Fractional Sums and Euler-like Identities

Markus Müller
Mathematisches Institut
Technische Universität Berlin
markus@physik.tu-berlin.de

Dierk Schleicher
School of Engineering and Science
International University Bremen
dierk@iu-bremen.de

Abstract

Sums of the form $\sum_{\nu=1}^{x} f(\nu)$ are classically only defined if $x \in \mathbb{N}$.
What if $x \in \mathbb{C}$? $\rightarrow$ fractional sums.

Overview

$$\sum_{\nu=1}^{x} f(\nu), \text{ what if } x \in \mathbb{C}\,? \rightarrow \text{ fractional sums.}$$

- Heuristics,
- formal definition of fractional sums,
- examples and theorems on fractional sums,
- application: classical identities.

Heuristics

Starting point: the continued summation identity

$$\sum_{\nu=a}^{b} f(\nu) + \sum_{\nu=b+1}^{c} f(\nu) = \sum_{\nu=a}^{c} f(\nu) \ .$$

- should remain valid for fractional sums.
  Let $n \in \mathbb{N}$, $x \in \mathbb{C} \ \rightarrow$

$$\sum_{\nu=1}^{n+x} = \sum_{\nu=1}^{n} + \sum_{\nu=n+1}^{n+x} = \sum_{\nu=1}^{x} + \sum_{\nu=x+1}^{x+n} \quad .$$

Remark: $\displaystyle\sum_{\nu=x+1}^{x+n} f(\nu) = \sum_{\nu=1}^{n} f(\nu + x)$ is classical.

- it then follows that

$$\sum_{\nu=1}^{x} f(\nu) = \sum_{\nu=1}^{n} f(\nu) - \sum_{\nu=x+1}^{x+n} f(\nu) + \sum_{\nu=n+1}^{n+x} f(\nu) \ .$$

- Regard $x \in \mathbb{C}$ as fixed. Then, for every $n \in \mathbb{N}$,

$$\sum_{\nu=1}^{x} f(\nu) = \sum_{\nu=1}^{n} \left(f(\nu) - f(\nu + x)\right) + \sum_{\nu=n+1}^{n+x} f(\nu) \ .$$

- Idea: approximate $\sum_{\nu=n+1}^{n+x} f(\nu) = \sum_{\nu=1}^{x} f(\nu + n)$ for $n \to \infty$. Consider two cases:

  – $f(\nu) \to 0$ as $\nu \to \infty$.
    It should also be that $\sum_{\nu=1}^{x} f(\nu + n) \to 0$ as $n \to \infty$. We can define:

$$\sum_{\nu=1}^{x} f(\nu) := \sum_{\nu=1}^{\infty} \left(f(\nu) - f(\nu + x)\right) \ .$$

For $x \in \mathbb{C}$ fixed, $n \in \mathbb{N}$:

$$\sum_{\nu=1}^{x} f(\nu) = \sum_{\nu=1}^{n} (f(\nu) - f(\nu + x)) + \sum_{\nu=n+1}^{n+x} f(\nu) \; .$$

– $f(\nu + n) \approx f(n)$ for every $\nu \in [0, x]$ as $n \to \infty$ ("approximately constant"). Then, it should be that $\sum_{\nu=1}^{x} f(\nu + n) \to x f(n)$ as $n \to \infty$. We can define:

$$\sum_{\nu=1}^{x} f(\nu) := \lim_{n\to\infty} \left(x f(n) + \sum_{\nu=1}^{n} (f(\nu) - f(\nu + x)) \right) \; .$$

  * Example: the logarithmic series. For fixed $x \in \mathbb{C}$, $\ln(\nu + n) \approx \ln(n)$ for every $\nu \in [0, x]$ as $n \to \infty$, so

$$\sum_{\nu=1}^{x} \ln(\nu) := \lim_{n\to\infty} \left(x \ln(n) + \sum_{\nu=1}^{n} \ln\left(\frac{\nu}{\nu + x}\right) \right) \; .$$

– approximation of $f(\nu + n)$ by polynomials.

Formal Definition of Fractional Sums

Definition:
Let $U \in \mathbb{C}$ be an admissible set. If $f : U \to \mathbb{C}$ is approximately polynomial of degree σ with approximating polynomials p_n, we define

$$\sum_{\nu=1}^{x} f(\nu) := \lim_{n\to\infty} \left(\sum_{\nu=n+1}^{n+x} p_n(\nu) + \sum_{\nu=1}^{n} (f(\nu) - f(\nu + x)) \right)$$

for every $x \in U^-$ for which this limit exists.

Definition of Fractional Products

$$\prod_{\nu=1}^{x} f(\nu) := \exp \left(\sum_{\nu=1}^{x} \ln(f(\nu)) \right)$$

whenever the fractional sum exists.

Examples and Theorems on Fractional Sums

Example: The Factorial

$$\to \sum_{\nu=1}^{x} \ln \nu = \lim_{n\to\infty} \left(x \ln n + \sum_{\nu=1}^{n} \ln \left(\frac{\nu}{\nu + x} \right) \right)$$

By definition for every $x \in \mathbb{C}$,

$$\prod_{\nu=1}^{x} \nu = \exp \left(\sum_{\nu=1}^{x} \ln \nu \right) = \lim_{n\to\infty} \left(n^x \prod_{\nu=1}^{n} \frac{\nu}{\nu + x} \right)$$

But this equals $\Gamma(x+1)$ by Gauss' Formula.

Polynomial Sums and the Geometric Series

For every $c \in \mathbb{C}$, $x \in \mathbb{C}$,

$$\sum_{\nu=1}^{x} c = cx \;\;, \;\; \sum_{\nu=1}^{x} \nu = \frac{x(x+1)}{2} \;\;, \;\; \ldots$$

For every $x \in \mathbb{C}$ and $q > 0$, $q \neq 1$,

$$\sum_{\nu=1}^{x} q^{\nu} = (q^x - 1) \frac{q}{q-1} \;.$$

The Binomial Series

For every $c \in \mathbb{C} \setminus \{-1, -2, -3, ...\}$ and $|x| \neq 1$,

$$(1+x)^c = \sum_{\nu=0}^{c} \binom{c}{\nu} x^\nu \ .$$

A Quadrature Formula

For every $x \in \mathbb{C}$,

$$\left[\sum_{\nu=1}^{x} f(\nu)\right] = \sum_{\nu=1}^{x} \left[f^2(\nu) + 2f(\nu)\sum_{k=1}^{\nu-1} f(k)\right] \ ,$$

whenever the left-hand side is an approximate polynomial in x.

Fractional Sums and the Zeta Functions

For every $x \in \mathbb{C} \setminus \{-1, -2, -3, \ldots\}$ and $z \in \mathbb{C} \setminus \{-1\}$,

$$\sum_{\nu=1}^{x} \nu^z = \zeta(-z) - \zeta(-z, x+1) \ ,$$

$$\sum_{\nu=1}^{-\frac{1}{2}} \nu^z = \left(2 - 2^{-z}\right) \zeta(-z) \ .$$

Riemann Zeta Function: $\zeta(s) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^s}$

Hurwitz Zeta Function: $\zeta(s, x) = \sum_{\nu=0}^{\infty} \frac{1}{(\nu + x)^s}$.

Identities involving well-known constants

$$\prod_{n=1}^{-\frac{1}{2}} n^{n^2} = e^{\left(\frac{7\zeta(3)}{16\pi^2}\right)}$$

$$\sum_{n=1}^{-\frac{1}{2}} \ln n \ln(n!) = \frac{\gamma^2}{4} + \frac{\gamma_1}{2} - \frac{\pi^2}{48} + \frac{\ln^2 2}{2} - \frac{\ln^2 \pi}{8} \ .$$

$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + ...$
$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln n\right) = .577...$
$\gamma_1 = \lim_{n\to\infty} \left(\frac{\ln 1}{1} + \frac{\ln 2}{2} + \frac{\ln 3}{3} + ... + \frac{\ln n}{n} - \frac{\ln^2 n}{2}\right)$

$$\sum_{\nu=1}^{-\frac{1}{2}} \frac{1}{\nu} = -2\ln 2 \ .$$

Leonhard Euler,"Dilucidationes in capita postrema mei differentialis de functionibus inexplicabilibus", 2nd ed. Commentatio 613 indicis enestroemiani, Memoires de l'academie des sciences de St.-Petersbourg 4 (1813), 88-119:

Application: Classical Identities

For every $a > 0$,

$$\prod_{n=1}^{\infty} \frac{1}{e} \left(1 + \frac{1}{an}\right)^{an+\frac{1}{2}} = \sqrt{\frac{\Gamma\left(1+\frac{1}{a}\right)}{2\pi}} \times$$

$$\times \exp\left[\frac{1}{2}\left(1+\frac{1}{a}\right) - a\zeta'\left(-1, 1+\frac{1}{a}\right) + a\zeta'(-1)\right] \ .$$

The case $a = 3$ is due to Gosper.

$$\sum_{\nu=1}^{\infty} (-1)^{\nu+1} \psi_2\left(\frac{\nu}{10} + \frac{1}{2}\right)$$

$$= -7\zeta(3) + \frac{50}{3}\pi^2 - \frac{36}{5}\sqrt{5 - 2\sqrt{5}}\pi^3 \ .$$

Polygamma function: $\psi_2(z) = \frac{d^3}{dz^3} \ln \Gamma(z)$.

Speculation / Outlook

Gosper, Ismail, Zhang: "On some strange summation formulas". They give:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\sqrt{b^2+\pi^2 n^2} = \frac{\pi^2}{4}\left(\frac{\sin b}{b} - \frac{\cos b}{3}\right) .$$

"Proof" by fractional sums: Why does this work here?

$$\frac{1}{4}\sum_{n=1}^{-\frac{1}{2}} \frac{1}{n^2}\cos\sqrt{b^2+4\pi^2 n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos\sqrt{b^2+\pi^2 n^2}$$

$$= \frac{1}{4}\sum_{n=1}^{-\frac{1}{2}}\left(\frac{\cos b}{n^2} - \frac{2\pi^2 \sin b}{b} + c_2 n^2 + c_4 n^4 + \ldots\right)$$

$$= \frac{1}{4}\left[\cos b \sum_{n=1}^{-\frac{1}{2}} \frac{1}{n^2} - \frac{2\pi^2\sin b}{b}\sum_{n=1}^{-\frac{1}{2}} 1 + c_2 \sum_{n=1}^{-\frac{1}{2}} n^2 + \ldots\right]$$

Remember that $\sum_{n=1}^{-\frac{1}{2}} n^z = \left(2-2^{-z}\right)\zeta(-z)$, so

$$\sum_{n=1}^{-\frac{1}{2}} \frac{1}{n^2} = -\frac{\pi^2}{3} \, , \sum_{n=1}^{-\frac{1}{2}} 1 = -\frac{1}{2} \, , \sum_{n=1}^{-\frac{1}{2}} n^{2k} = 0 \, .$$

$$= \frac{1}{4}\cos b\left(-\frac{\pi^2}{3}\right) - \frac{2\pi^2 \sin b}{4b}\left(-\frac{1}{2}\right) = \frac{\pi^2}{4}\left(\frac{\sin b}{b} - \frac{\cos b}{3}\right) .$$