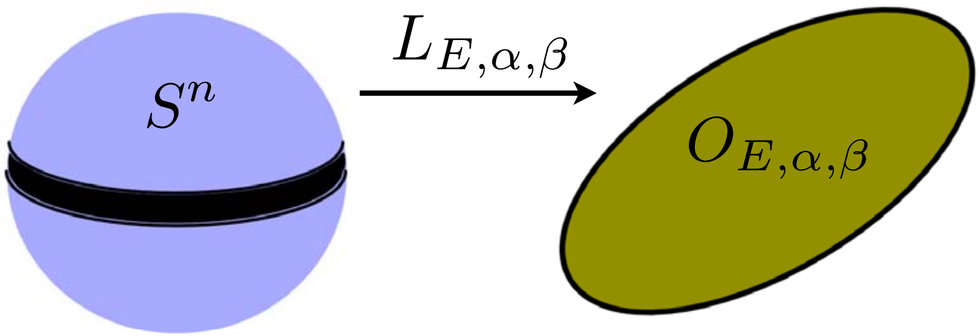
- Choose $x = (x_0, \ldots, x_n) \in \mathbb{R}^n$ with components $d\mu(x_i) \sim e^{-cx_i^2}$.

2. Random States with Fixed Energy

Emergence of Gibbs States

Way out: look at an *norm-energy-ellipsoid* instead:

$$O_{E,\alpha,\beta} := \{\psi \in \mathbb{C}^n \mid \alpha \|\psi\|^2 + \beta \langle\psi|H|\psi\rangle = \alpha + \beta E\}$$



$\psi \in \mathbb{C}^{(n+1)/2} \leftrightarrow x \in \mathbb{R}^{n+1}$

$\langle\psi|H|\psi\rangle =: E(x)$

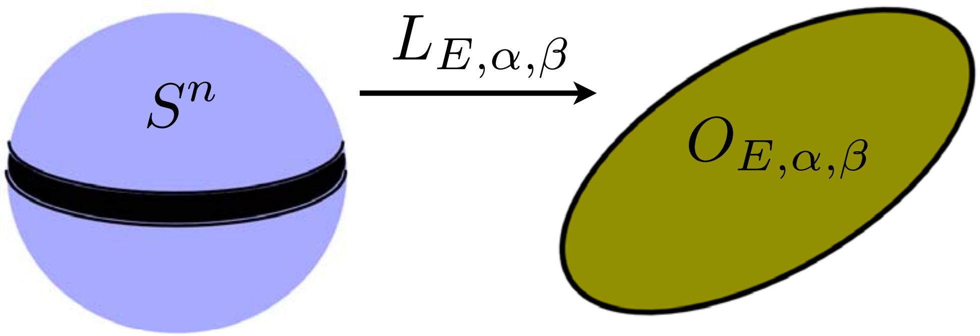
Sampling the n-Sphere:

- Choose $x = (x_0, \ldots, x_n) \in \mathbb{R}^n$ with components $d\mu(x_i) \sim e^{-cx_i^2}$.
- Resulting Gaussian measure $d\mu(x) \sim \prod_i e^{-cx_i^2} = e^{-c\|x\|^2}$.

2. Random States with Fixed Energy
Emergence of Gibbs States

Way out: look at an *norm-energy-ellipsoid* instead:

$$O_{E,\alpha,\beta} := \{\psi \in \mathbb{C}^n \mid \alpha\|\psi\|^2 + \beta\langle\psi|H|\psi\rangle = \alpha + \beta E\}$$



$\psi \in \mathbb{C}^{(n+1)/2} \leftrightarrow x \in \mathbb{R}^{n+1}$

$\langle\psi|H|\psi\rangle =: E(x)$

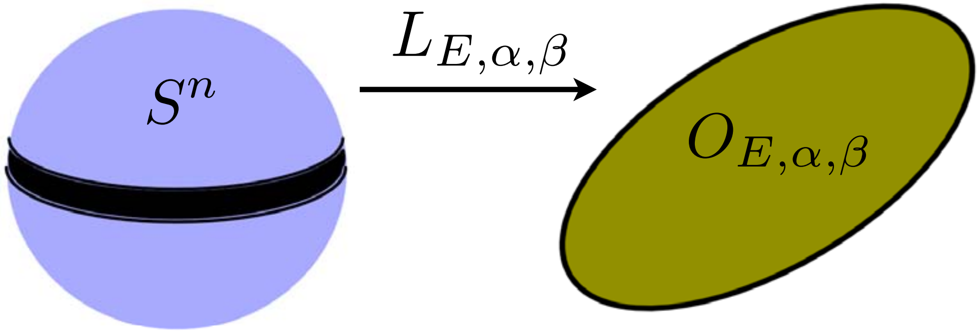
Sampling the n-Sphere:

- Choose $x = (x_0, \ldots, x_n) \in \mathbb{R}^n$ with components $d\mu(x_i) \sim e^{-cx_i^2}$.
- Resulting Gaussian measure $d\mu(x) \sim \prod_i e^{-cx_i^2} = e^{-c\|x\|^2}$.
- For $n \to \infty$, resulting measure very close to μ_n on sphere S^n.

2. Random States with Fixed Energy
Emergence of Gibbs States

Way out: look at an *norm-energy-ellipsoid* instead:

$$O_{E,\alpha,\beta} := \{\psi \in \mathbb{C}^n \mid \alpha\|\psi\|^2 + \beta\langle\psi|H|\psi\rangle = \alpha + \beta E\}$$



$\psi \in \mathbb{C}^{(n+1)/2} \leftrightarrow x \in \mathbb{R}^{n+1}$

$\langle\psi|H|\psi\rangle =: E(x)$

Sampling the n-Sphere:

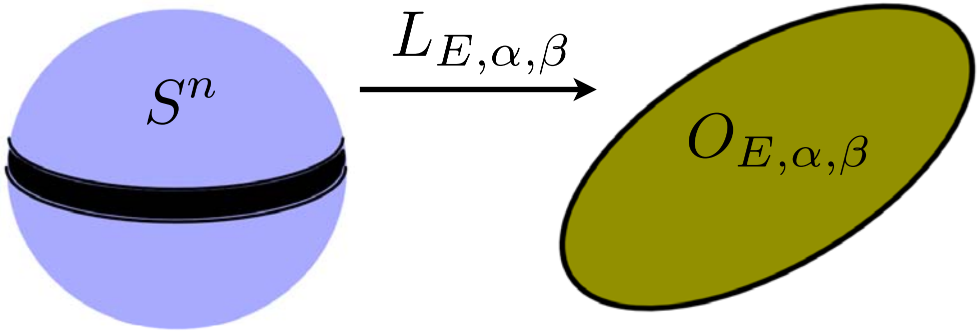
- Choose $x = (x_0, \ldots, x_n) \in \mathbb{R}^n$ with components $d\mu(x_i) \sim e^{-cx_i^2}$.
- Resulting Gaussian measure $d\mu(x) \sim \prod_i e^{-cx_i^2} = e^{-c\|x\|^2}$.
- For $n \to \infty$, resulting measure very close to μ_n on sphere S^n.
- $L_{E,\alpha,\beta}$ pushes it forward: sampling $O_{E,\alpha,\beta}$ via $e^{-\alpha\|x\|^2 - \beta E(x)}$

2. Random States with Fixed Energy

Emergence of Gibbs States

Way out: look at an *norm-energy-ellipsoid* instead:

$$O_{E,\alpha,\beta} := \{\psi \in \mathbb{C}^n \mid \alpha\|\psi\|^2 + \beta\langle\psi|H|\psi\rangle = \alpha + \beta E\}$$



$\psi \in \mathbb{C}^{(n+1)/2} \leftrightarrow x \in \mathbb{R}^{n+1}$

$\langle\psi|H|\psi\rangle =: E(x)$

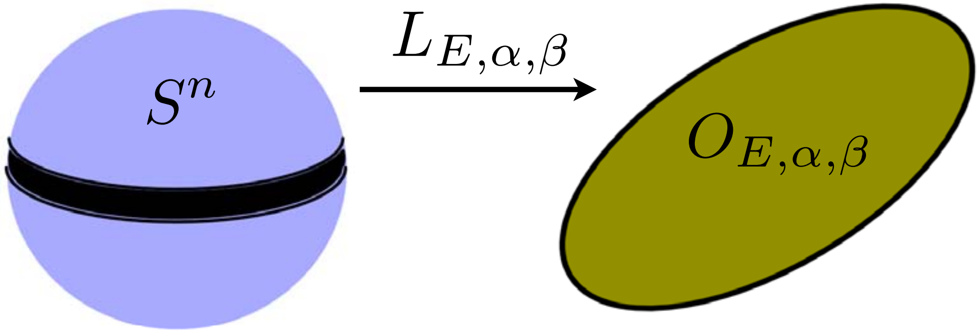
Sampling the n-Sphere:

- Choose $x = (x_0, \ldots, x_n) \in \mathbb{R}^n$ with components $d\mu(x_i) \sim e^{-cx_i^2}$.
- Resulting Gaussian measure $d\mu(x) \sim \prod_i e^{-cx_i^2} = e^{-c\|x\|^2}$.
- For $n \to \infty$, resulting measure very close to μ_n on sphere S^n.
- $L_{E,\alpha,\beta}$ pushes it forward: sampling $O_{E,\alpha,\beta}$ via $e^{-\alpha\|x\|^2 - \beta E(x)}$
- If spectrum of H is unbounded for large *n*, then $\beta \gg \alpha$.

2. Random States with Fixed Energy
Emergence of Gibbs States

Way out: look at an *norm-energy-ellipsoid* instead:

$$O_{E,\alpha,\beta} := \{\psi \in \mathbb{C}^n \mid \alpha\|\psi\|^2 + \beta\langle\psi|H|\psi\rangle = \alpha + \beta E\}$$



$\psi \in \mathbb{C}^{(n+1)/2} \leftrightarrow x \in \mathbb{R}^{n+1}$

$\langle\psi|H|\psi\rangle =: E(x)$

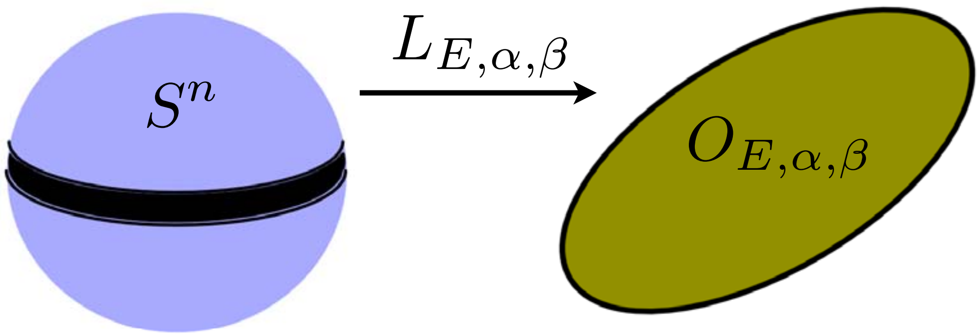
Sampling the n-Sphere:

- Choose $x = (x_0, \ldots, x_n) \in \mathbb{R}^n$ with components $d\mu(x_i) \sim e^{-cx_i^2}$.
- Resulting Gaussian measure $d\mu(x) \sim \prod_i e^{-cx_i^2} = e^{-c\|x\|^2}$.
- For $n \to \infty$, resulting measure very close to μ_n on sphere S^n.
- $L_{E,\alpha,\beta}$ pushes it forward: sampling $O_{E,\alpha,\beta}$ via $e^{-\alpha\|x\|^2 - \beta E(x)}$
- If spectrum of H is unbounded for large *n*, then $\beta \gg \alpha$.
- Thus, measure behaves essentially like $\exp(-\beta E(x))$ (Gibbs).

2. Random States with Fixed Energy
Emergence of Gibbs States

Way out: look at an *norm-energy-ellipsoid* instead:

$$O_{E,\alpha,\beta} := \{\psi \in \mathbb{C}^n \mid \alpha\|\psi\|^2 + \beta\langle\psi|H|\psi\rangle = \alpha + \beta E\}$$



$\psi \in \mathbb{C}^{(n+1)/2} \leftrightarrow x \in \mathbb{R}^{n+1}$

$\langle\psi|H|\psi\rangle =: E(x)$

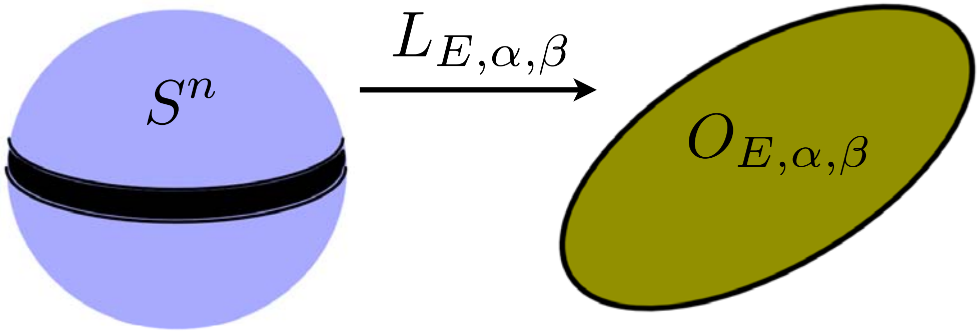
Two more gimmicks:

- If spectrum of *H* does not diverge, then we get exponential concentration, but not on Gibbs state (counterexamples).

2. Random States with Fixed Energy
Emergence of Gibbs States

Way out: look at an *norm-energy-ellipsoid* instead:

$$O_{E,\alpha,\beta} := \{\psi \in \mathbb{C}^n \mid \alpha\|\psi\|^2 + \beta\langle\psi|H|\psi\rangle = \alpha + \beta E\}$$



$\psi \in \mathbb{C}^{(n+1)/2} \leftrightarrow x \in \mathbb{R}^{n+1}$

$\langle\psi|H|\psi\rangle =: E(x)$

Two more gimmicks:

- If spectrum of *H* does not diverge, then we get exponential concentration, but not on Gibbs state (counterexamples).
- Formally efficient algorithm for sampling energy submanifold.

Conclusions

- concentration of measure: n-dimensional unit spheres have "observable diameter" $1/\sqrt{n}$.
- consequences: most quantums states very entangled; idea for app. in statistical physics

Conclusions

- concentration of measure: n-dimensional unit spheres have "observable diameter" $1/\sqrt{n}$.
- consequences: most quantums states very entangled; idea for app. in statistical physics
- what we have shown: measure concentration under energy constraint

$$\langle\psi|H|\psi\rangle = E.$$

- typical states with fixed energy behave as Gibbs states if energy spectrum diverges.

Conclusions

- concentration of measure: n-dimensional unit spheres have "observable diameter" $1/\sqrt{n}$.
- consequences: most quantums states very entangled; idea for app. in statistical physics
- what we have shown: measure concentration under energy constraint

$$\langle\psi|H|\psi\rangle = E.$$

- typical states with fixed energy behave as Gibbs states if energy spectrum diverges.

Literature: Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*
Milman, Schechtman, *Asymptotic Theory of Finite-Dimensional Normed Spaces*
Ledoux, *The Concentration of Measure Phenomenon*
Barvinok, *Measure Concentration (Math 710 Lecture Notes)*