Undecidability in quantum measurements

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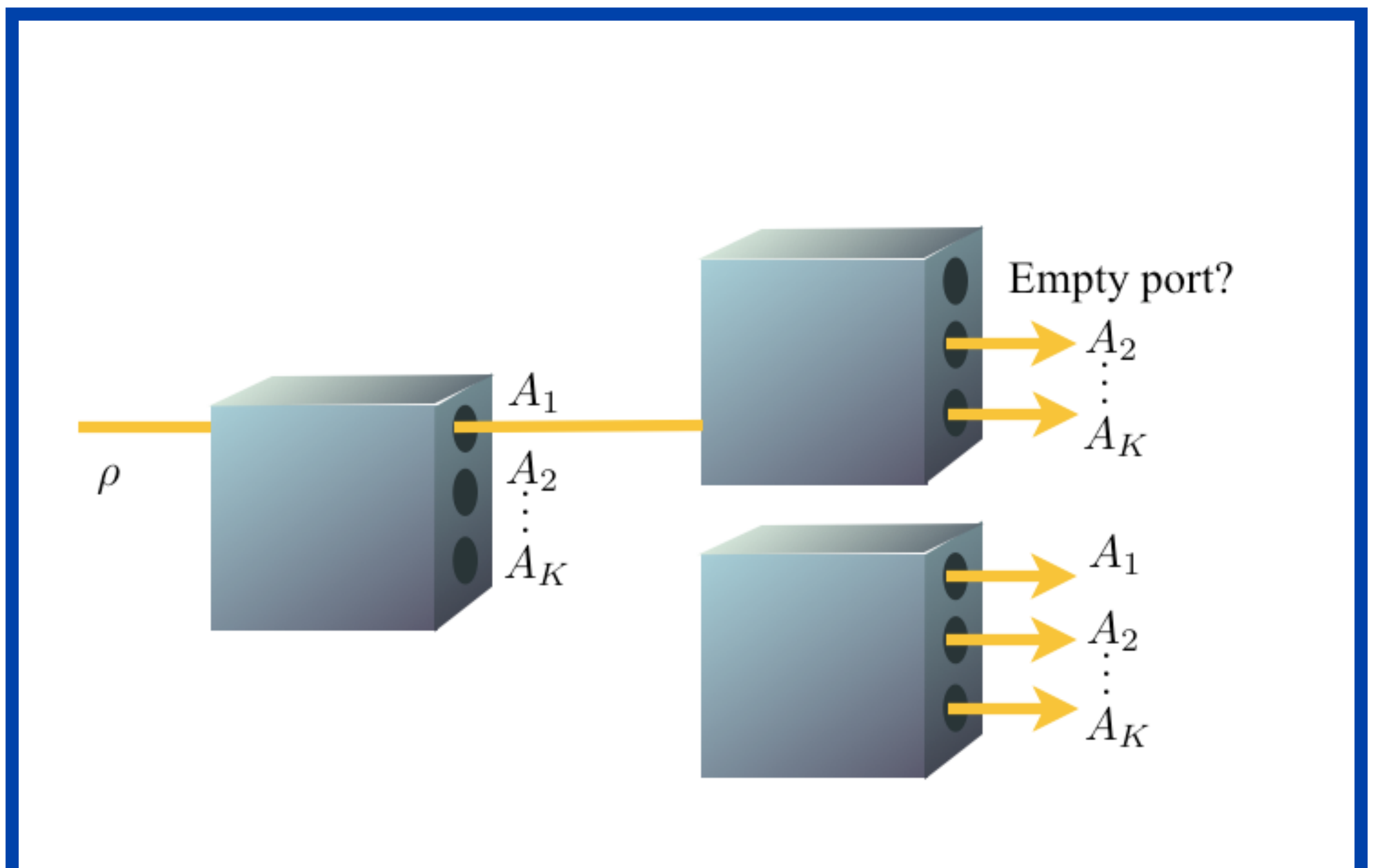
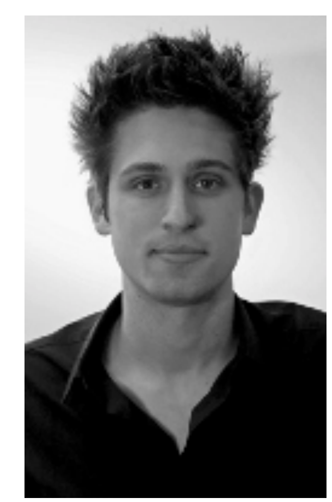


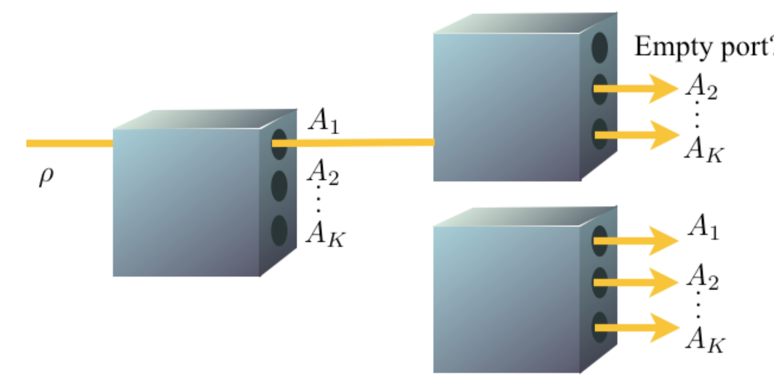
Joint work with Jens Eisert & Christian Gogolin (FU Berlin)

arXiv: 1111.3965



Outline

1. Motivation / undecidability in general
2. The "measurement occurrence problem"



3. Undecidability of the quantum problem
4. Decidability of the classical problem
5. Outlook

1. Motivation / undecidability in general

Quantum computers are believed to be more powerful than classical computers (Shor's algorithm, ...).

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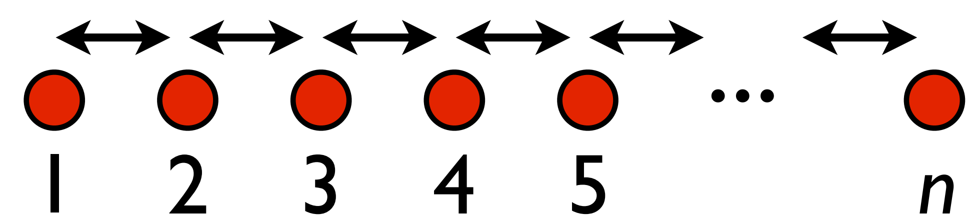
- Quantum complexity theory. Example: The 2-local Hamiltonian problem. Given $a < b$, and

$$H = \sum_{j=1}^{r} H_j \,,$$



where all H_j act on at most 2 qubits, $r, \|H_j\| \leq poly(n)$, decide if the smallest eigenvalue is $< a$ or $> b$.
This problem is QMA-complete.

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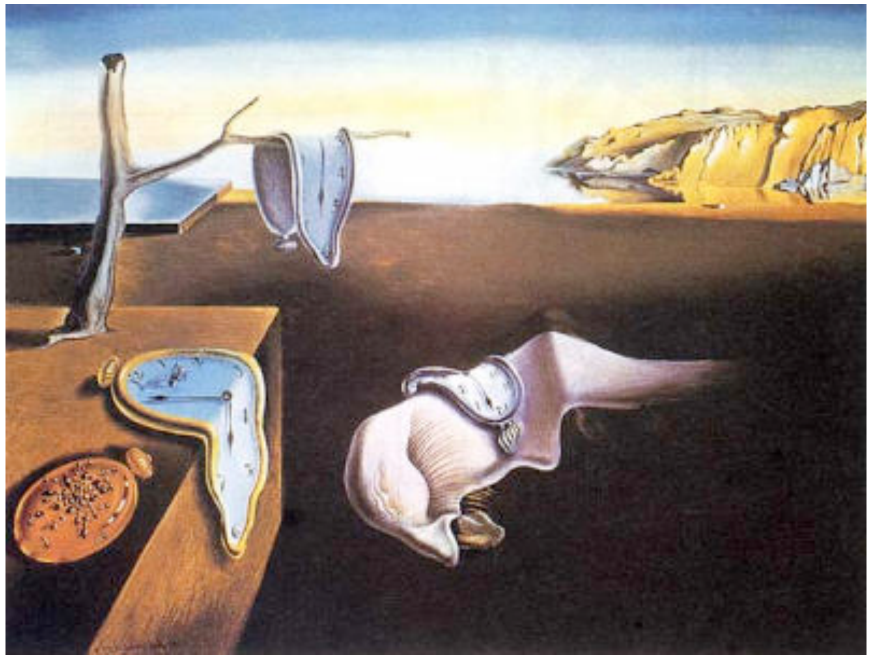

No algorithm will solve the problem... ever!

Origin: the Halting Problem.

1. Motivation / undecidability in general

Fix a universal Turing machine which takes natural numbers $x \in \mathbb{N}$ as input.

Halting problem: Given input $x \in \mathbb{N}$, will the TM eventually halt on that input, or will it run forever?

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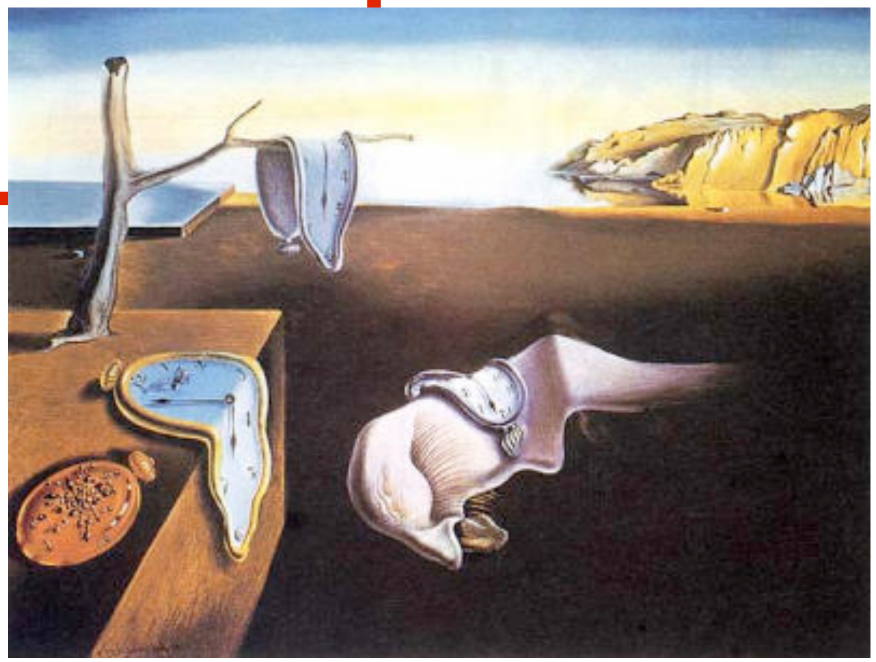
Fix a universal Turing machine which takes natural numbers $x \in \mathbb{N}$ as input.

Halting problem: Given input $x \in \mathbb{N}$, will the TM eventually halt on that input, or will it run forever?

Alan Turing 1936: The halting problem is undecidable. That is, there is no single algorithm which, for every input *x*, decides in finite time whether the TM halts on input *x* or not.



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$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} = (a_{ij})$$

Matrix mortality problem: Given some finite set of integer matrices $\{M_1, \ldots, M_k\}$, is there any finite matrix product $M_{i_1} M_{i_2} \ldots M_{i_n}$ which equals the zero matrix?

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Paterson 1970; Halava, Harju 2001: The matrix mortality problem is undecidable, even for eight 3x3 integer matrices.

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Inspiration: Michael M. Wolf, Toby S. Cubitt, David Perez-Garcia, *Are problems in Quantum Information Theory (un)decidable?*, arXiv:1111.5425

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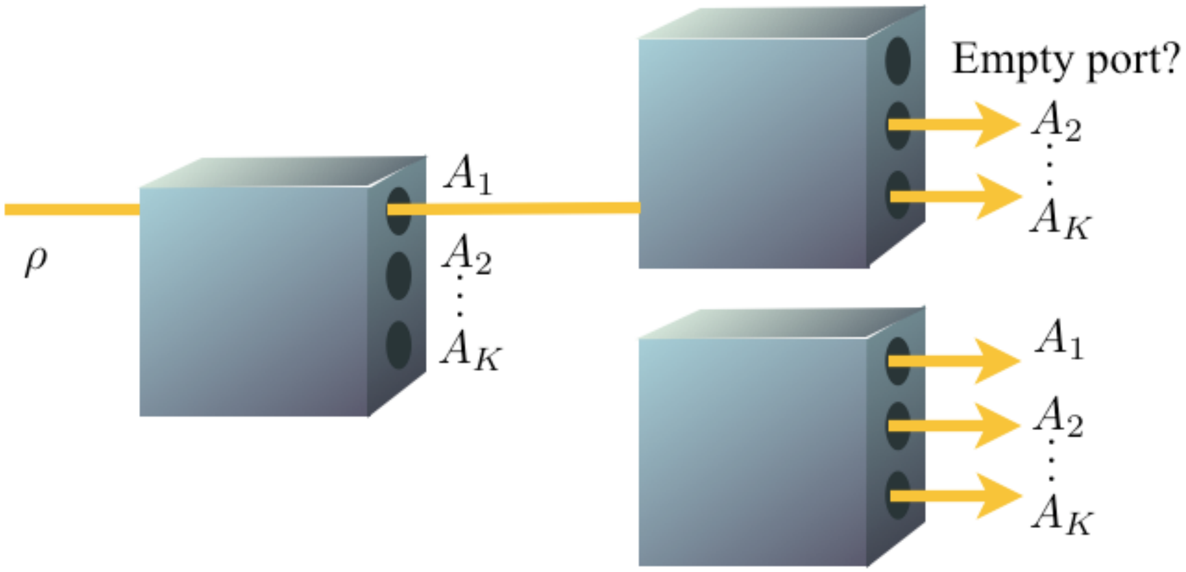
Earlier works in similar spirit:

• V. Blondel, E. Jeandel, P. Koiran, and N. Portier, *Decidable and undecidable problems about quantum automata*, SIAM J Comp. **34**, 1464-1473 (2005).
• H. Derksen, E. Jeandel, and P. Koiran, *Quantum automata and algebraic groups*, J. Symb. Comp. **39**, 357-371 (2005)
• M. Hirvensalo, *Various aspects of finite quantum automata*, Developments of Language Theory, vol. 5257, Lecture Notes in Computer Science, Springer (2008).

2. The "measurement occurrence problem"

The Setting



Many copies of the *same* measurement device. Output is repeatedly fed into device as input → sequence of outcomes $(j_1, \ldots, j_n)$.

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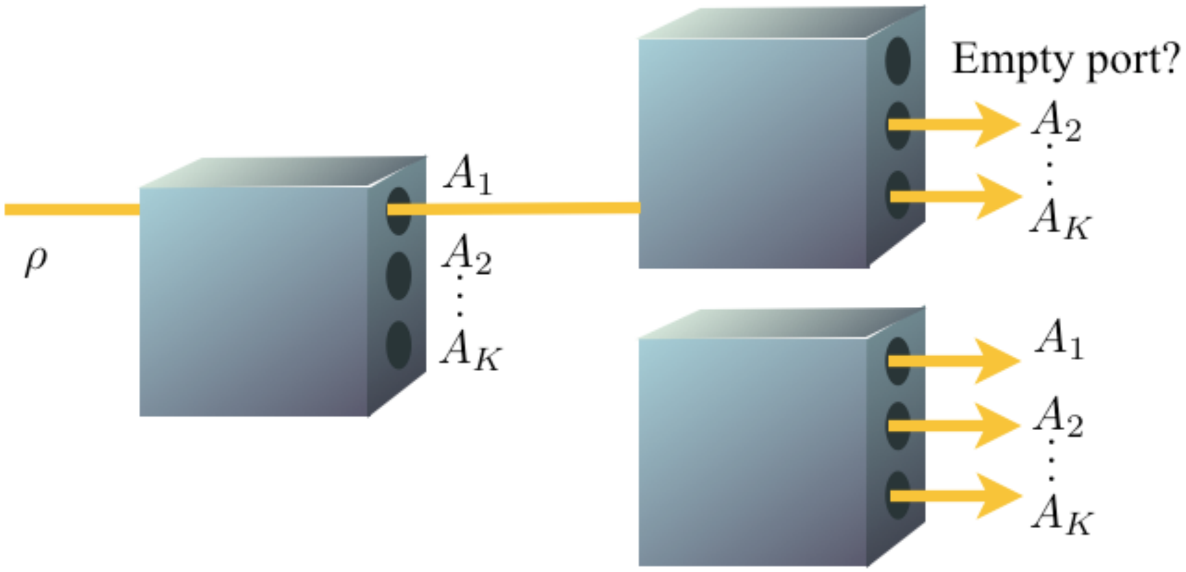

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- **Input:** quantum state $\rho \in \mathbb{C}^{d\times d}$, $\rho \geq 0$, $\mathrm{Tr}\rho = 1$.
- **Device:** specified by *K* "Kraus operators" $\{A_j\}_{j=1}^K \subset \mathbb{C}^{d\times d}$.

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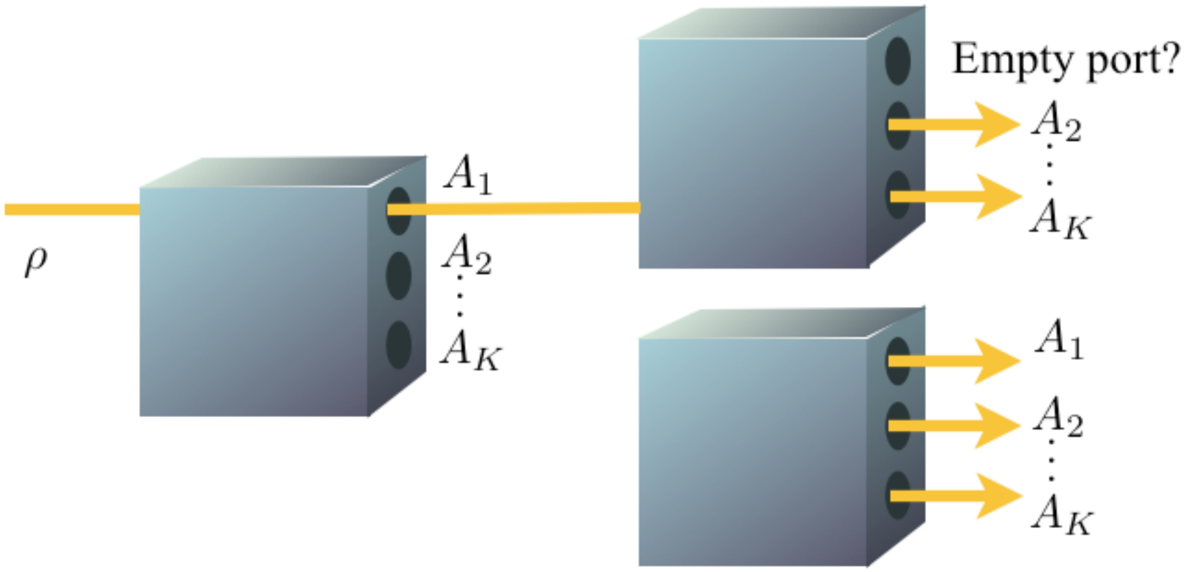

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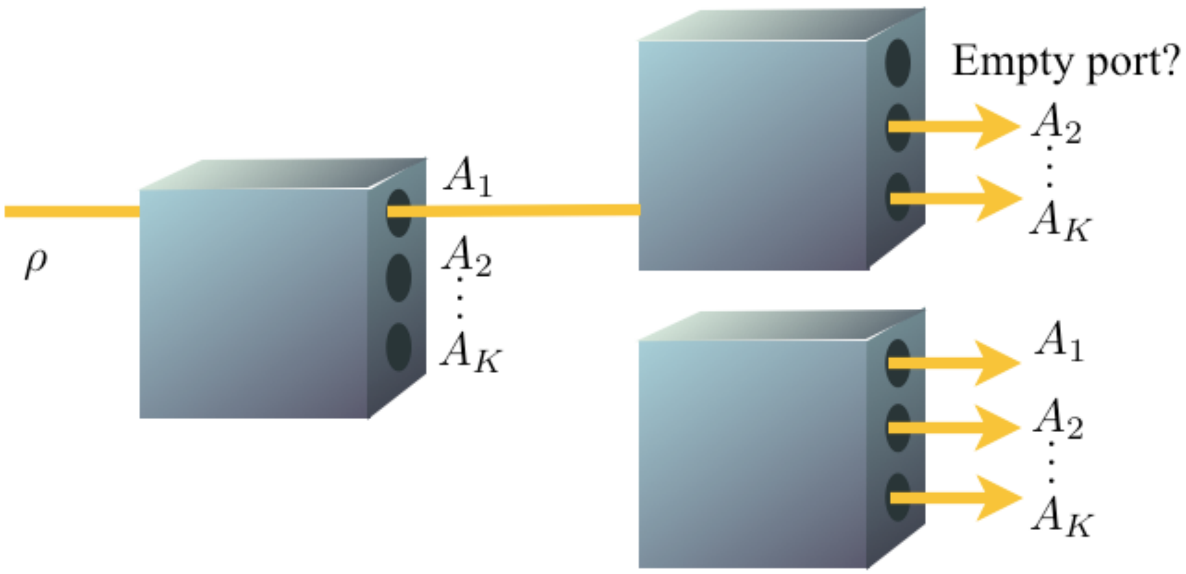

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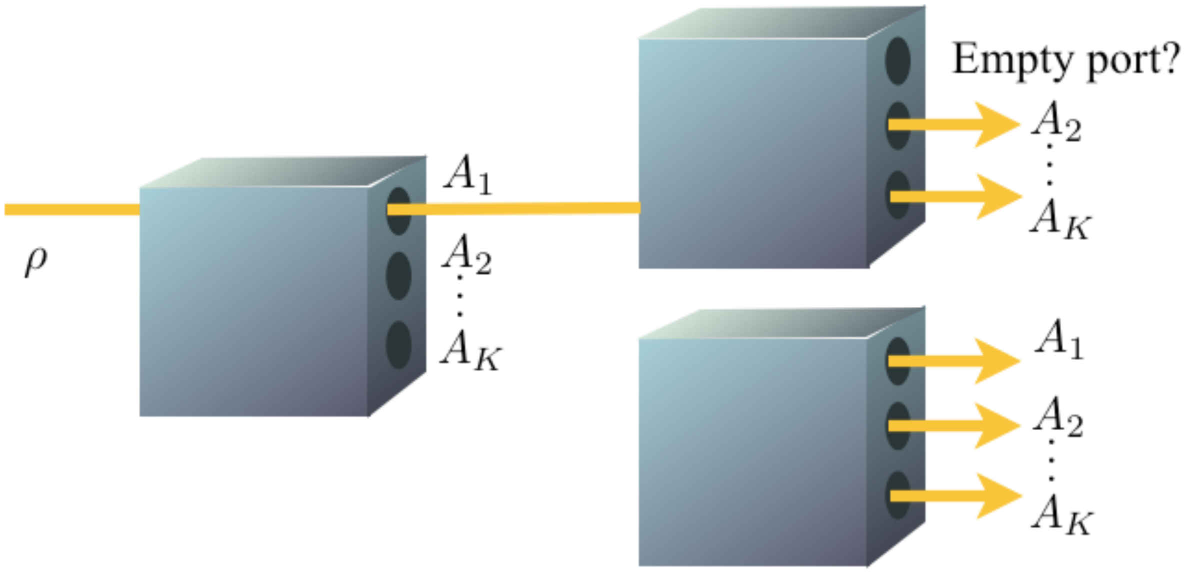

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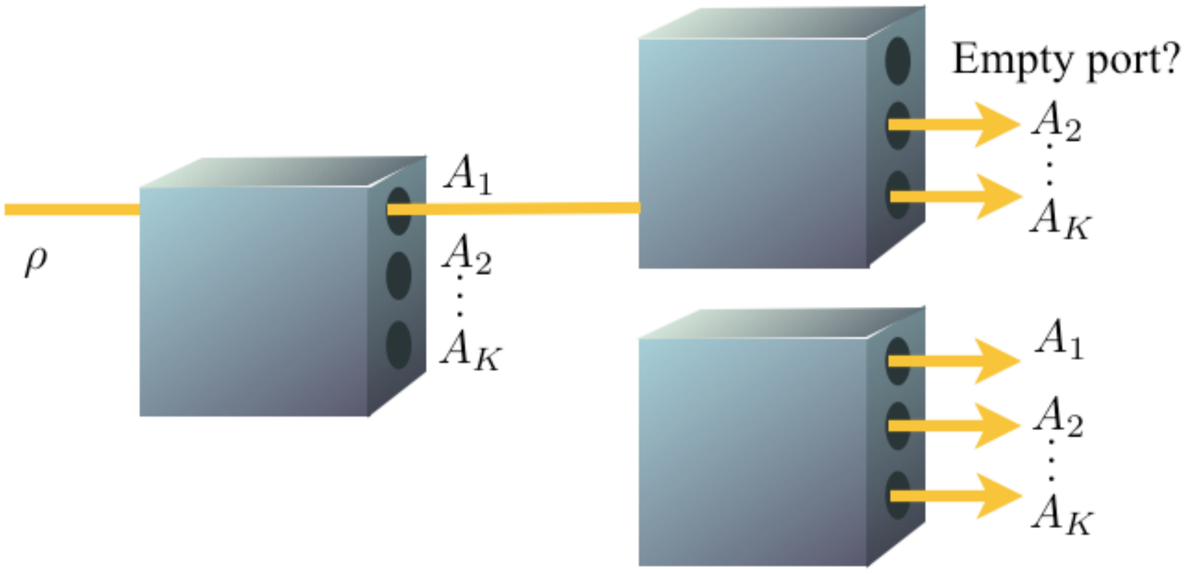

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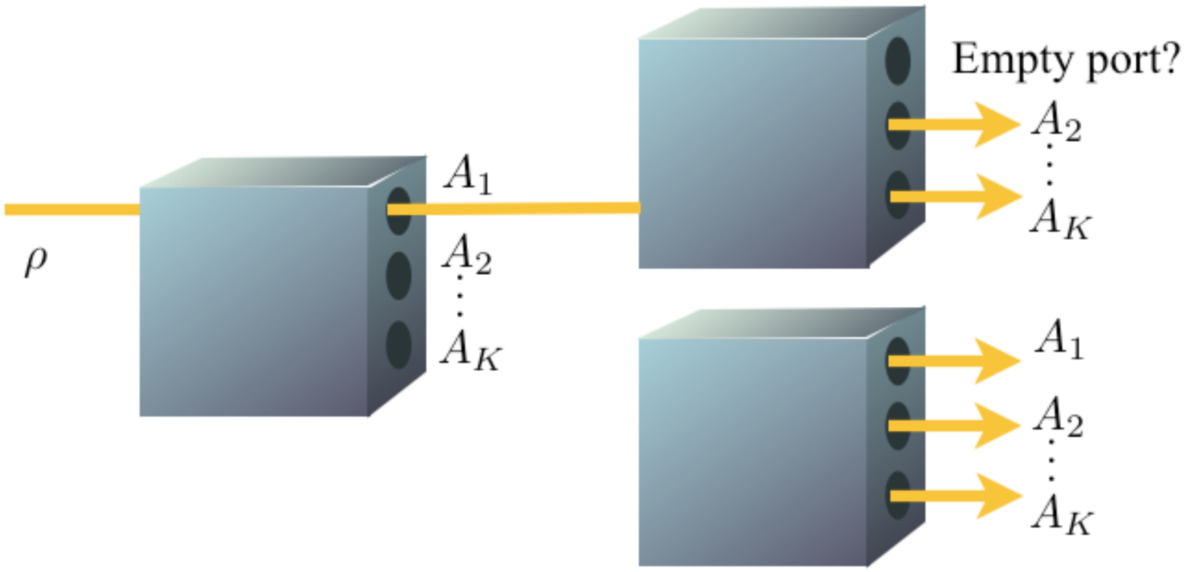

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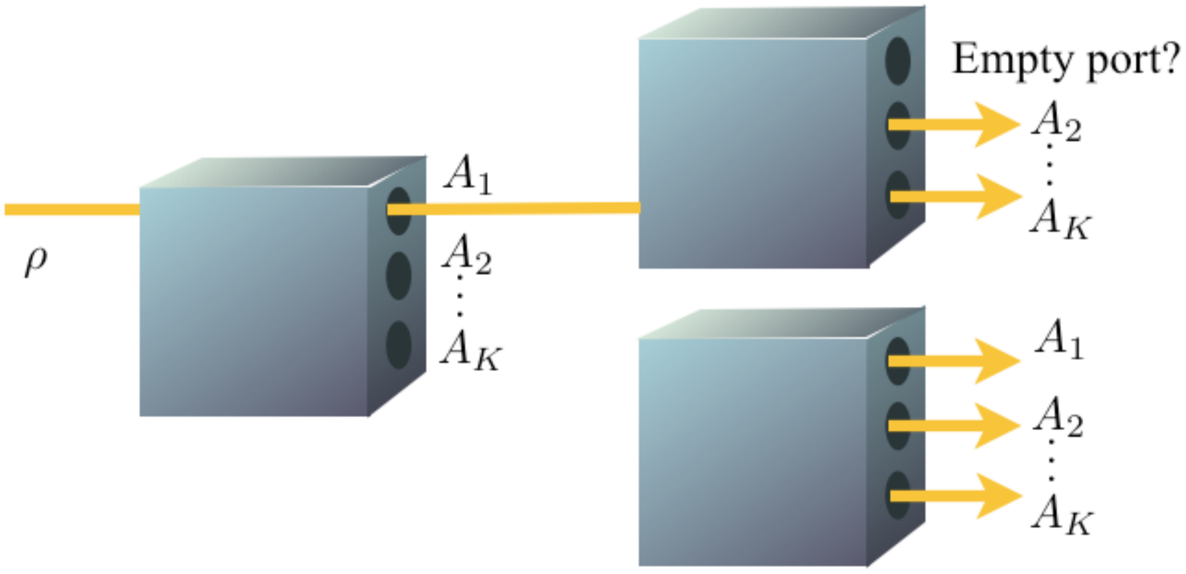

Quantum Measurement occurrence problem (QMOP):
Given a description of a quantum measurement device in terms of K Kraus operators $A_1, \ldots, A_K \in \mathbb{Q}^{d\times d}$, decide whether there is any finite sequence $j_1, \ldots, j_n$ which can never be observed, even if the input state has full rank.

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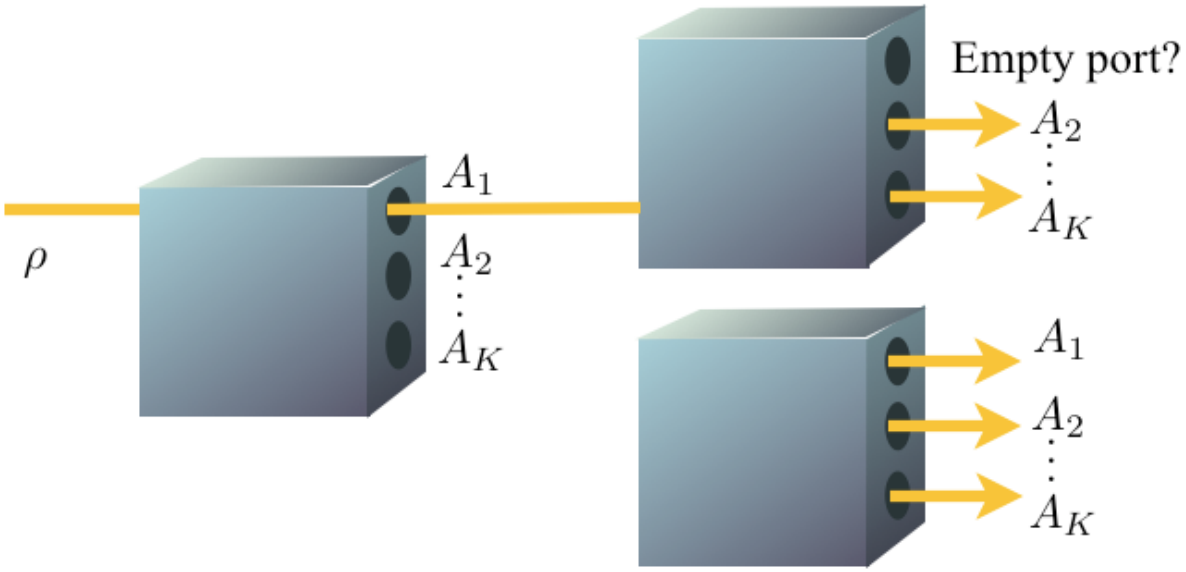

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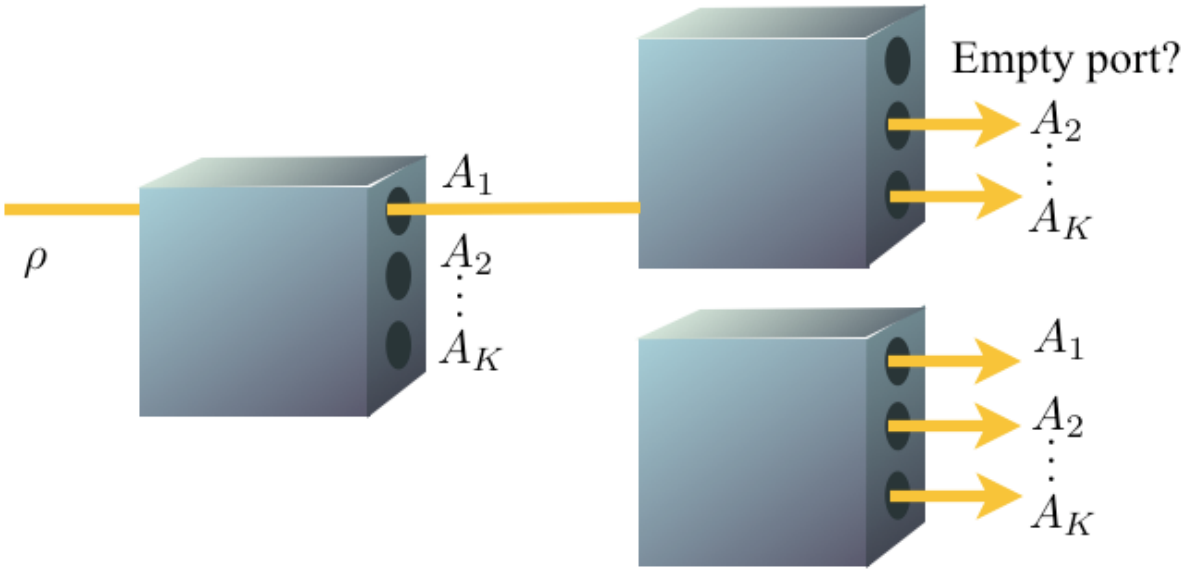

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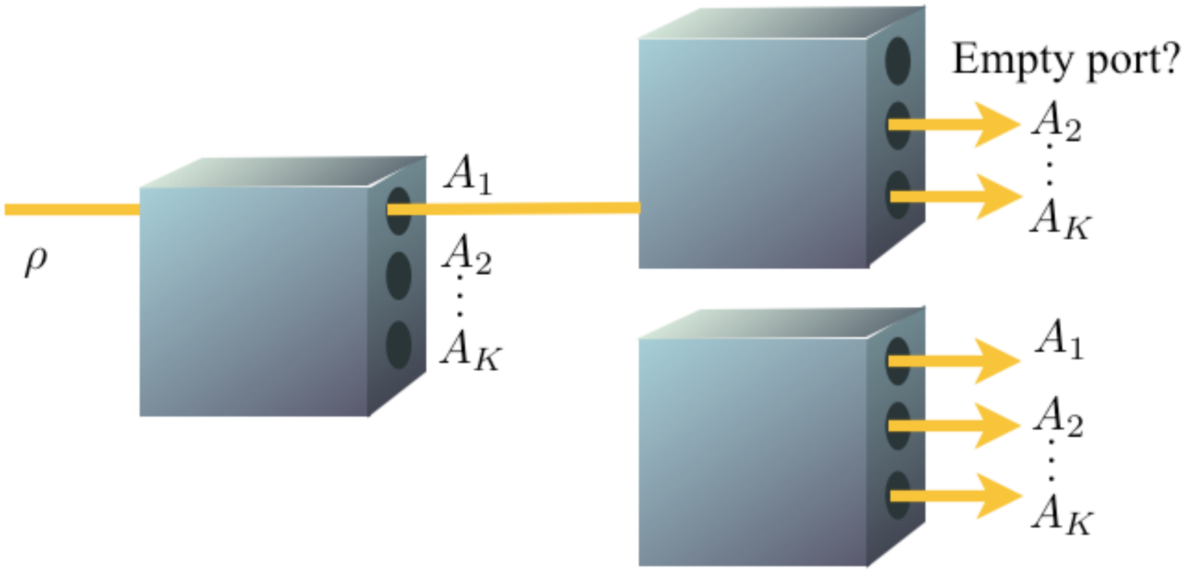

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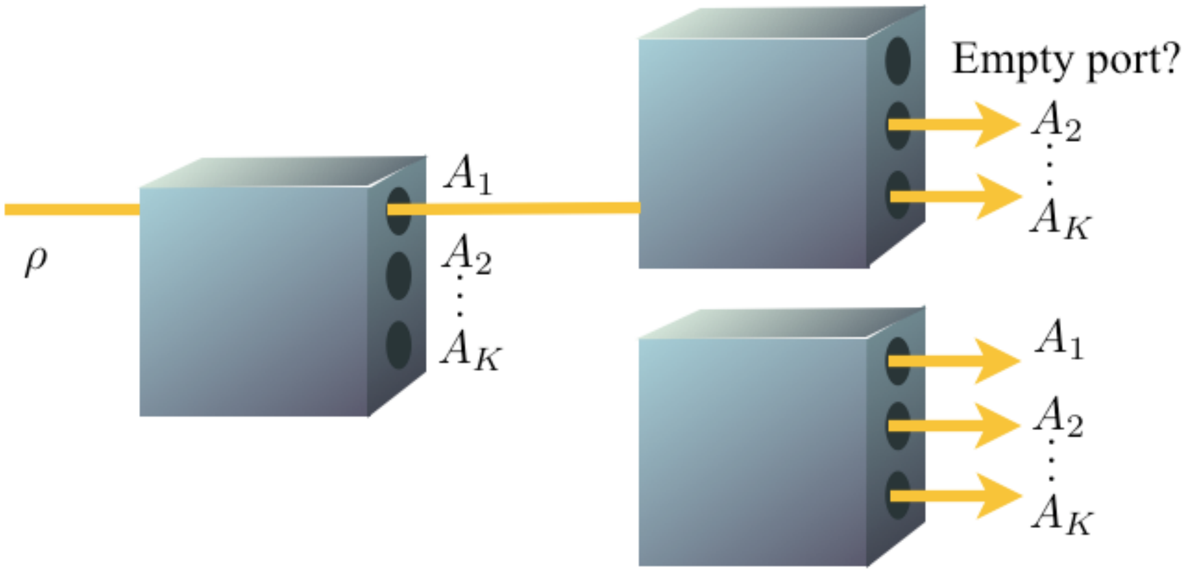

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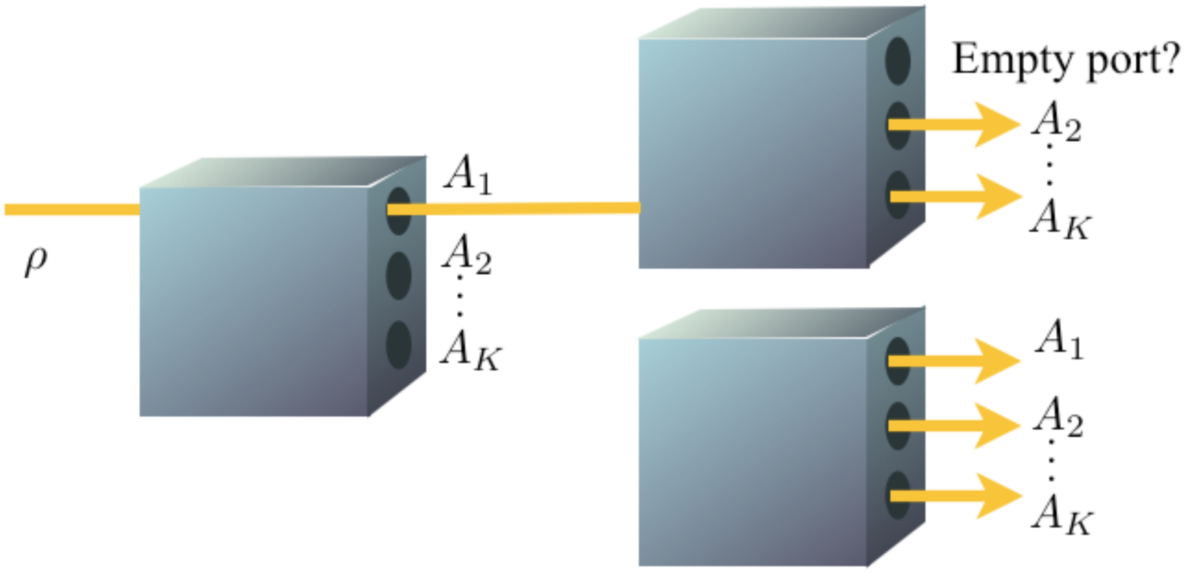

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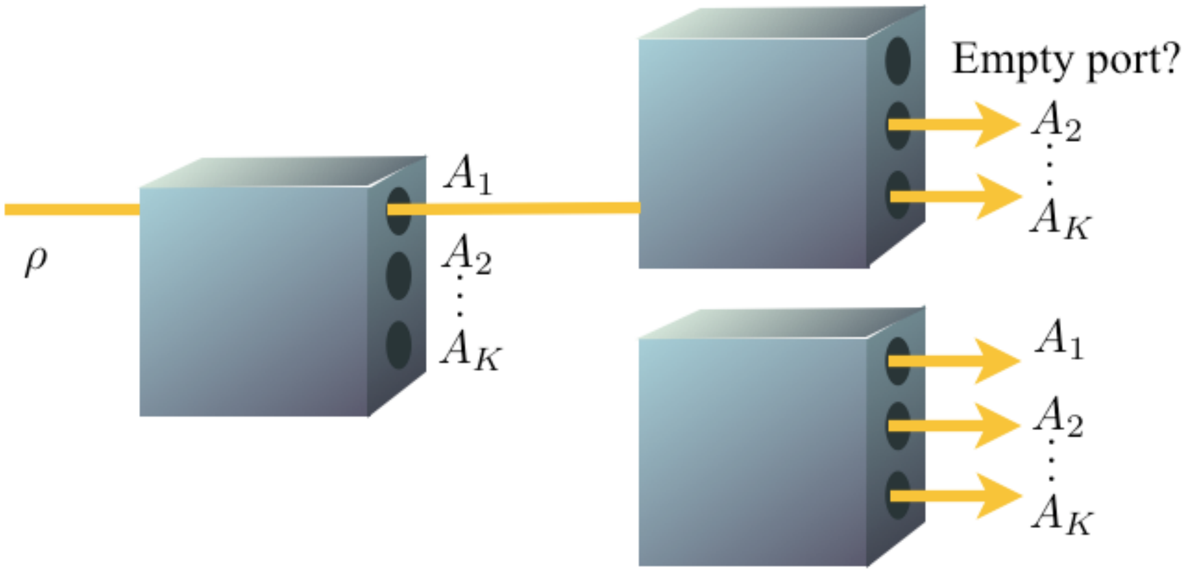

Classical measurement occurrence problem (CMOP):
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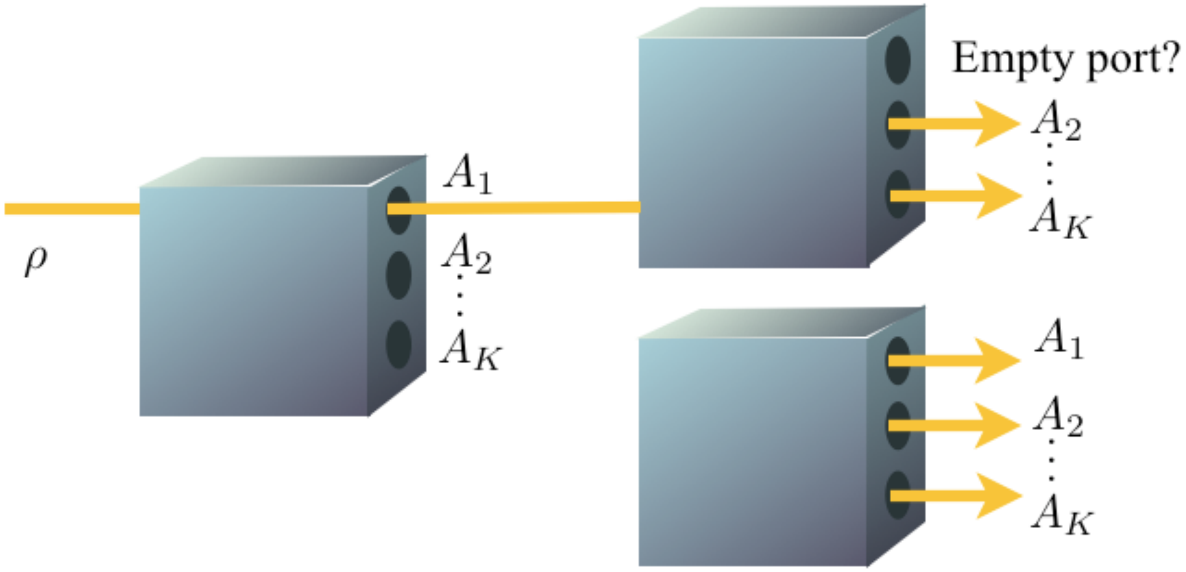

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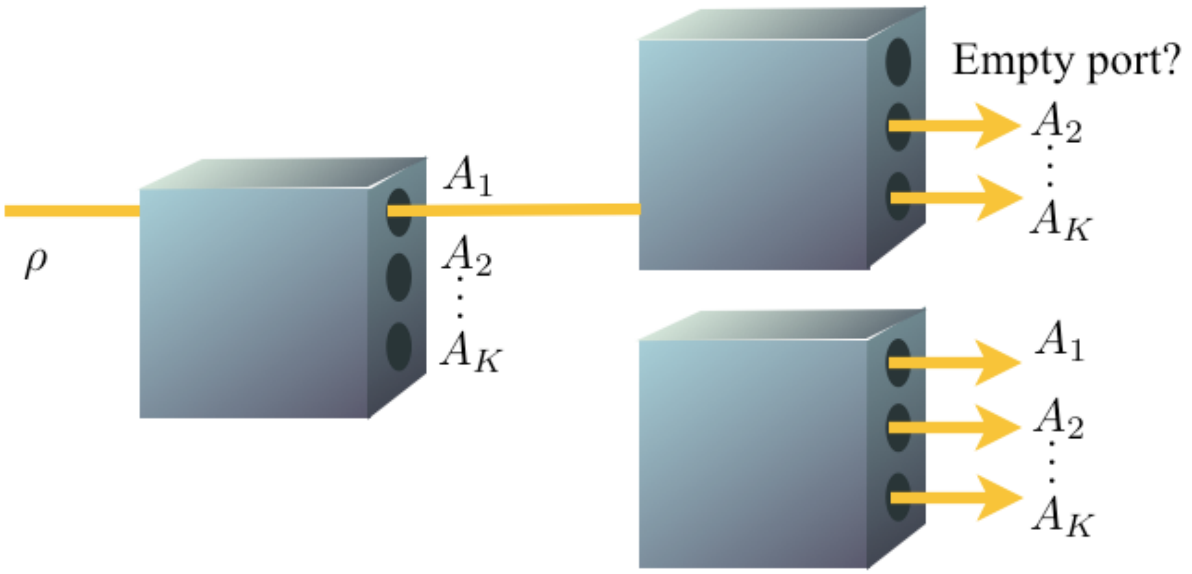

$$\mathrm{Prob}(j_1, \ldots, j_n) = \mathrm{Tr}(A_{j_n} \ldots A_{j_1} \rho A_{j_1}^\dagger \ldots A_{j_n}^\dagger) = 0$$

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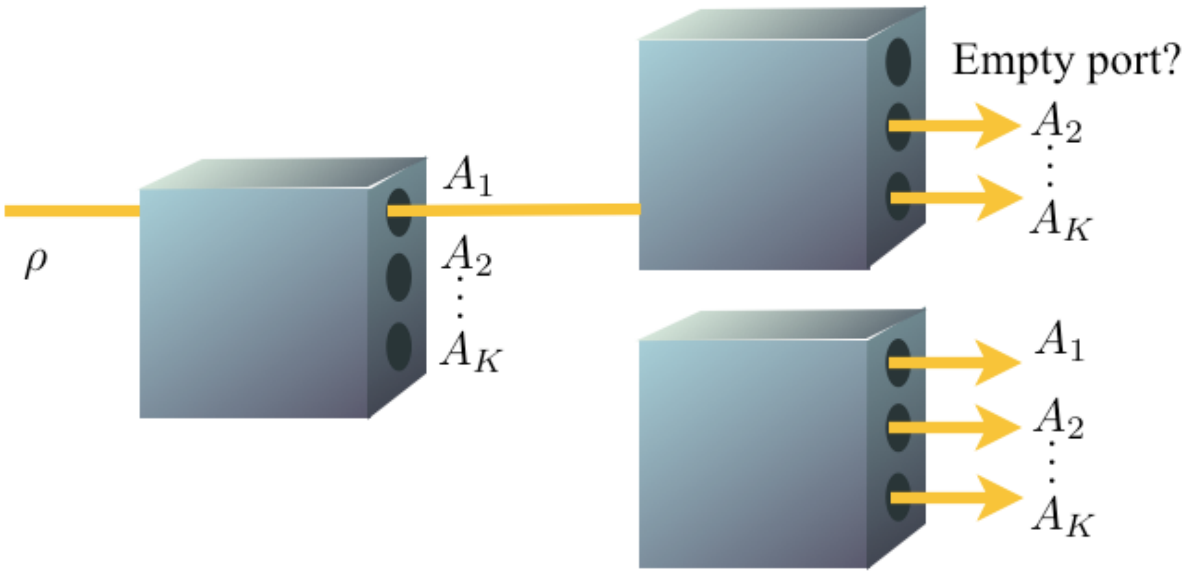

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Instance of the matrix mortality problem!

Undecidability of MMP $\Rightarrow$ undecidability of QMOP **?**

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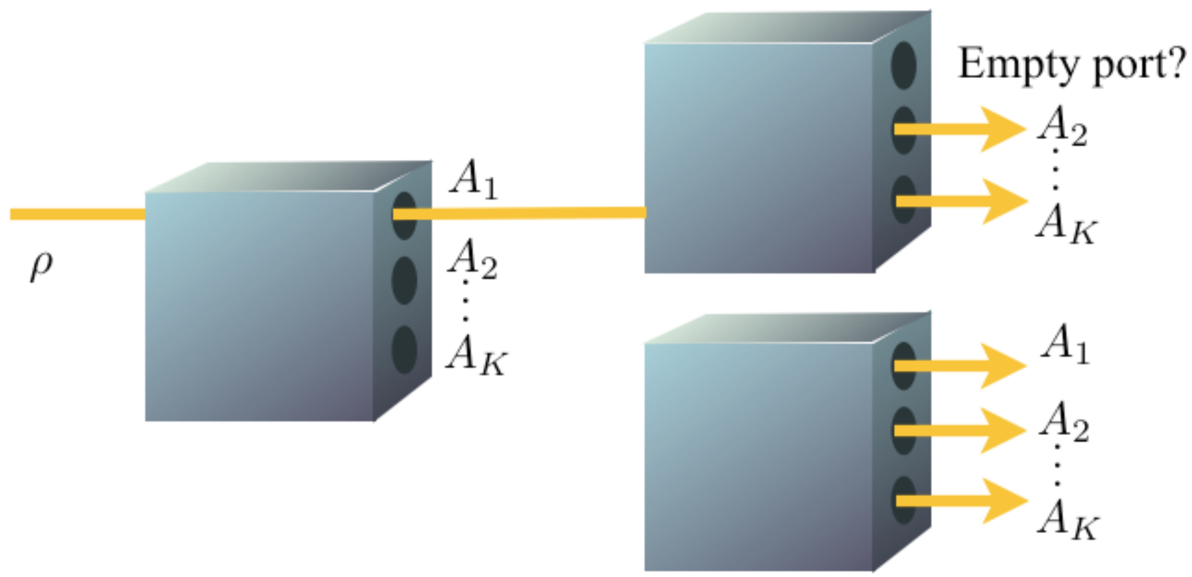

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Not quite! Normalization $\sum_j A_j^\dagger A_j = \mathbf{1}$ gives additional information.

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Encoding MMP-instances into QMOP:

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- MMP undecidable already for eight integer 3x3 matrices.
- Take $\{M_1, \ldots, M_8\} \subset \mathbb{Z}^{3\times 3}$, then $T := \sum_{j=1}^{8} M_j^\dagger M_j \neq \mathbf{1}$.

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- Take $\{M_1, \ldots, M_8\} \subset \mathbb{Z}^{3\times 3}$, then $T := \sum_{j=1}^{8} M_j^\dagger M_j \neq \mathbf{1}$.
- First, add some more matrices:

$$P_1 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, P_2 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, P_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix},$$

$$M_{8+j} = M_j P_1, \quad M_{16+j} = M_j P_2, \quad M_{24+j} = M_j P_3.$$

$$\Rightarrow \sum_{j=1}^{32} M_j^\dagger M_j = \begin{pmatrix} 4T_{11} & 0 & 0 \\ 0 & 4T_{22} & 0 \\ 0 & 0 & 4T_{33} \end{pmatrix}.$$

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$$+ \underbrace{\begin{pmatrix} ? & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & ? \end{pmatrix}}_{M_{35}^\dagger M_{35}} + \underbrace{\begin{pmatrix} ? & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & ? \end{pmatrix}}_{M_{36}^\dagger M_{36}} = \begin{pmatrix} c^2 & & \\ & c^2 & \\ & & c^2 \end{pmatrix}$$

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$$\begin{pmatrix} \boxed{4T_{11}} & 0 & 0 \\ 0 & 4T_{22} & 0 \\ 0 & 0 & 4T_{33} \end{pmatrix} + \underbrace{\begin{pmatrix} \blacksquare & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & ? \end{pmatrix}}_{M_{33}^\dagger M_{33}} + \underbrace{\begin{pmatrix} \blacksquare & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & ? \end{pmatrix}}_{M_{34}^\dagger M_{34}}$$

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Lagrange: $c^2 - 4T_{ii}$ can be written as sum of four integer squares!

3. Undecidability of the quantum problem (QMOP)

Encoding MMP-instances into QMOP:

$$\Rightarrow \sum_{j=1}^{\mathbf{36}} M_j^\dagger M_j = c^2 \mathbf{1}.$$

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$$\underbrace{A_j}_{j=1,\ldots,8} := \frac{4}{5c} \left[\begin{array}{c|c} M_j & \\ M_{8+j} & \\ M_{16+j} & 0_{15\times 12} \\ M_{24+j} & \\ M_{32+j} & \end{array} \right], \quad A_9 := \frac{3}{5} \mathbf{1}_3 \oplus \mathbf{1}_{12}. \Rightarrow \sum_{j=1}^{9} A_j^\dagger A_j = \mathbf{1}.$$

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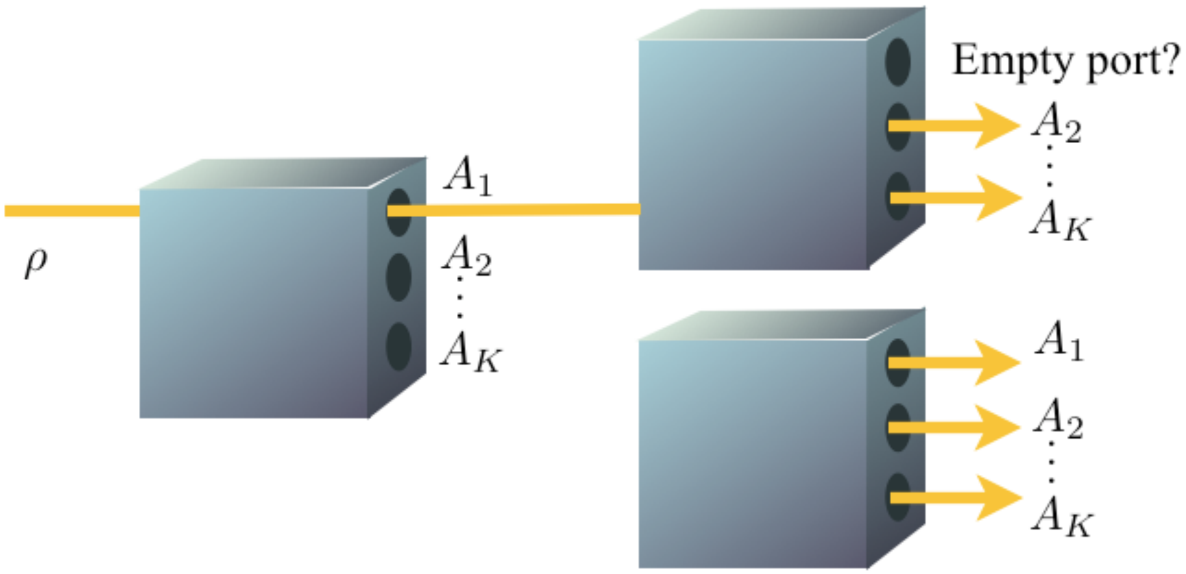

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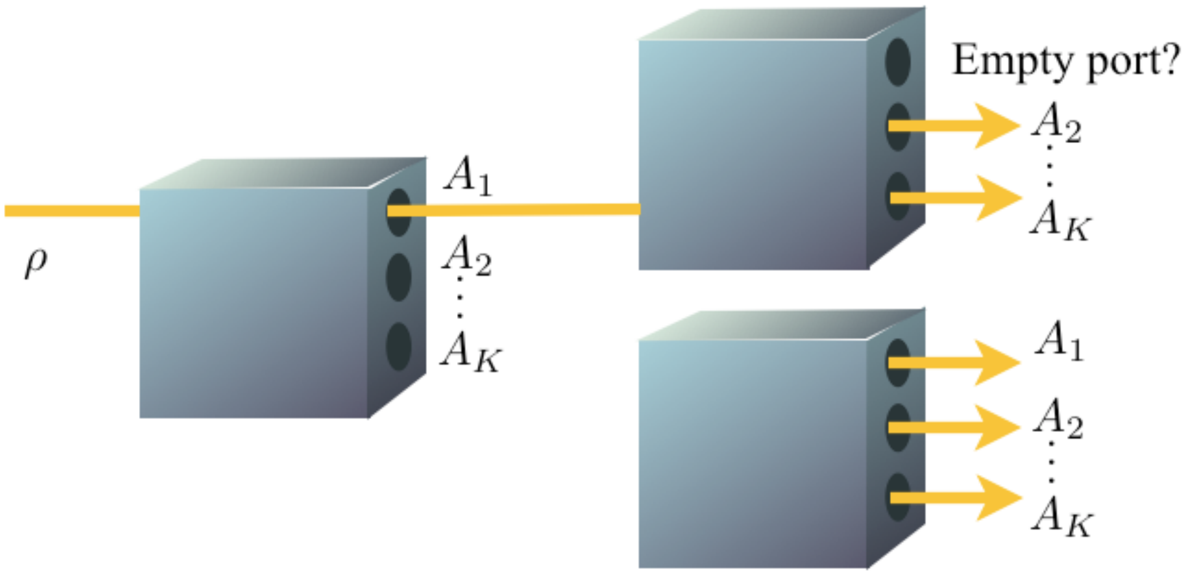

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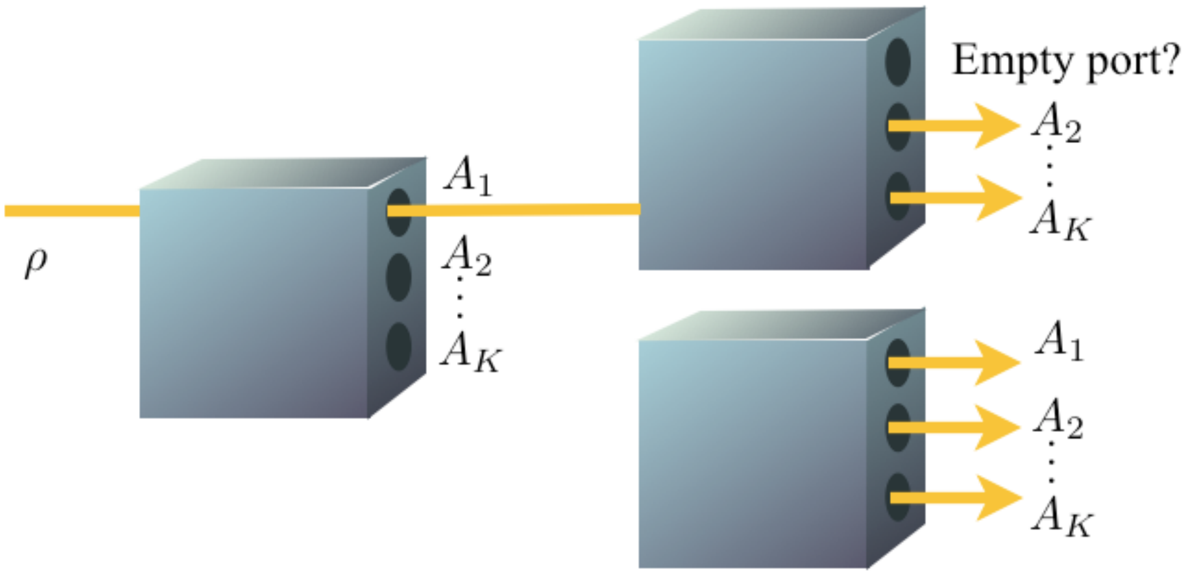

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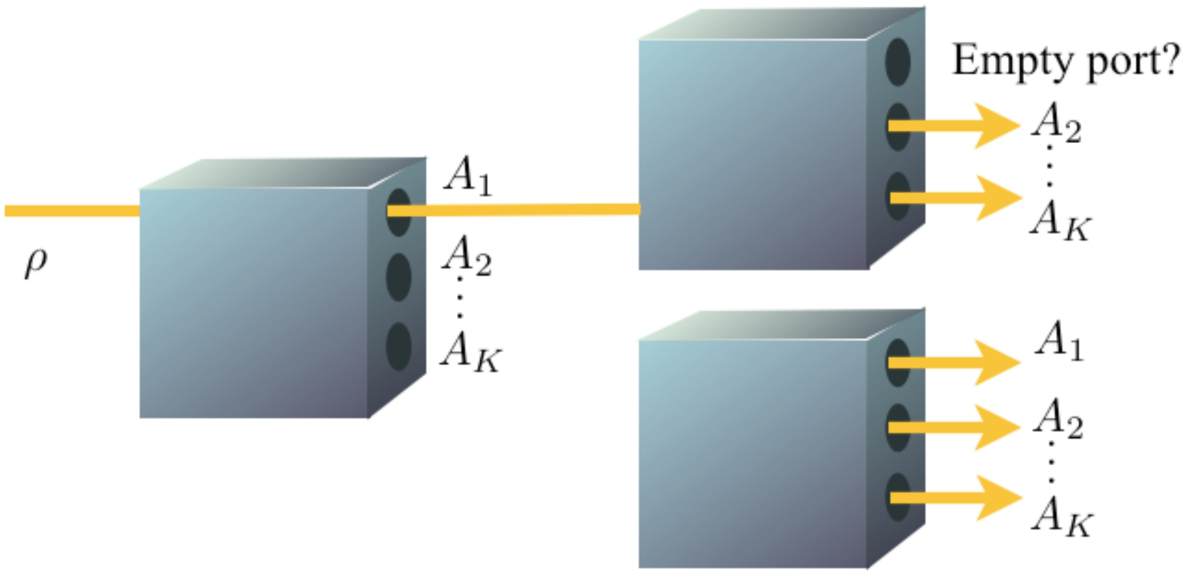

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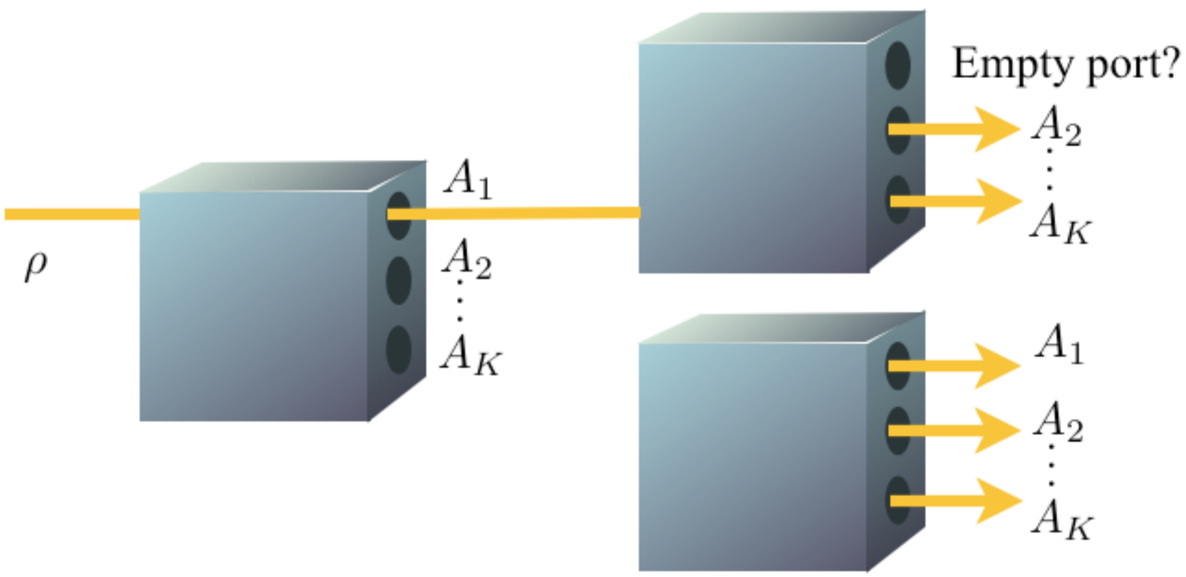

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-decidable-

Summary: quantum cs. classical MOP

Quantum MOP	Classical MOP
MMP	$MMP_{\geq 0}$
Destructive interference	Only constructive interference
undecidable	decidable

5. Outlook

Are further natural quantum problems undecidable?

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Paradigm of a non-computable number: Chaitin‘s Omega.
Let U be a prefix-free universal Turing machine. Set

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Let *U* be a prefix-free universal Turing machine. Set

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- There is an algorithm which, on input *n*, computes an approximation Ω_n such that $\Omega_n \leq \Omega_{n+1}$ and $\lim_{n\to\infty} \Omega_n = \Omega.$
- But: There is *no* algorithm which, on input *n*, computes an approximation Ω'_n such that $|\Omega - \Omega'_n| < 1/n.$ Ω is not computable.

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$$C(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n}\chi(\mathcal{N}^{\otimes n})$$

where $\chi(\mathcal{M}) = \max_{p_i,\varphi_i}\left[S\left(\mathcal{M}\left(\sum_i p_i|\varphi_i\rangle\langle\varphi_i|\right)\right) - \sum_i p_i S\left(\mathcal{M}(|\varphi_i\rangle\langle\varphi_i|)\right)\right]$

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The quest for a single-letter formula:

- <2008: maybe $C(\mathcal{N}) = \chi(\mathcal{N})$?
- Hastings 2008: no!

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c_n is a computable, increasing sequence with $\lim_{n\to\infty} c_n = C(\mathcal{N})$.

But: maybe $C(\mathcal{N})$ is not computable in general?

This would prove - once and for all - that there cannot be any single-letter formula.

Conclusions

- Undecidability in quantum measurements:

Quantum MOP ↕ MMP ↕ Destructive interference: undecidable

Classical MOP ↕ $MMP_{\geq 0}$ ↕ Only constructive interference: decidable

- Speculation: are quantum channel capacities noncomputable?

arXiv:1111.3965