Concentration of measure and the mean energy ensemble

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see also **arXiv:1003.4982**

Outline of the talk

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1. Motivation from statistical mechanics
    - Problem: single instances vs. ensembles?
    - Concentration of measure

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2. Typicality in mean energy ensemble
   - Main result: Concentration of measure
   - Typical reduced density matrix
   - No concentration in Ising model

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   - Main result: Concentration of measure
   - Typical reduced density matrix
   - No concentration in Ising model

3. Conclusions

1. Motivation from statistical mechanics
Foundational questions on statistical physics

Two kinds of missing information:

- Observer's lack of knowledge: knows only volume, temperature, ...
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"Postulate of equal apriori probabilities":

=

=...

Why does it work?

1. Motivation from statistical mechanics
What about ergodicity?

Idea: Time evolution explores all accessible phase space uniformly.

Problems:

- Proven only for some special systems.
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Is there another justification?

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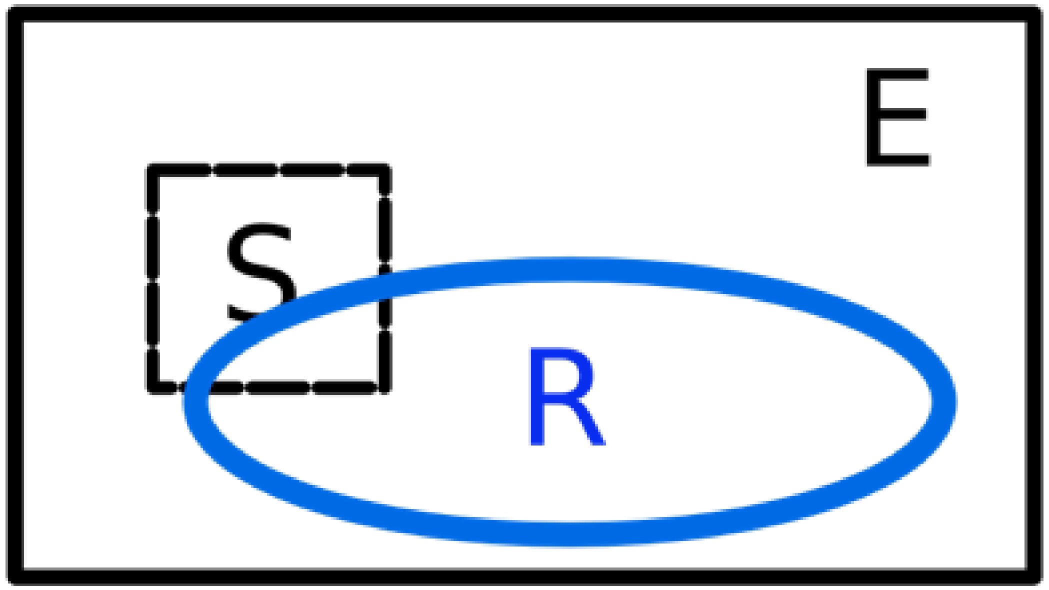
S. Popescu, A. J. Short, A. Winter, Nature Physics 2(11), 2006

$$\mathcal{H}_R \subset \mathcal{H}_S \otimes \mathcal{H}_E$$



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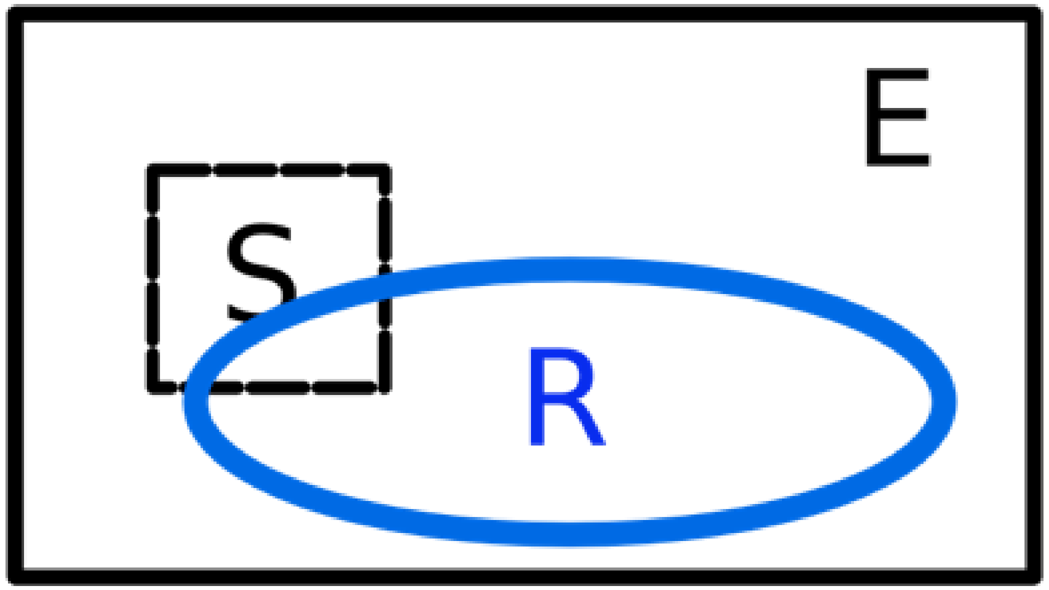

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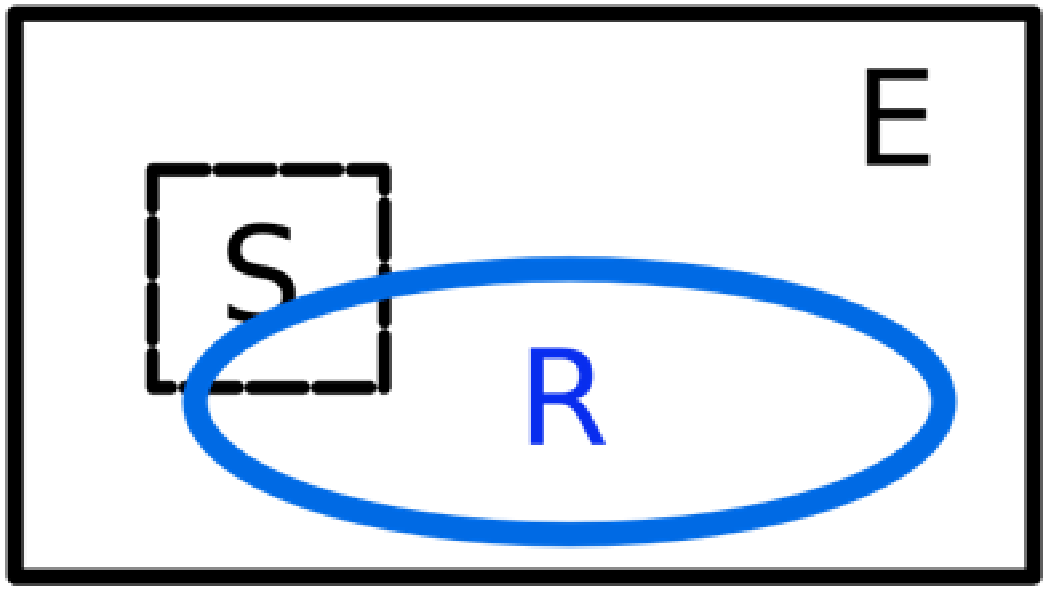

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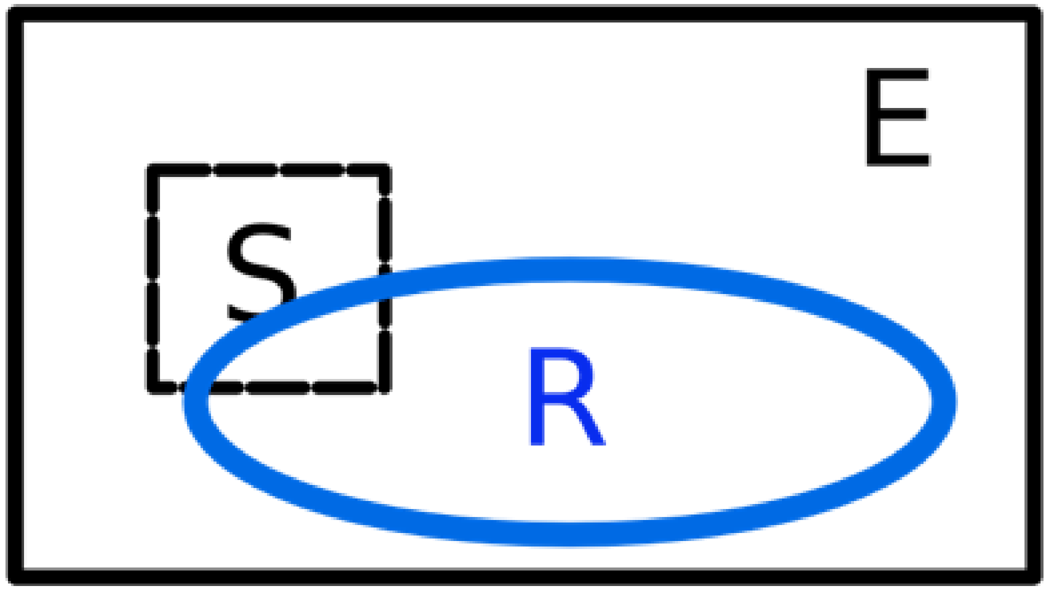

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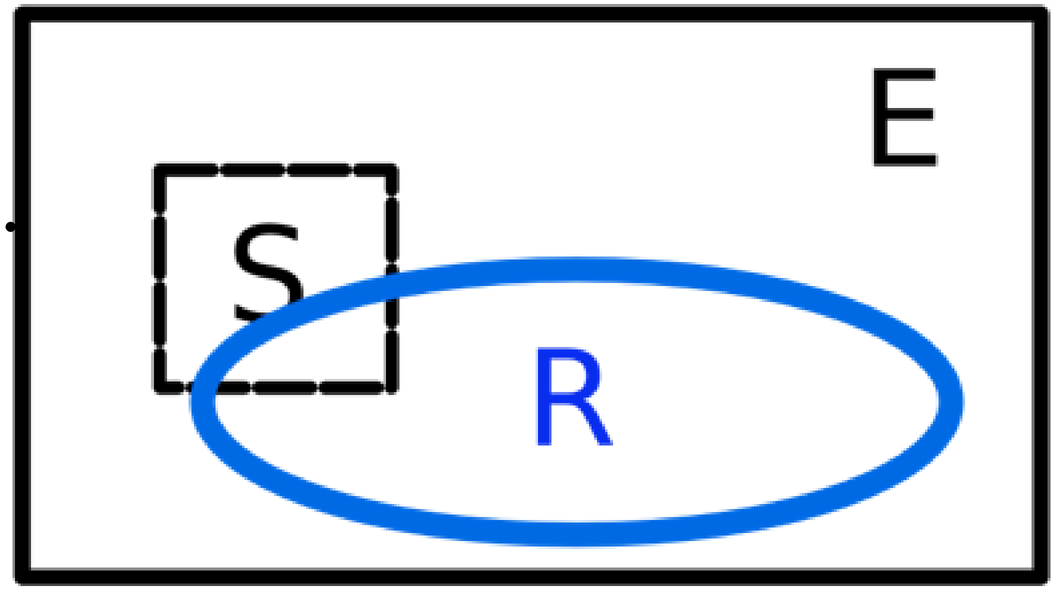

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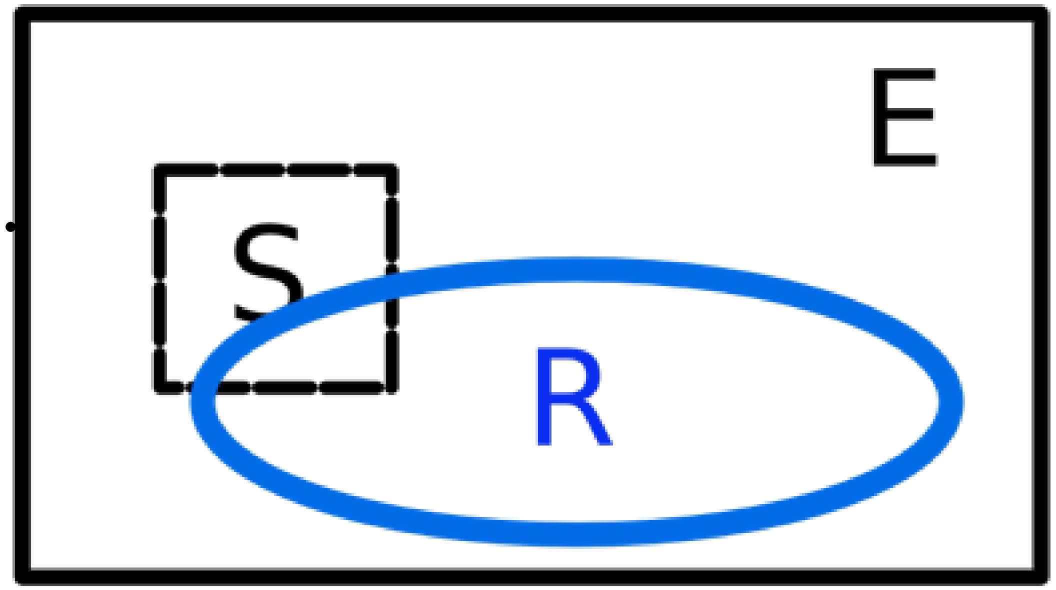

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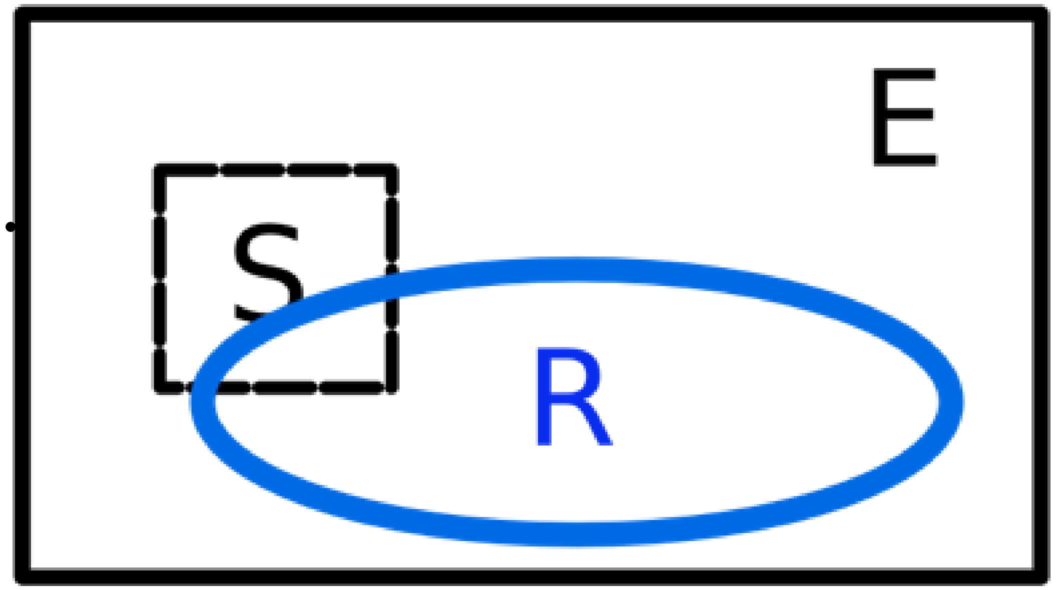

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Theorem (Concentration of measure): Draw $|\psi\rangle \in \mathcal{H}_R$ randomly acc. to unitarily invariant measure. Then,

$$\mathrm{Prob}\left[\|\rho_S - \Omega_S\|_1 \geq \varepsilon + \frac{d_S}{\sqrt{d_R}}\right] \leq 2\exp\left(-C d_R \varepsilon^2\right),$$

where $C = 1/18\pi^3$, $d_R = \dim \mathcal{H}_R$, $d_S = \dim \mathcal{H}_S$, $\Omega_S = \mathrm{Tr}_E\left(\mathbf{1}_S/d_S\right)$.

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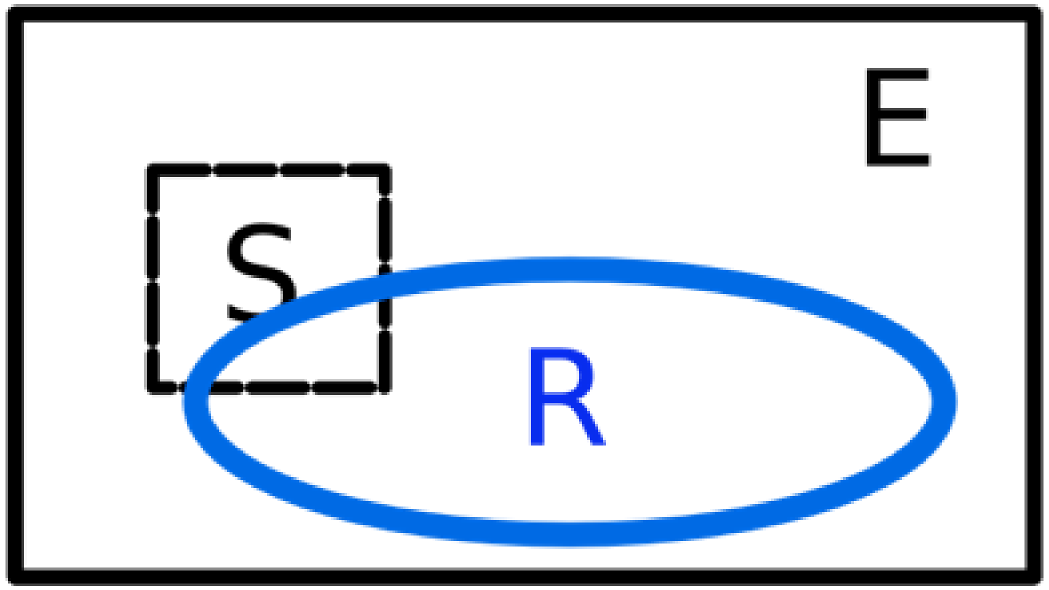

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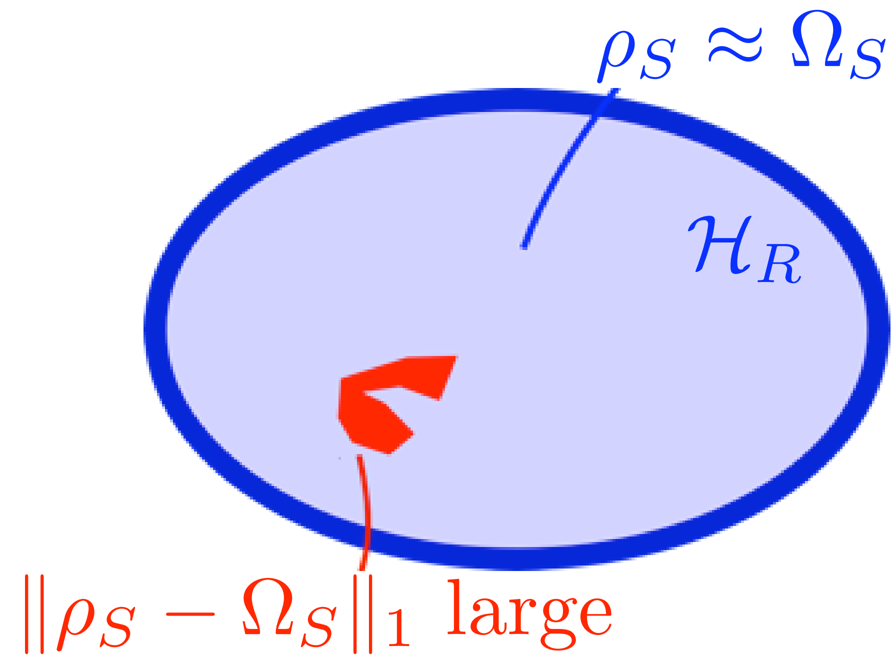

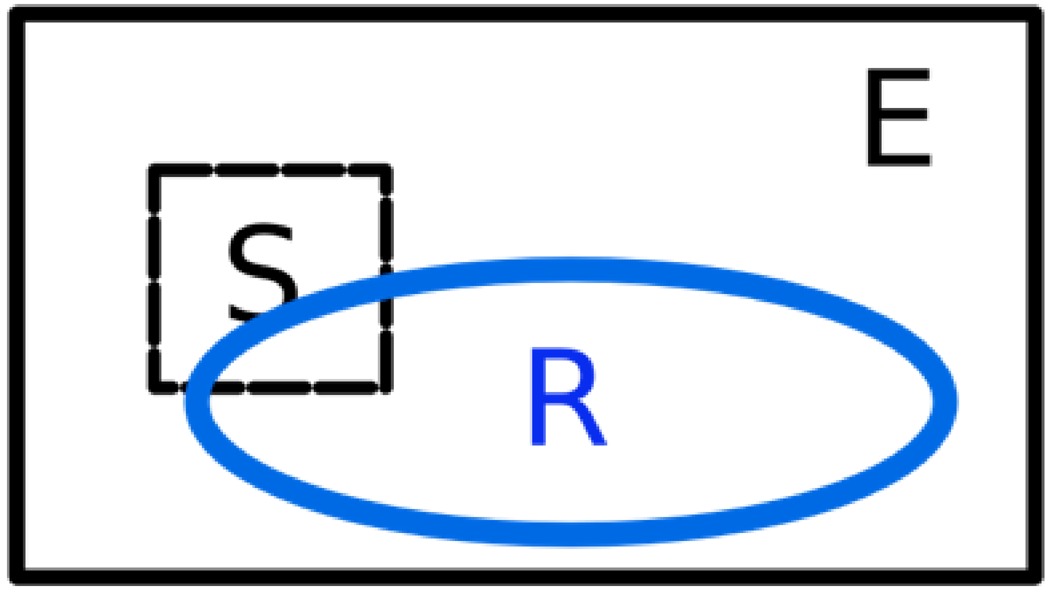

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1. Motivation from statistical mechanics

The perfect coffee machine

$|\psi\rangle \Bigg\{$ 

$n = 1$

$|\psi\rangle \Bigg\{$ 

$n = 2$

$|\psi\rangle \Bigg\{$ 

$n = 3$

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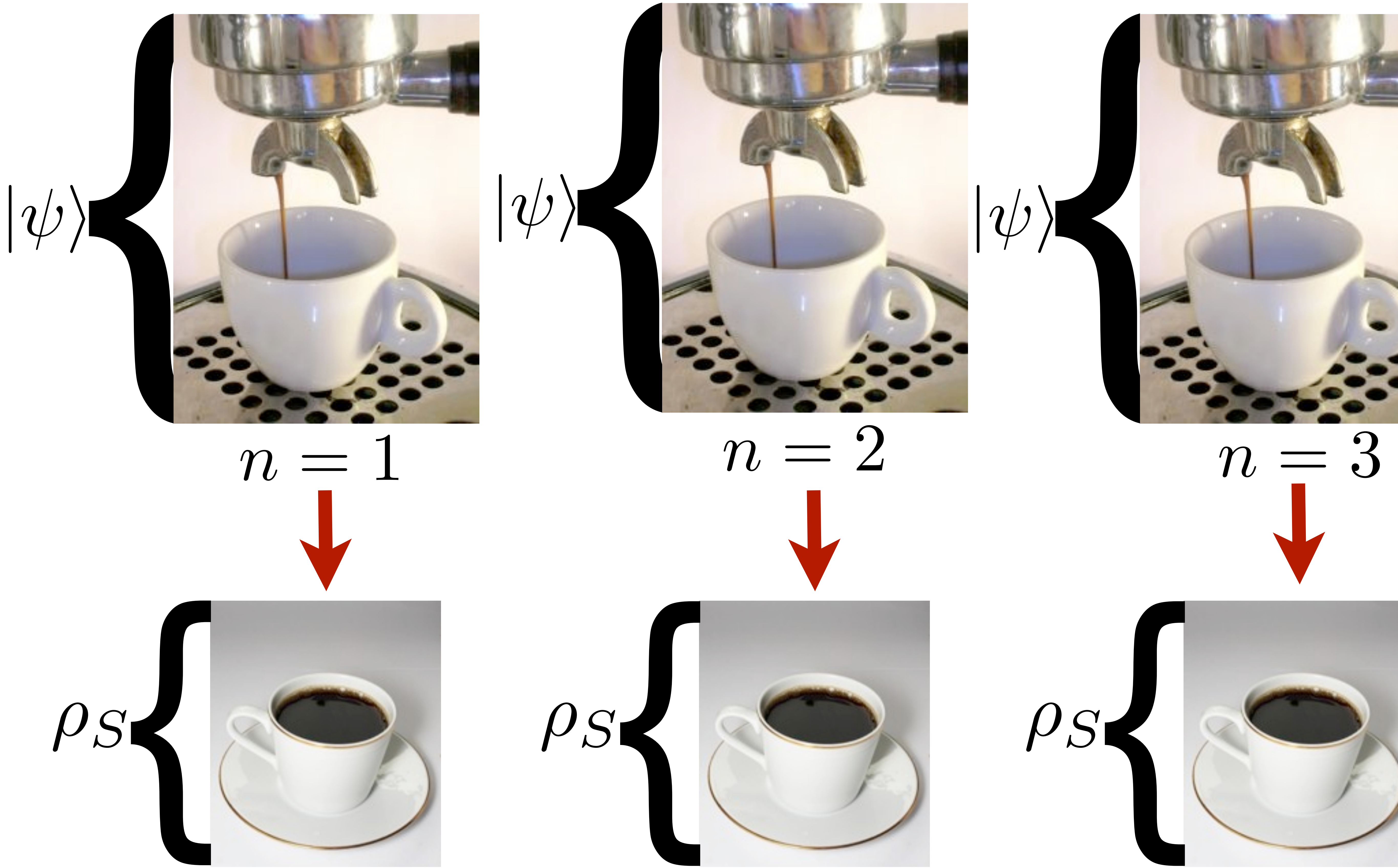

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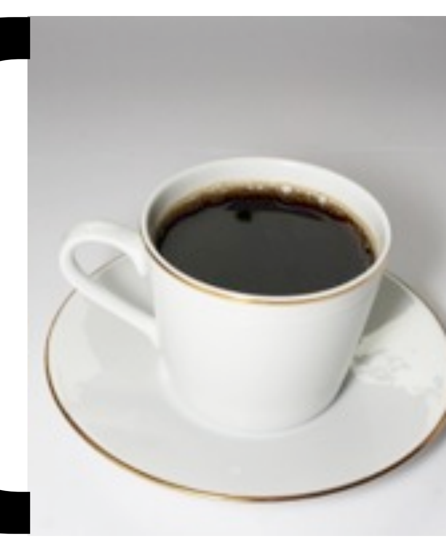
ρ_S 

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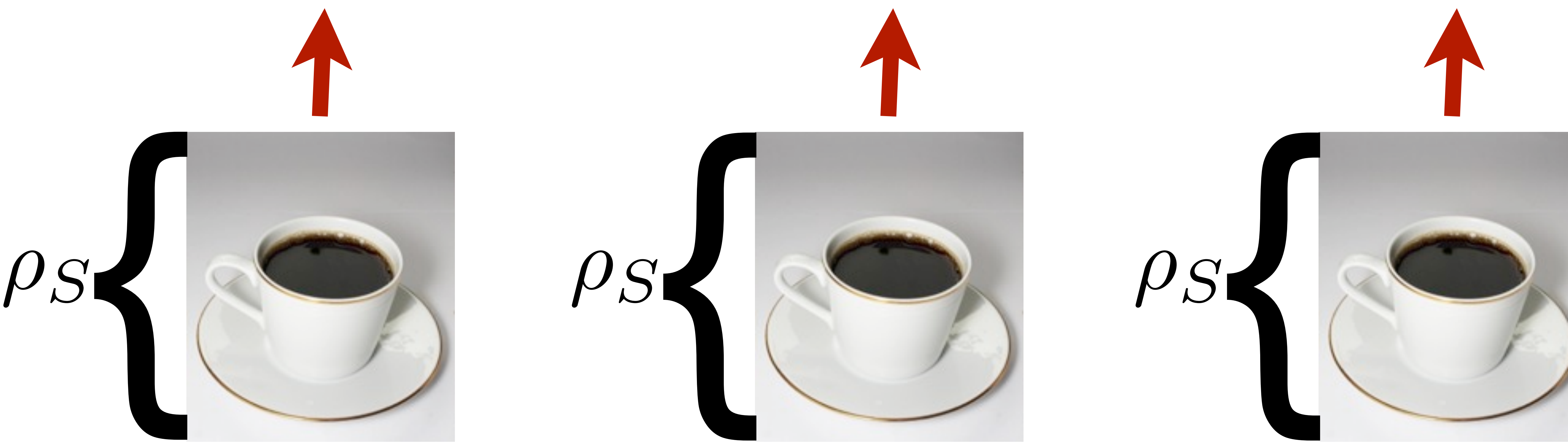

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The perfect coffee machine

Reveals ρ_S. But $\rho_S \approx \Omega_S$ (microcanonical ensemble) for "almost all" $|\psi\rangle \in \mathcal{H}_R$.
Hence, almost all coffee machines (compatible with restrictions) prepare the microcanonical ensemble.

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ρ_S 

ρ_S 

ρ_S 

1. Motivation from statistical mechanics

Form of the reduced density matrix

- Exact form of Ω_S is not given by Popescu et al. (generality!).

- Goldstein, Lebowitz, Tumulka, Zanghi, PRL **96** (2006):
no interaction $H = H_S + H_{env}$, fixed energy E,
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What if the constraint is *not* given by a *subspace*?

2. Typicality in mean energy ensemble

Going beyond subspaces

- Observers may have knowledge on systems that is different from "*being element of a subspace*".
- Example: given Hamiltonian H, the energy expectation value $\langle\psi|H|\psi\rangle = E$ might be known instead.

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$$M_E = \{|\psi\rangle \in \mathbb{C}^n \mid \langle\psi|H|\psi\rangle = E, \ \ \|\psi\| = 1\}$$

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Goal of our work:

- Prove typicality (=measure concentration) for *m*.e.e.,
- analyze its role in quantum statistical mechanics.

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Our result

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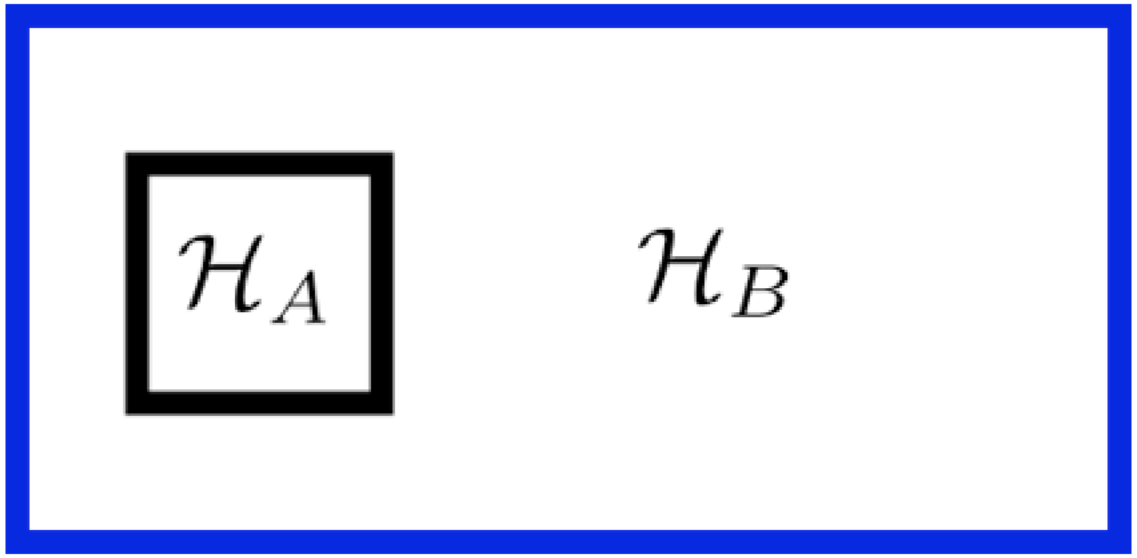
$$\dim \mathcal{H}_A = 3, \;\; \dim \mathcal{H}_B = n$$



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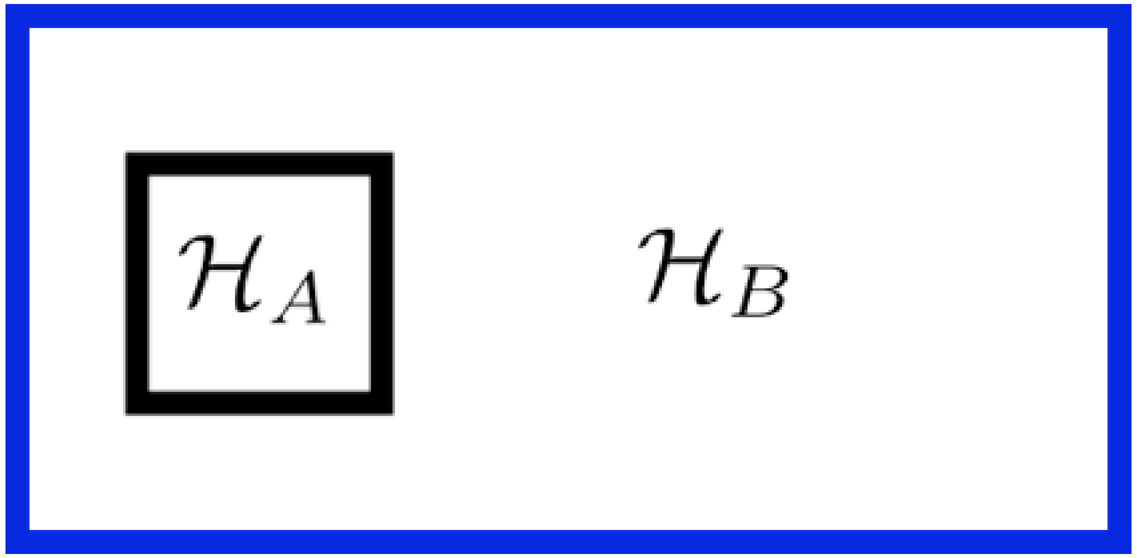

Hamiltonian $H = H_A + H_B$

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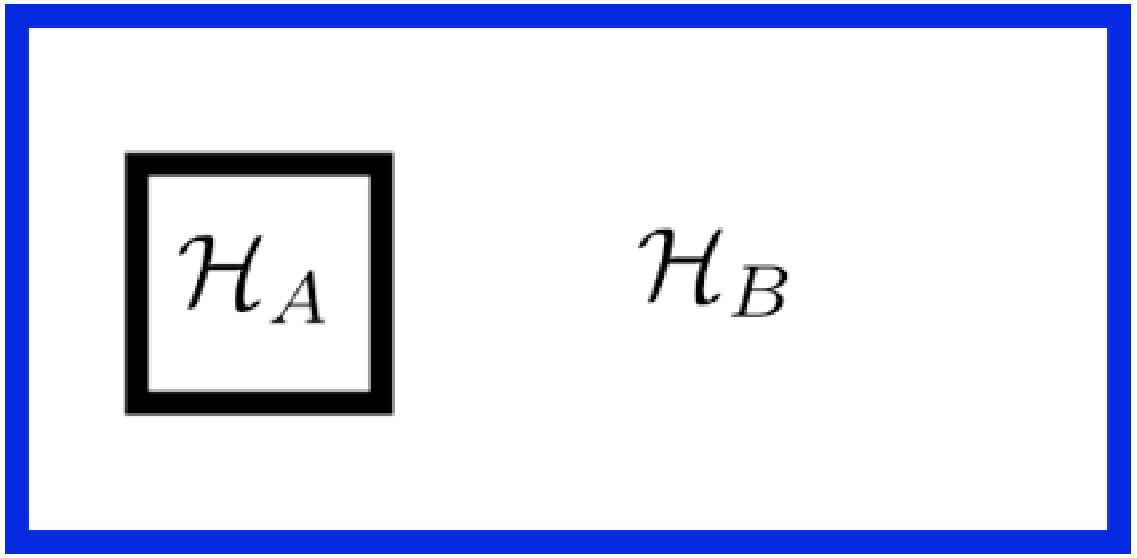

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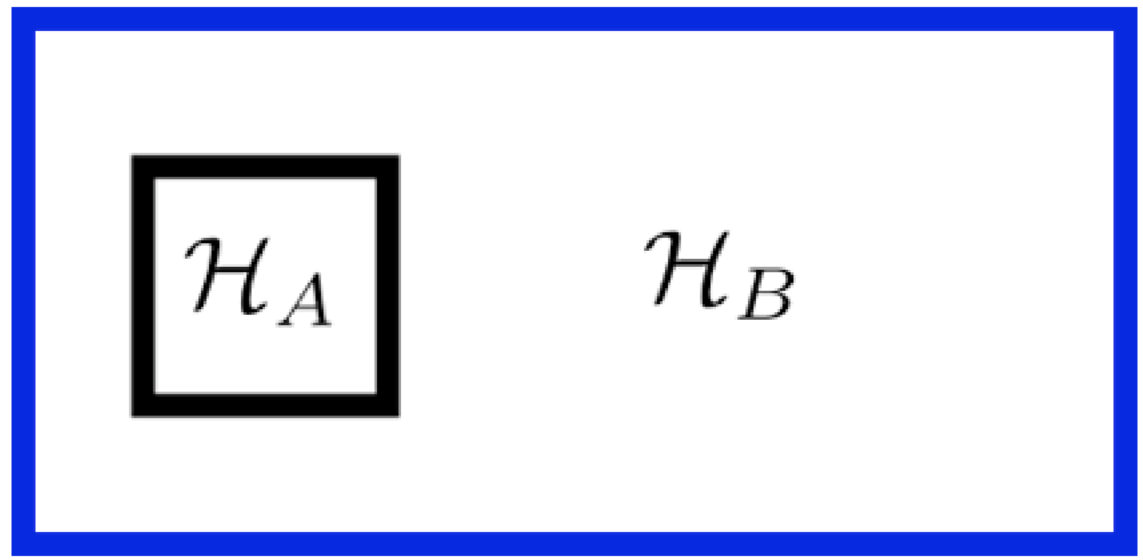

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Draw $|\psi\rangle \in \mathcal{H}$ randomly under $\|\psi\| = 1$ and $\langle\psi|H|\psi\rangle = 3/2$ and compute $\psi^A := \mathrm{Tr}_B |\psi\rangle\langle\psi|$. Then, with high probability,

$$\psi^A \approx \frac{1}{12} \begin{pmatrix} 5+\sqrt{7} & 0 & 0 \\ 0 & 2(4-\sqrt{7}) & 0 \\ 0 & 0 & -1+\sqrt{7} \end{pmatrix}$$

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More in detail,

$$\mathrm{Prob}\left\{\left\|\psi^A - \rho_c\right\|_2 > 3\sqrt{8}\left(\varepsilon + \frac{59}{\sqrt[4]{n}}\right)\right\} \leq 369960\, n^{\frac{3}{2}}\, e^{-\frac{3}{64}n\left(\varepsilon - \frac{1}{4n}\right)^2 + 4\sqrt{n}}.$$

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- Concentration of measure = typicality for energy ensemble
- Note that $[\psi^A, H_A] = 0$ but $\psi^A \neq \exp(-\beta H_A)$. Not Gibbs!

2. Typicality in mean energy ensemble

Our result

General result (arXiv:1003.4982): On a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with Hamiltonian $H = H_A + H_B$, draw a pure state $|\psi\rangle \in \mathcal{H}$ randomly under $\|\psi\| = 1$ and $\langle\psi|H|\psi\rangle = E$. Compute $\psi^A := \mathrm{Tr}_B|\psi\rangle\langle\psi|$. Then, with high prob. (made precise)

$$\psi^A \approx \rho_c \quad \text{where} \quad \rho_c = \frac{1}{\dim \mathcal{H}} \sum_{k=1}^{\dim \mathcal{H}_B} \frac{E + s}{H_A + E_k^B + s}$$

where $s \in \mathbb{R}$ is given by an algebraic equation, and E_k^B are the eigenvalues of H_B.

The amount of concentration and s depend on the spectrum!

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This follows from an even more general result:

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<u>Main Theorem</u> (arXiv:1003.4982): Let *H* be any observable on $\mathbb{C}^n$, and draw a pure normalized state $|\psi\rangle \in \mathbb{C}^n$ randomly under the constraint $\langle\psi|H|\psi\rangle = E.$

If *f* is any real function (on states) with $|f(x) - f(y)| \leq \lambda \|x - y\|$

then $\mathrm{Prob}\left\{|f(\psi) - \bar{f}| > \lambda\varepsilon\right\} \leq a \cdot n^{\frac{3}{2}} e^{-c\,n\left(\varepsilon - \frac{1}{4n}\right)^2 + 2\delta\sqrt{n}}$

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The median $\bar{f}$ can be approximated by integration over a high-dimensional ellipsoid.

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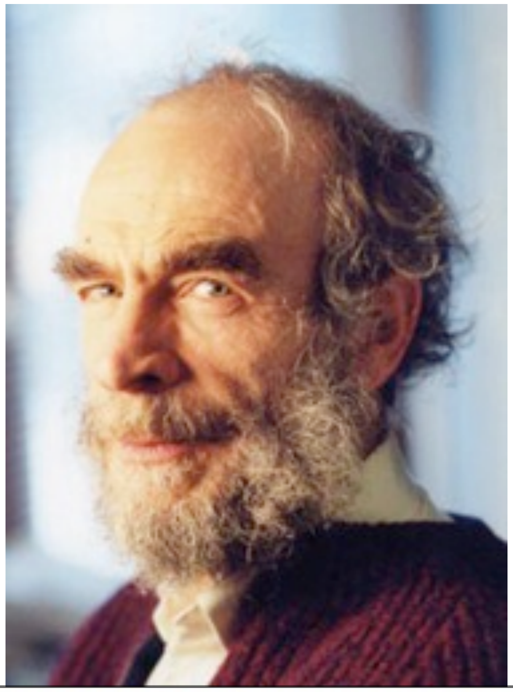
For some spectra, this result can be trivial (e.g. $c \approx 0$)!

2. Typicality in mean energy ensemble

Idea of proof: integral geometry



M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces* (Birkhäuser '01).

GROMOV AWARDED 2009 ABEL PRIZE
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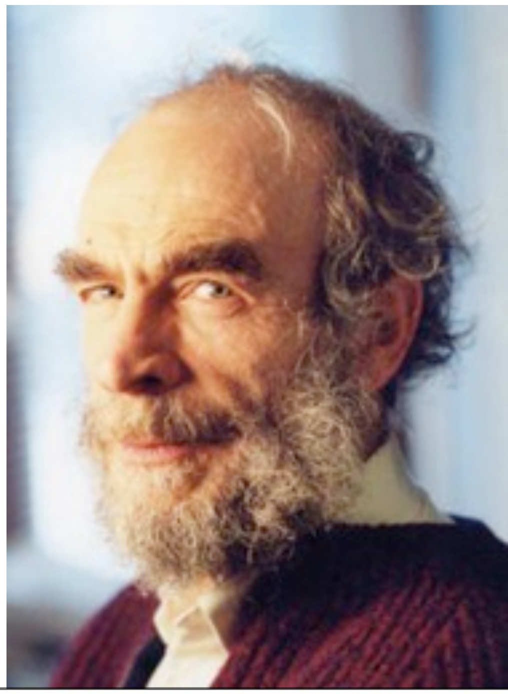
2. Typicality in mean energy ensemble

Idea of proof: integral geometry



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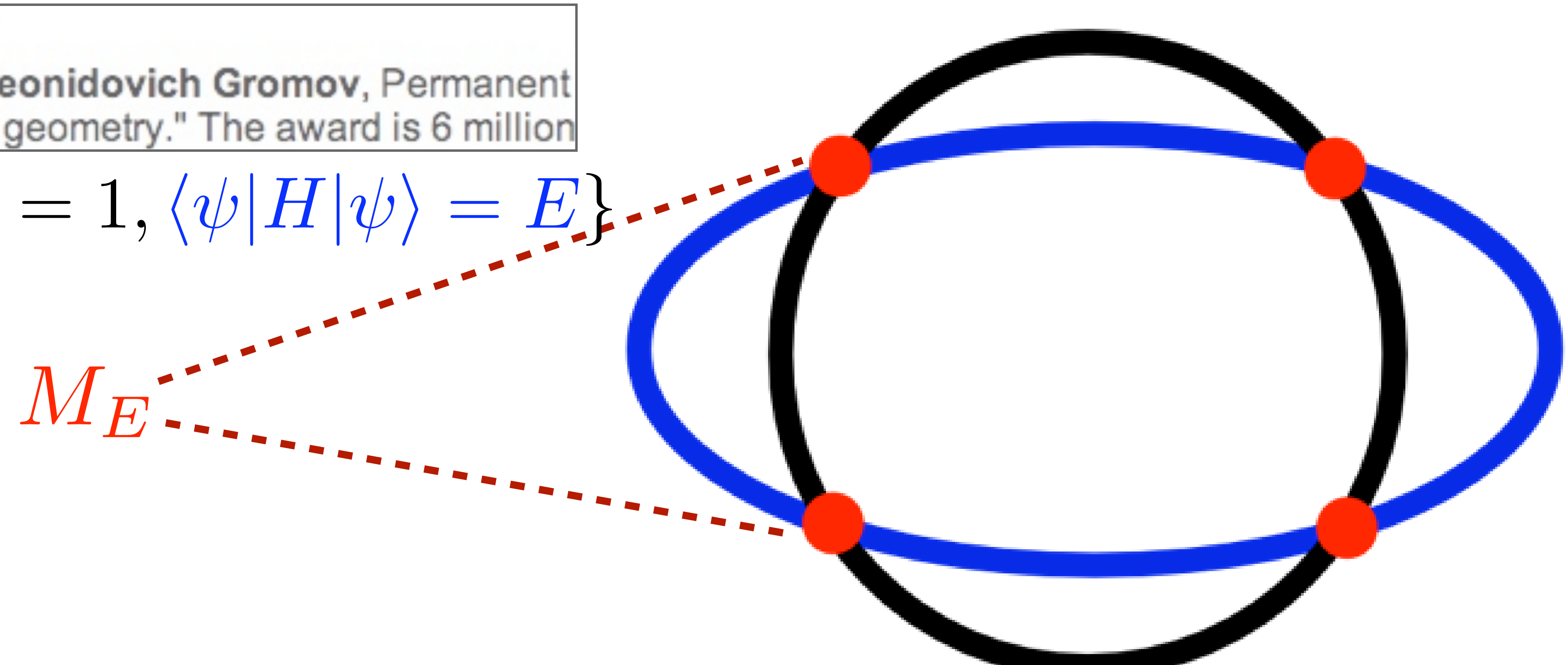

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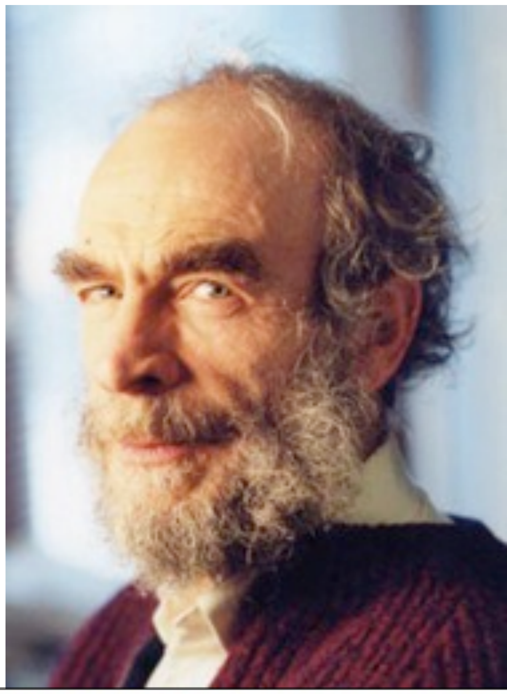

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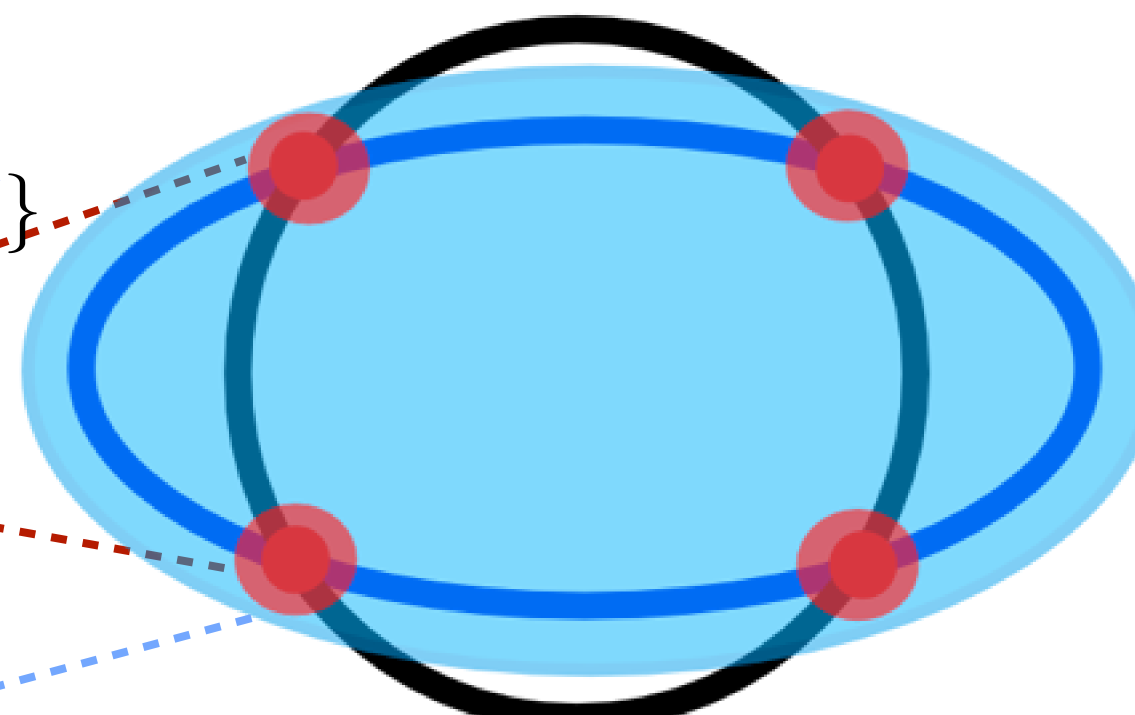

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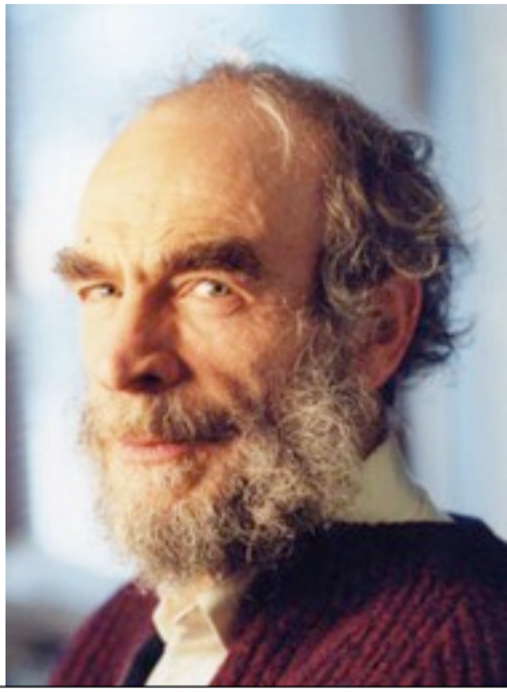

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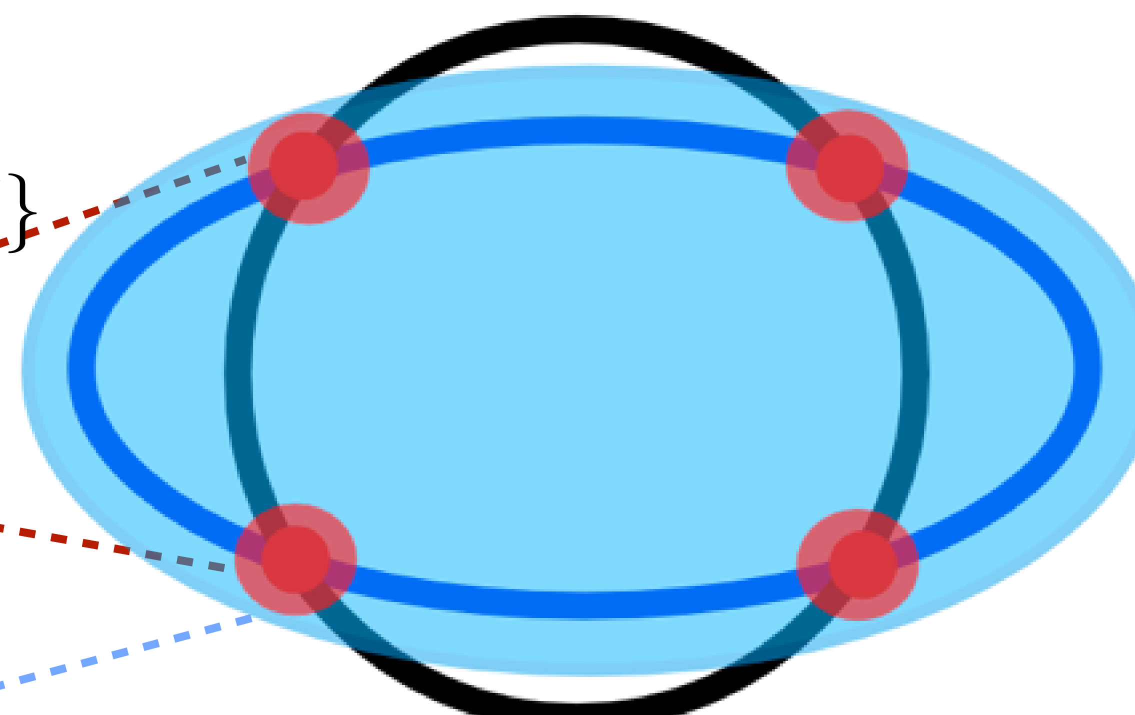

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covers a large part of *N* if
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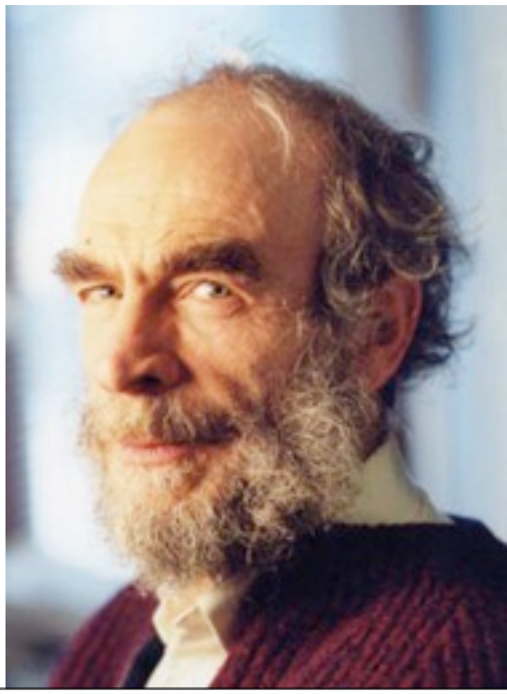

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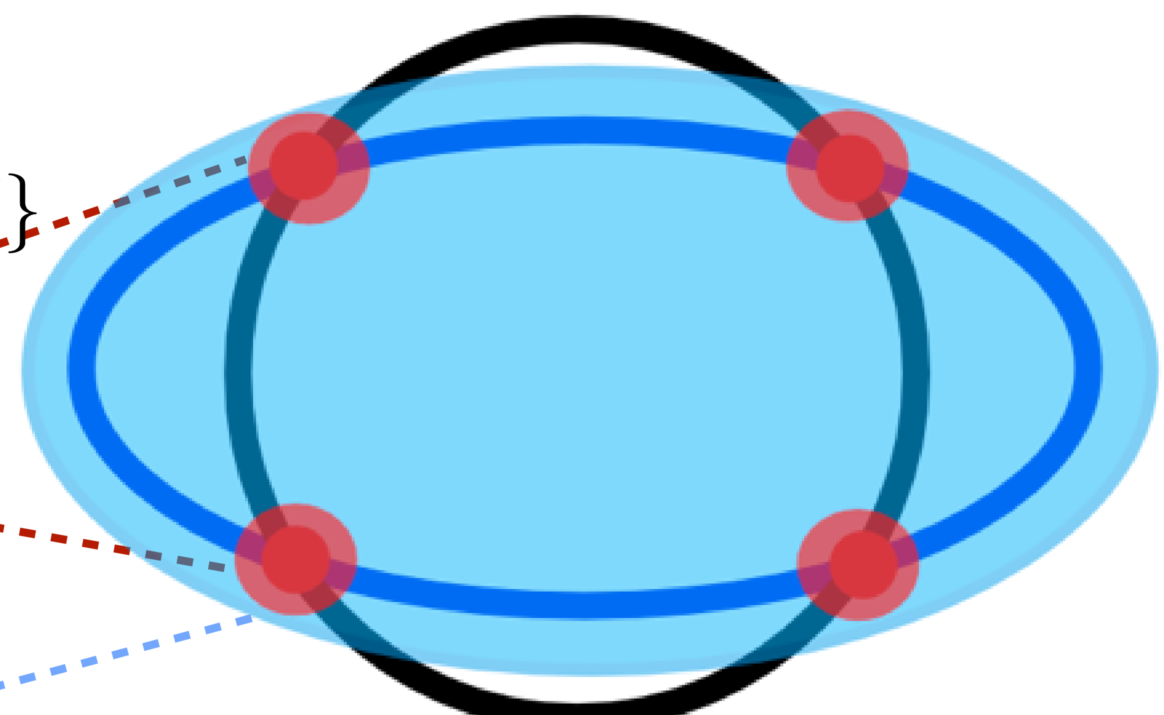

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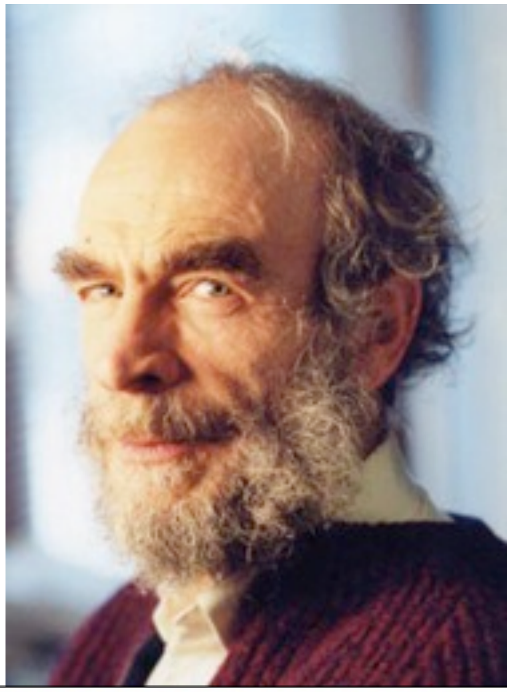

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Mean energy manifold inherits concentration of measure from surrounding ellipsoid.

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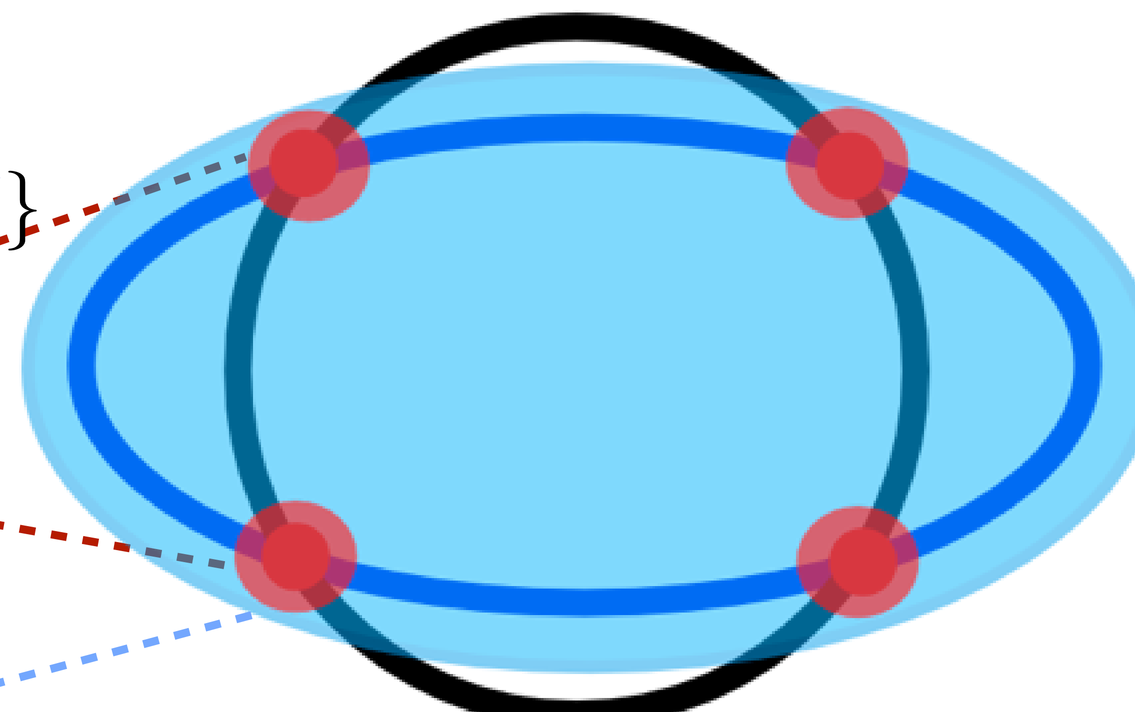

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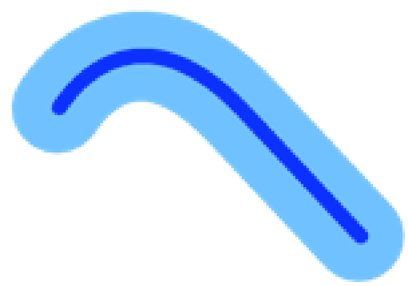
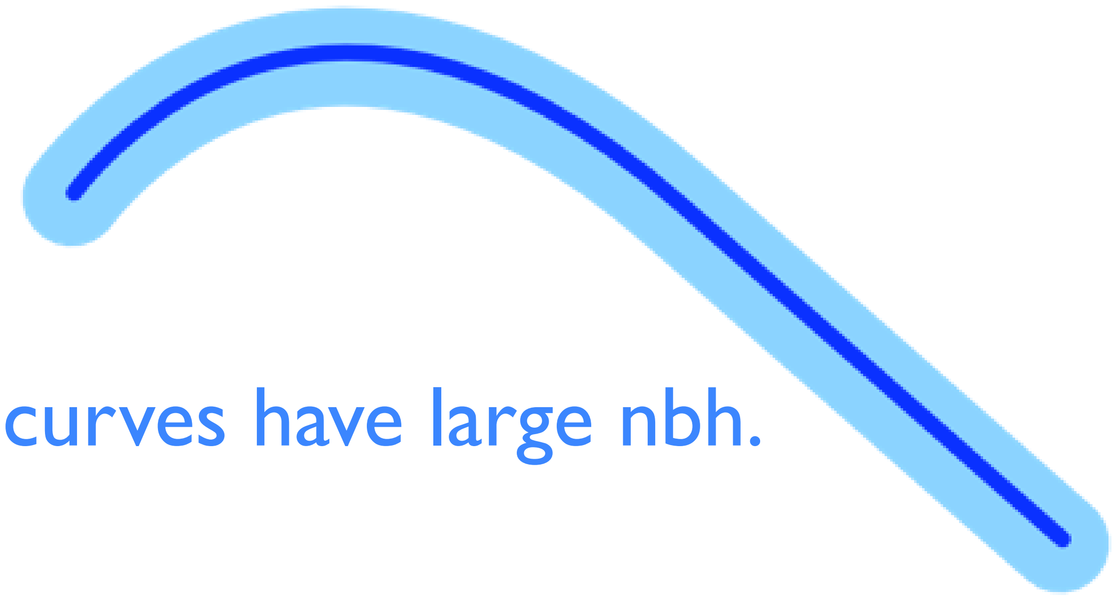
Proof: how to estimate neighborhood volume

Intuition:



short curves have small nbh...



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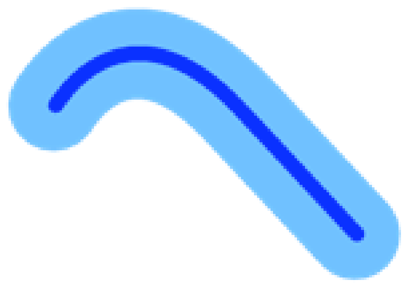

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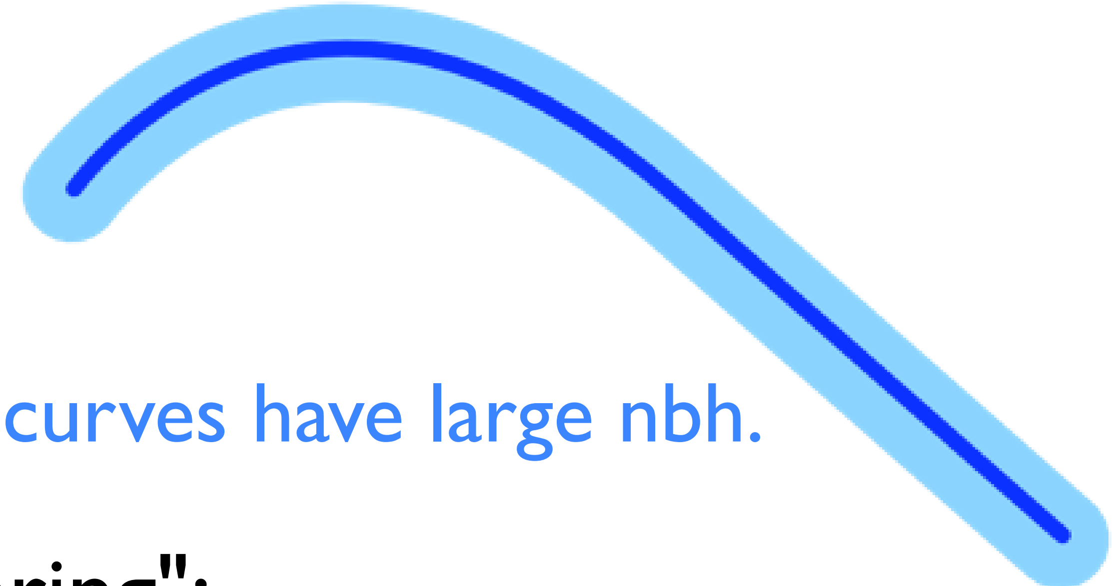

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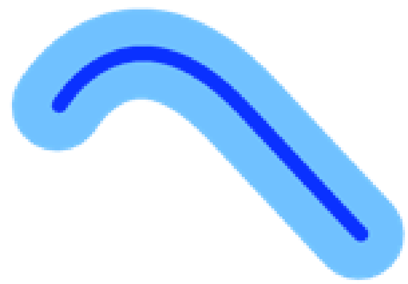

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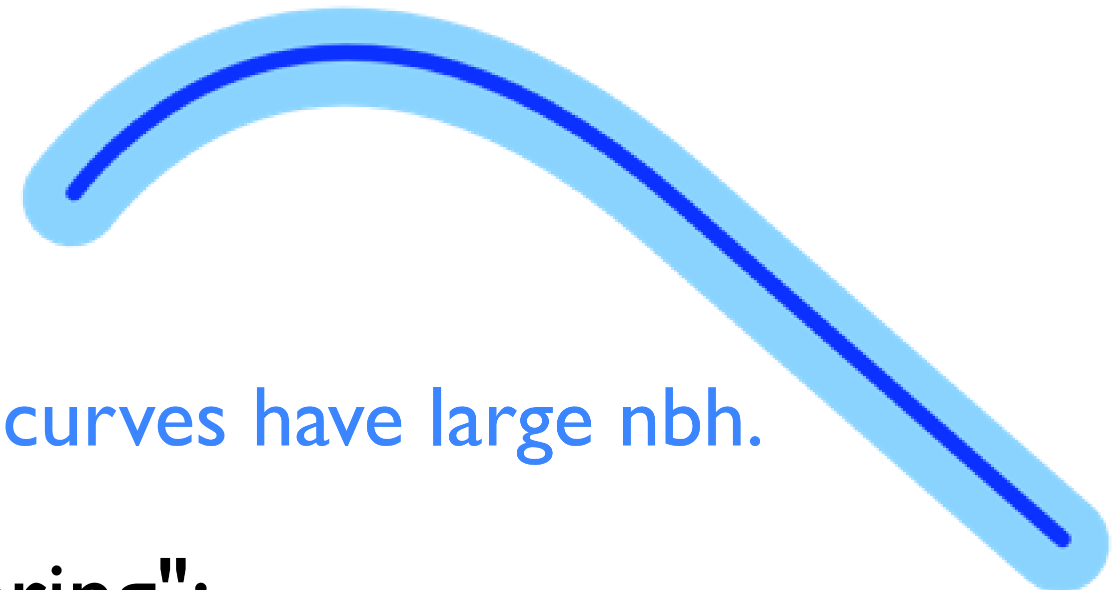

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Cauchy-Crofton formula ("Buffon's needle experiment"):

C: curve, D: domain (e.g. $D = U_\varepsilon(C)$)

$$\int_{\text{lines } L} \#(L \cap C)\, dL = 2 \cdot \text{length}(C)$$

$$\int_{\text{lines } L} \text{length}(L \cap D)\, dL = \pi \cdot \text{area}(D)$$

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No concentration in the Ising model

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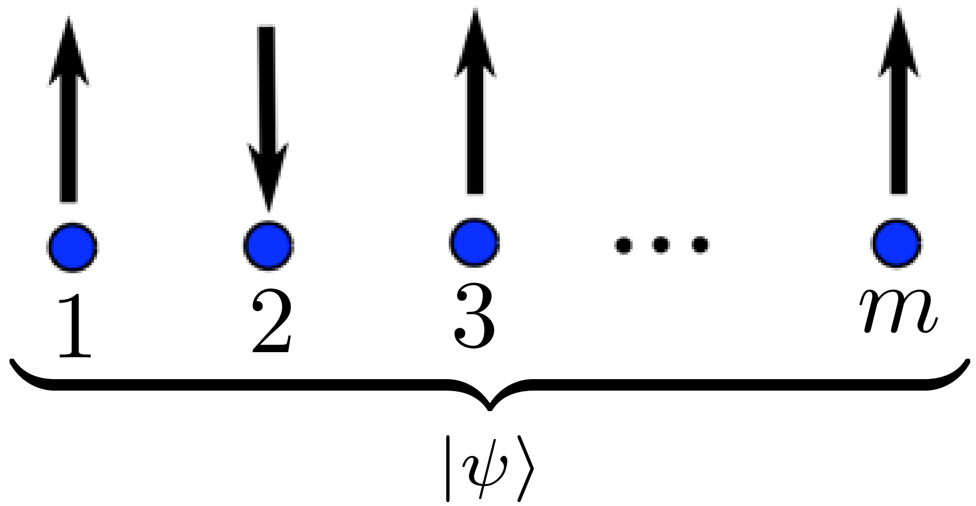

$$H = \frac{1}{2}\left(m\mathbf{1} + \sum_{i=1}^{m} Z_i\right)$$

Ground state energy 0, infinite temperature: energy $m/2$.

$\dim \mathcal{H} = 2^m =: n$. Draw $|\psi\rangle$ randomly under $\langle\psi|H|\psi\rangle \stackrel{!}{=} \alpha \cdot m$

where $0 \leq \alpha \leq 1/2$.

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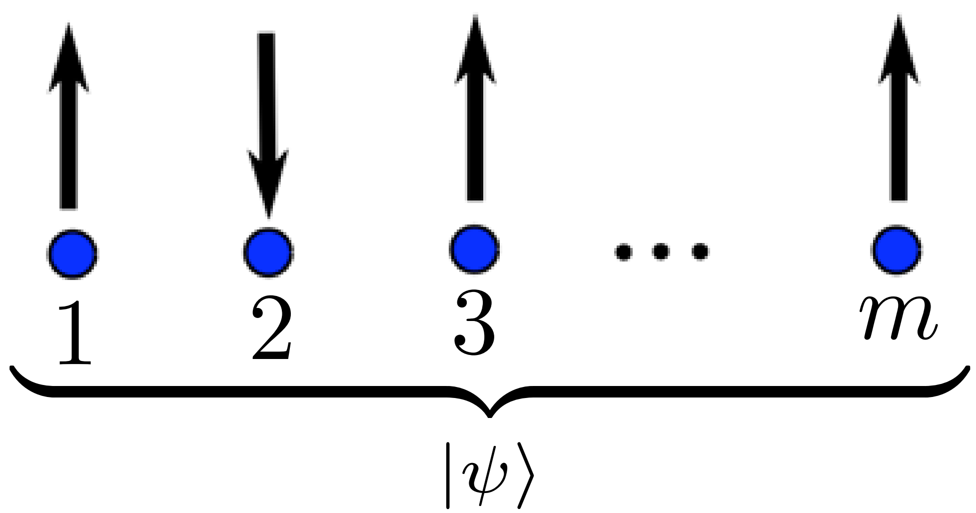

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Observation: Bound from our theorem gets useless:

$$\mathrm{Prob}\left\{ |f(\psi) - \bar{f}| > \lambda\varepsilon \right\} \lesssim \exp\left(-c\, n\varepsilon^2 + 2\delta\sqrt{n}\right)$$

For Ising spectrum, we get $c \approx 1/n.$ Why is that?

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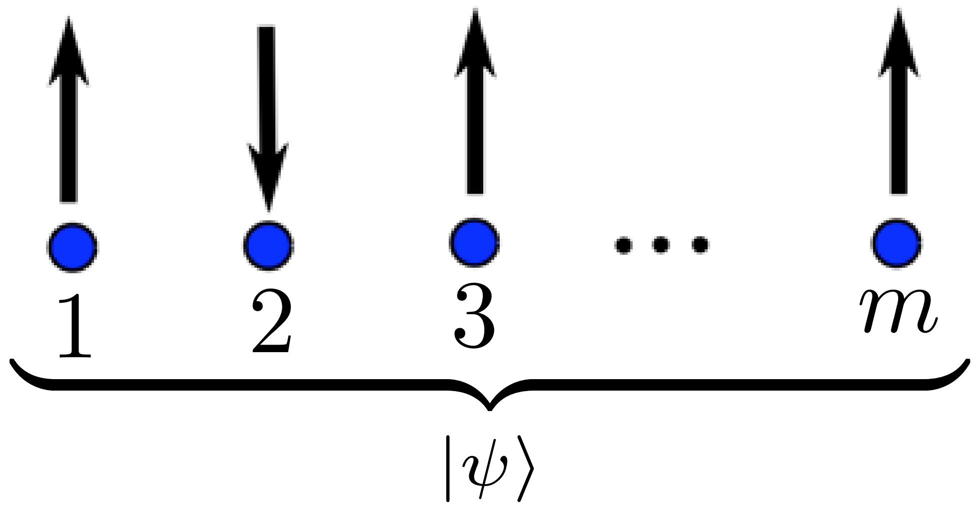

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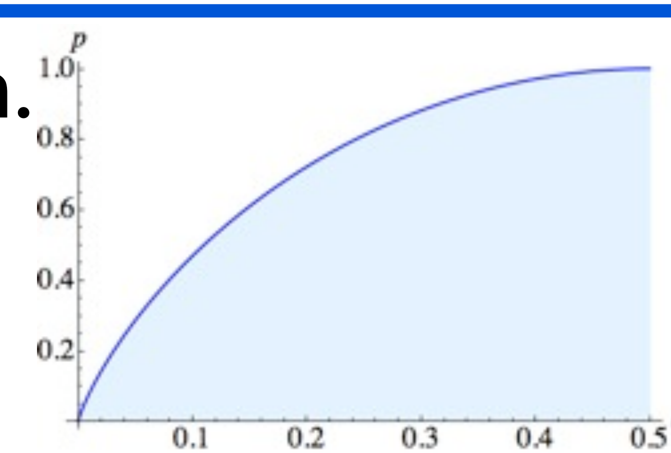
where $0 \le \alpha \le 1/2$.

<u>Theorem</u>: There *is no* exponential concentration.

Best possible concentration bound is

$$\text{Prob}\left\{|f - \bar{f}| > \lambda\varepsilon\right\} \lesssim \exp\left(-c\, n^{p} \varepsilon^2\right)$$

with $p \equiv p(\alpha) < 1$, see graph.



2. Typicality in mean energy ensemble

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- We have proven typicality (exponential concentration) in the mean energy ensemble (*m.e.e.*) for large class of H's.
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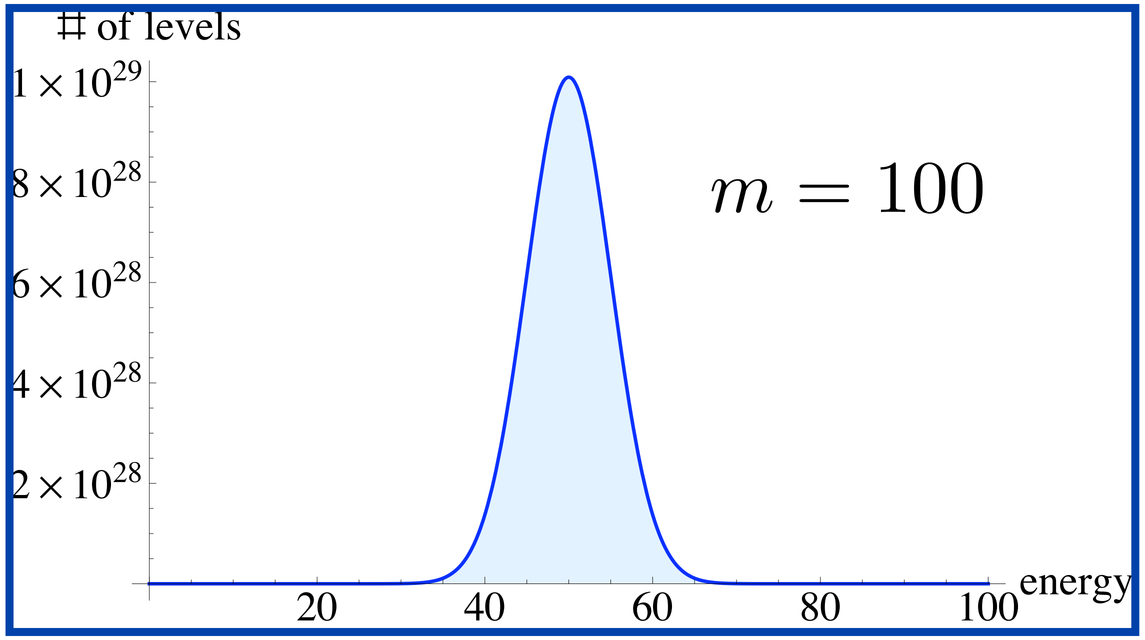

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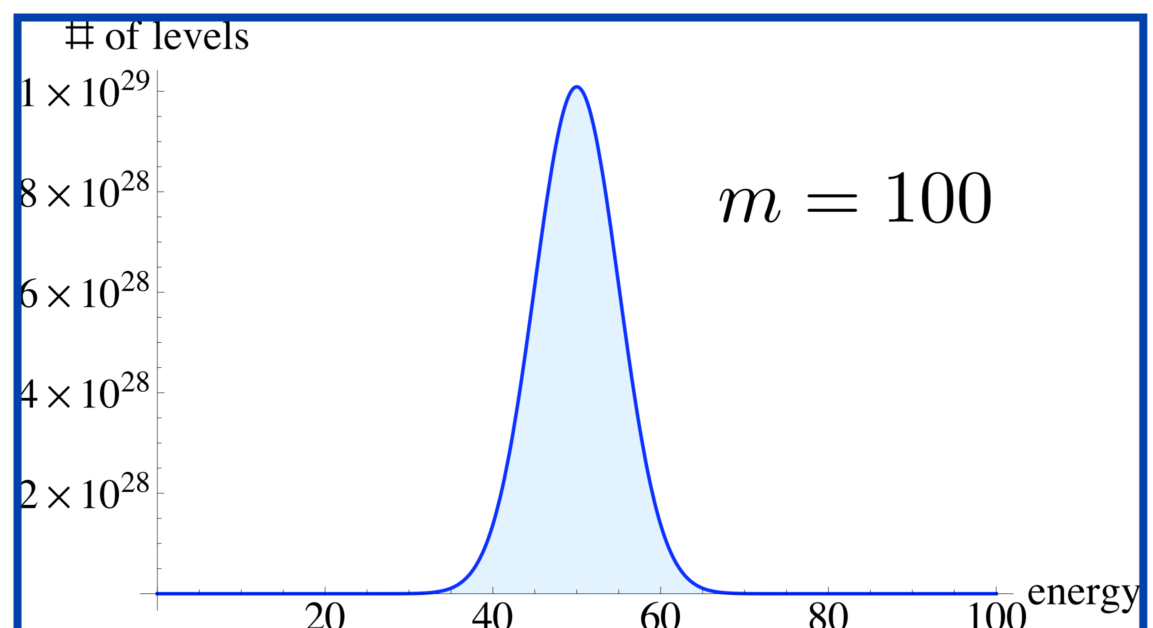
- Interpretation: "almost all" of the $n = 2^m$ energy levels are close to energy value $m/2$.
- If $|\psi\rangle$ is to have much smaller energy, then it "does not see" most of the levels $\Rightarrow$ effectively lives in smaller dim.



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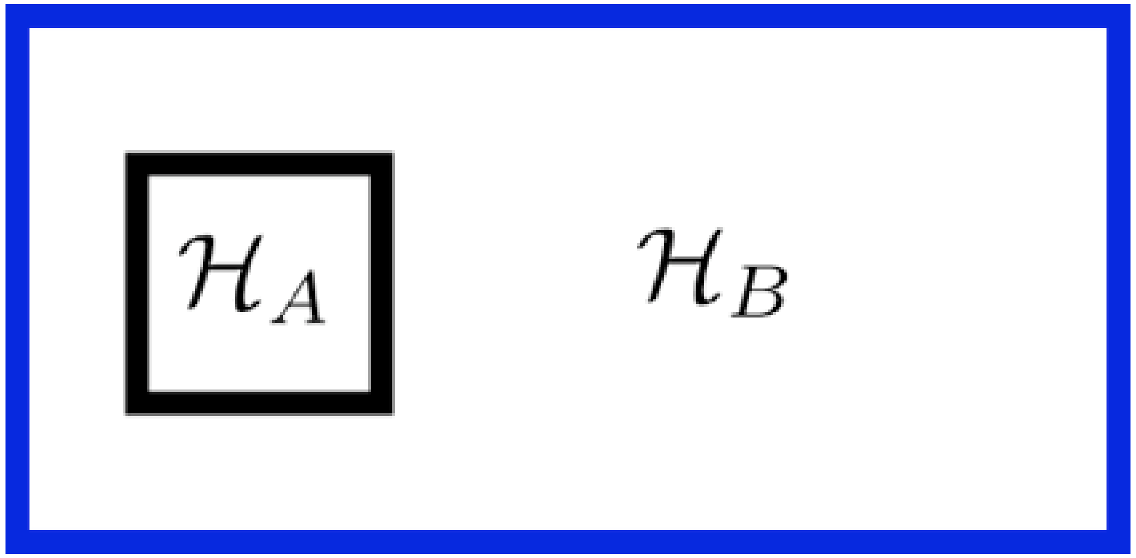
- In those cases where *m.e.e.* concentrates, typical reduced density matrix is not of Gibbs form. Instead, a sum of terms $(H_A + s)^{-1}$ with some $s \in \mathbb{R}$.

Why?



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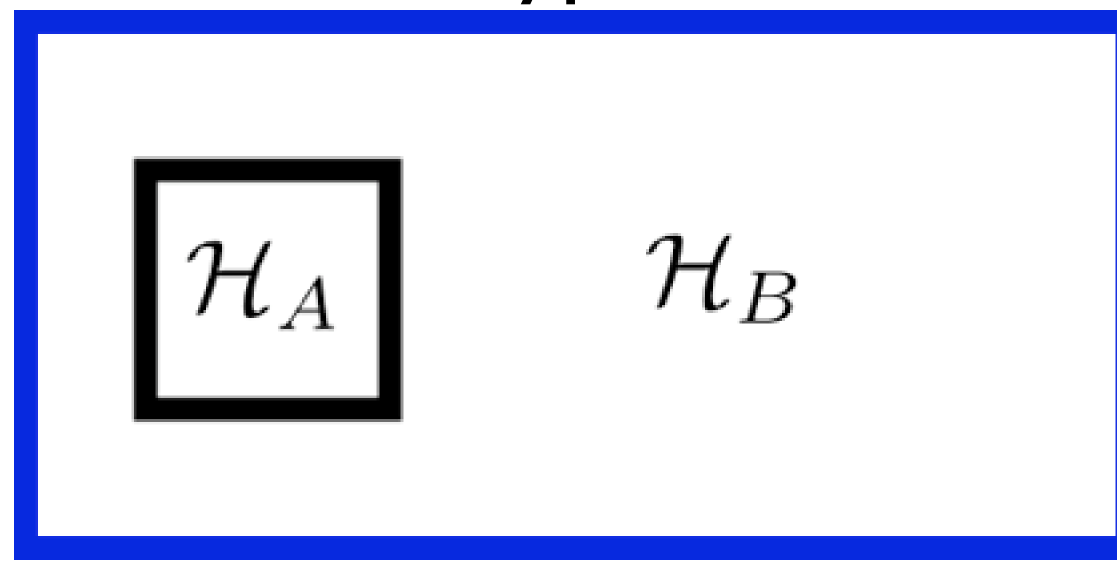

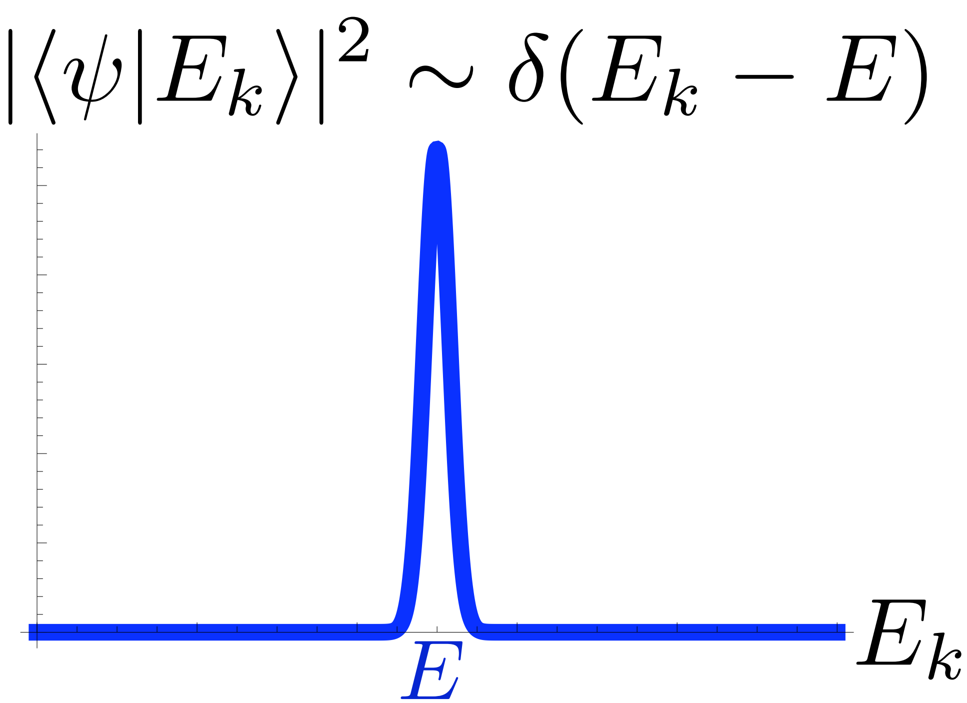

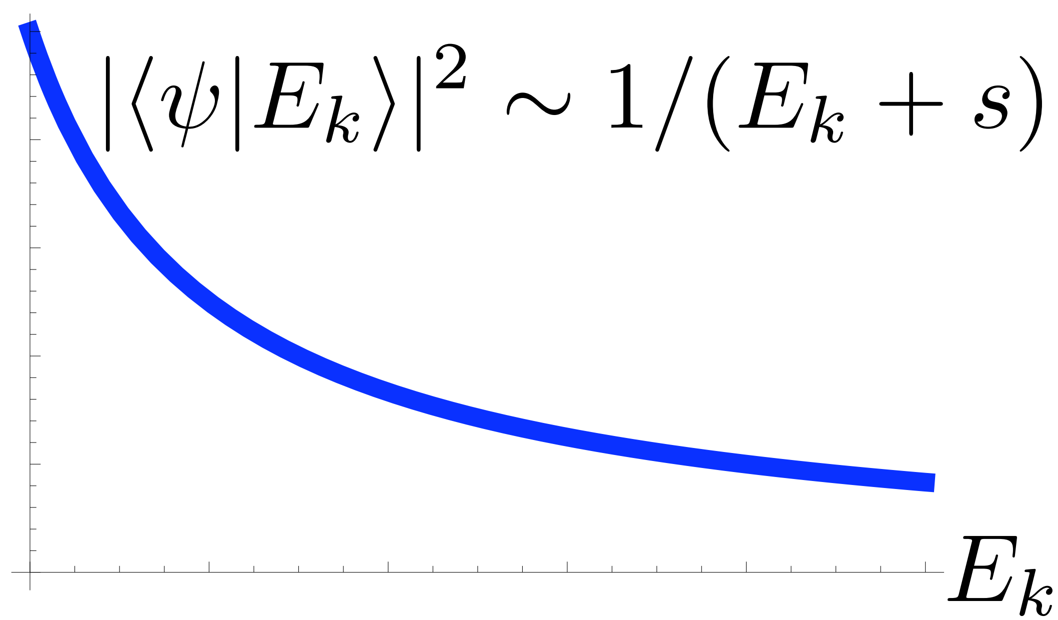

Gibbs situation:
Small spectral window.

Mean energy ensemble:
"Schrödinger cat state".

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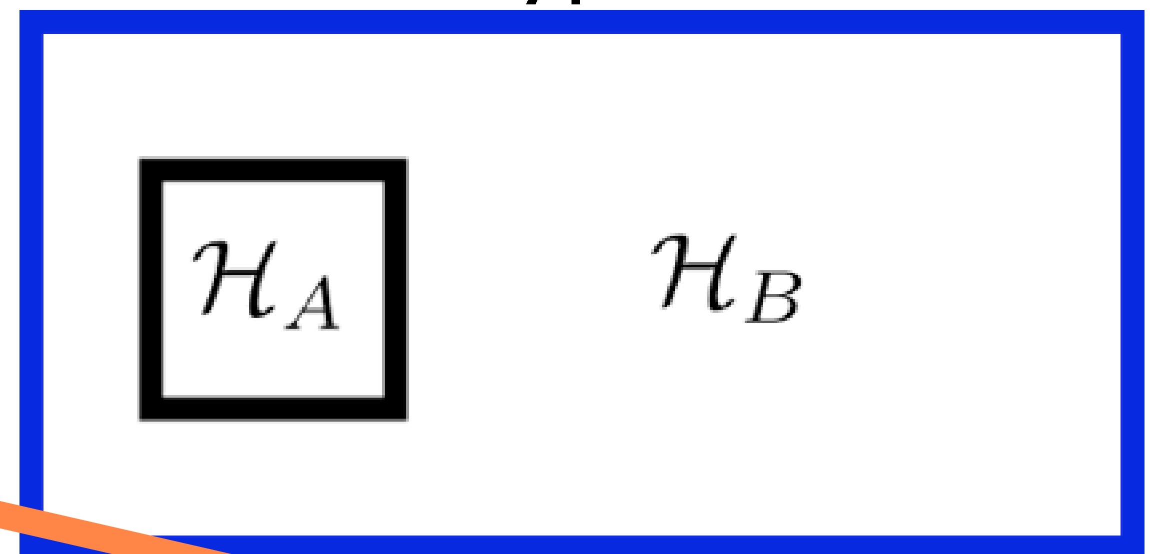

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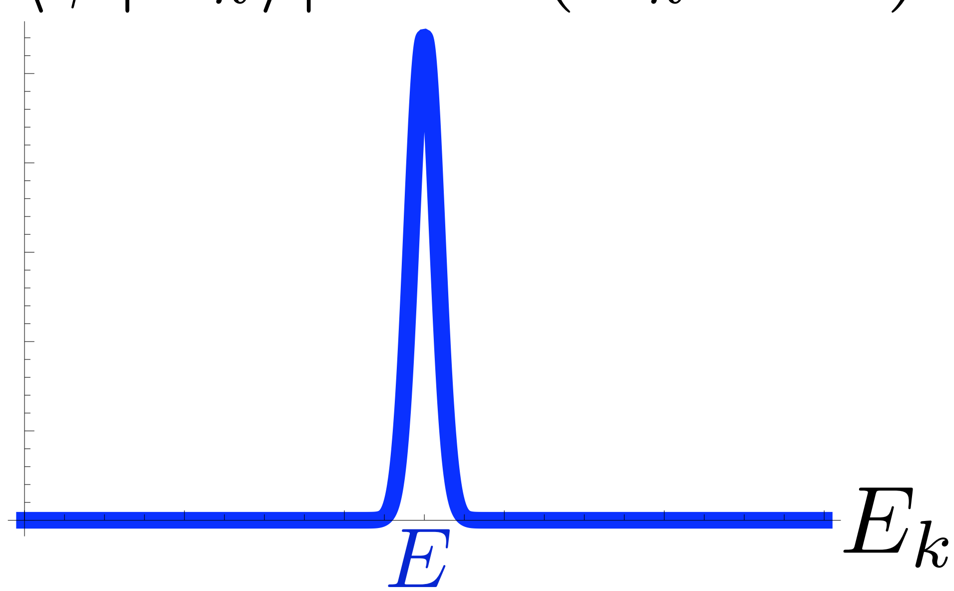

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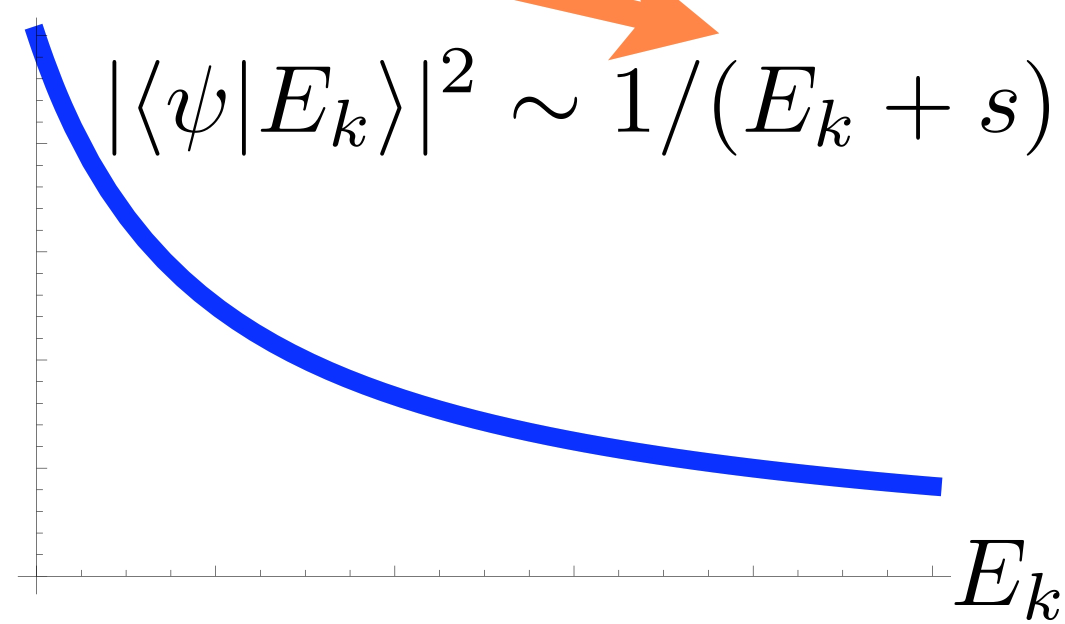
$$|\langle\psi|E_k\rangle|^2 \sim 1/(E_k + s)$$



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Positive and negative results on typicality in *m.e.e.*

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  (Applications in quantum information theory?)

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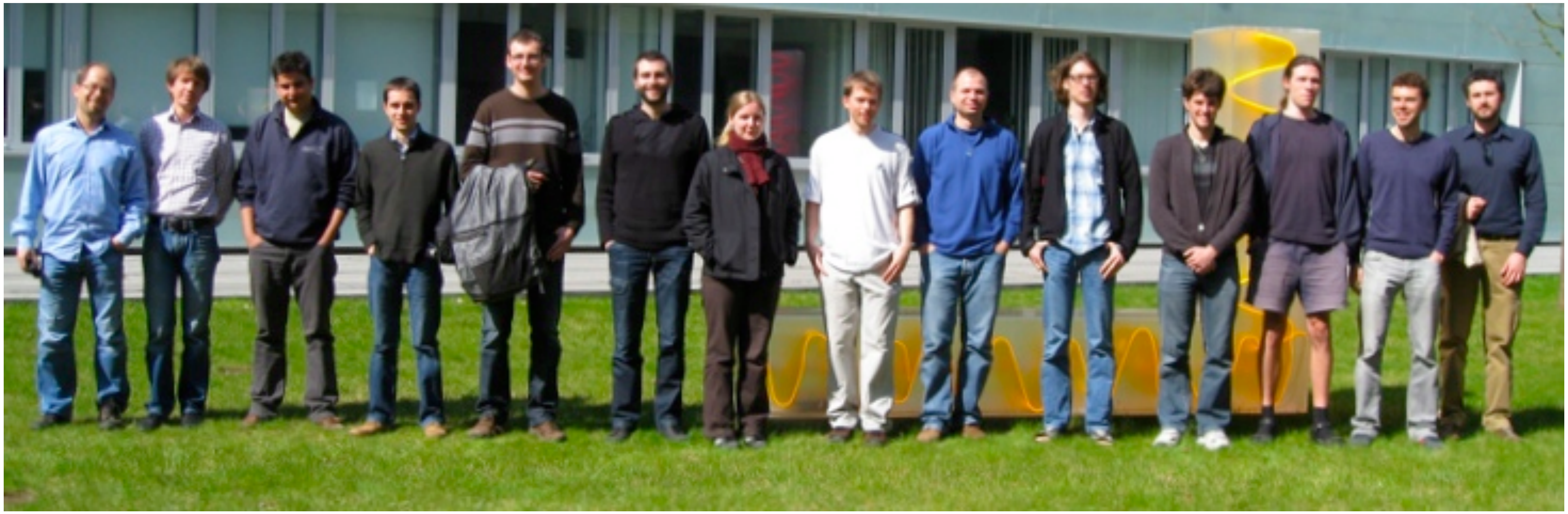
- Analytic results on exponential concentration in *m*.e.e. for many Hamiltonians ⟶ typicality.
- Fundamental mathematical result:
  New proof tools for "typical states under constraints"
  (Applications in quantum information theory?)

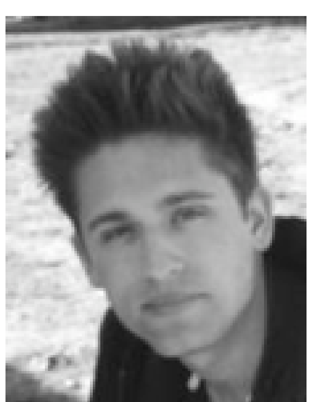
- Proof that *m*.e.e. does not concentrate in Ising model.
  Typicality does not hold for *m*.e.e. of many-body systems.
- Computed typical reduced state ⟶ not a Gibbs state.
  M.e.e. not directly useful to describe statistical physics.
  Does it describe more exotic, but still physical situations?

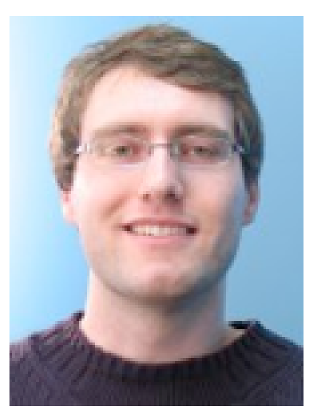
Our group in Potsdam (Prof. Jens Eisert)

Institute for Physics and Astronomy, Potsdam University



Jens Eisert, Tomaz Prosen, Carlos Pineda, Andrea Mari, Holger Bernigau, Arnau Riera, Inka Benthin, Martin Kliesch, Thomas Barthel, Matthias Ohliger, Niel de Beaudrap, Konrad Kieling, Markus Müller, Tommaso Gagliardoni





David Gross (now ETH Zürich), Christian Gogolin (coming soon), Robert Hübener

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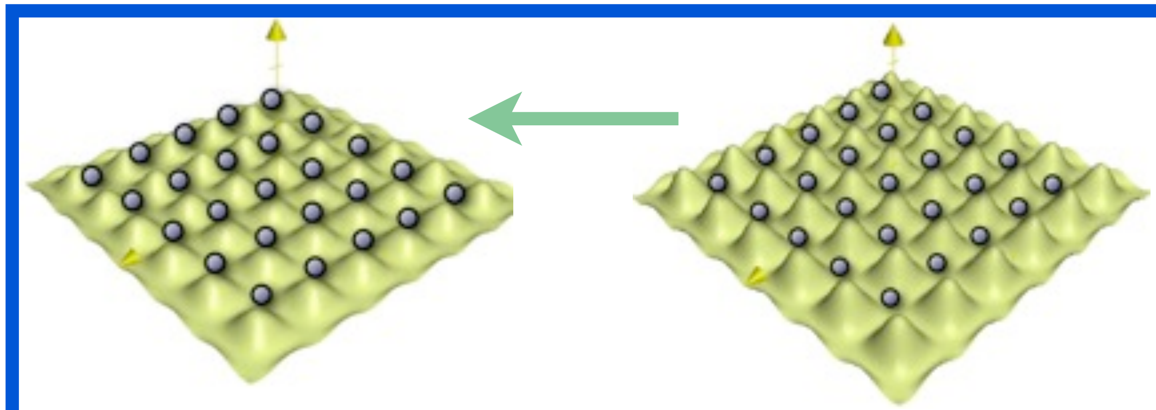

1. Rigorous relaxation theorems:

$$\|\rho_s(t) - \rho_G\|_1 < \varepsilon$$

Closeness to maximum entropy states for subsystems in 1-norm for arbitrarily long times and any given error

Methods: (i) Non-commutative Lindeberg central limit theorems (ii) Lieb-Robinson bounds (iii) Ideas of concentration of measure

- Cramer, Eisert, New J. Phys. **12**, 055020 (2010)
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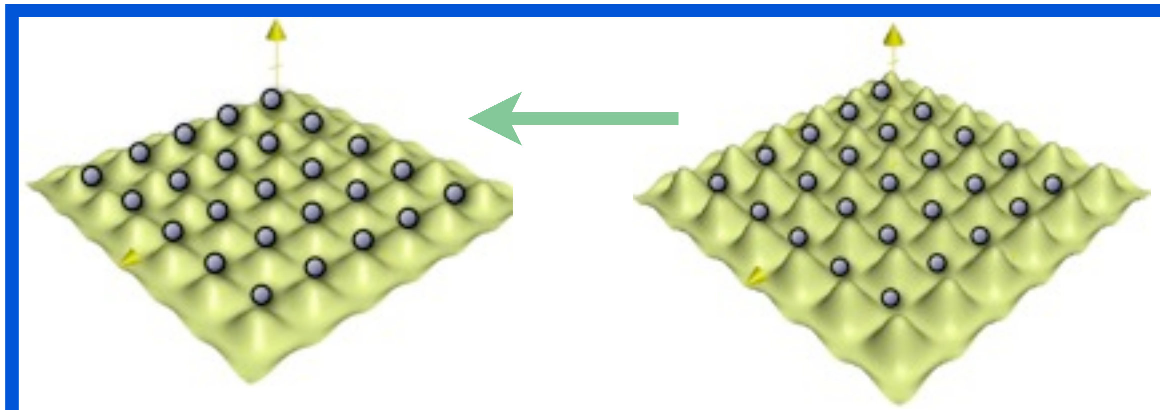

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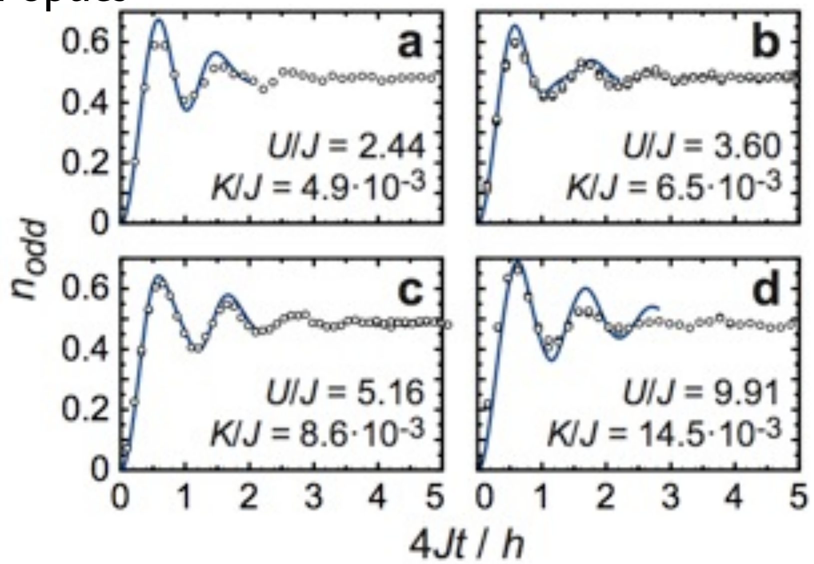
Do theory work probing non-equilibrium dynamics of strongly correlated systems using cold atoms in optical superlattices (joint work with Immanuel Bloch's (MPQ) and Uli Schollwock's (LMU) groups

Methods: DMRG, quantum optics



- Trotzky, Chen, Flesch, Schollwock, Eisert, Bloch, to be submitted (2010)
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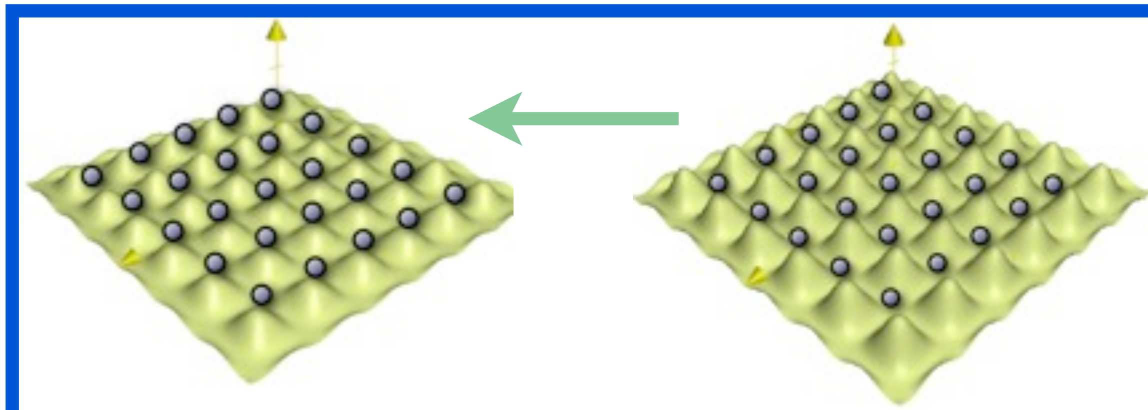

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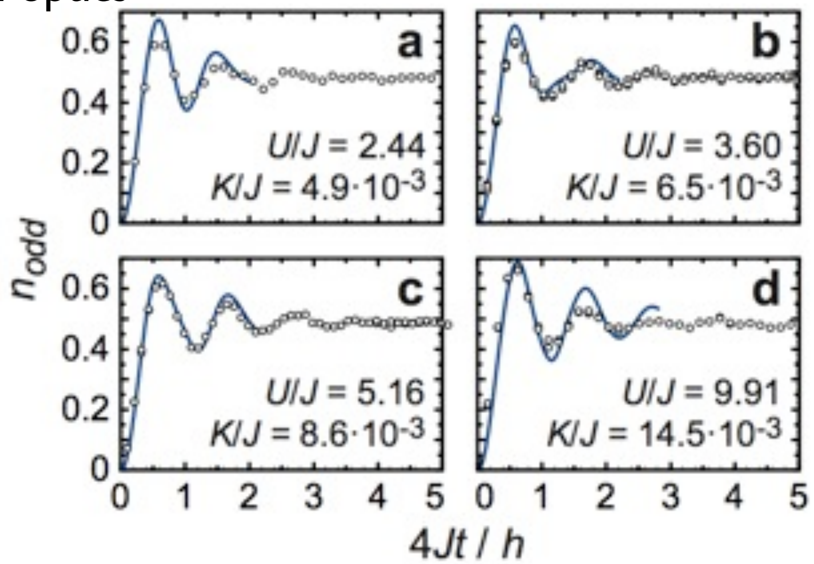

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3. Absence of thermalization in non-integrable models:

Study role of entanglement and locally conserved quantities in relaxation dynamics in integrable and non-integrable models

- Gogolin, Müller, Eisert, arXiv:1009.2493 (today)