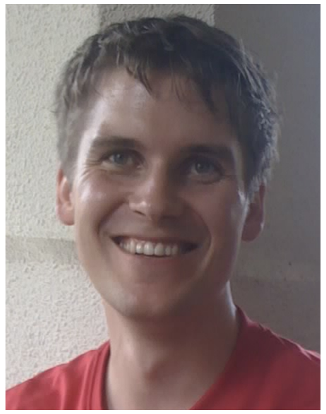
Typical entanglement, coin tossing, and a general-probabilistic decoupling theorem

Markus Müller
Perimeter Institute for Theoretical Physics, Waterloo (Canada)



Jointly with Oscar Dahlsten, Vlatko Vedral
arXiv: 1007.6029

Part on black holes with Jonathan Oppenheim

Outline

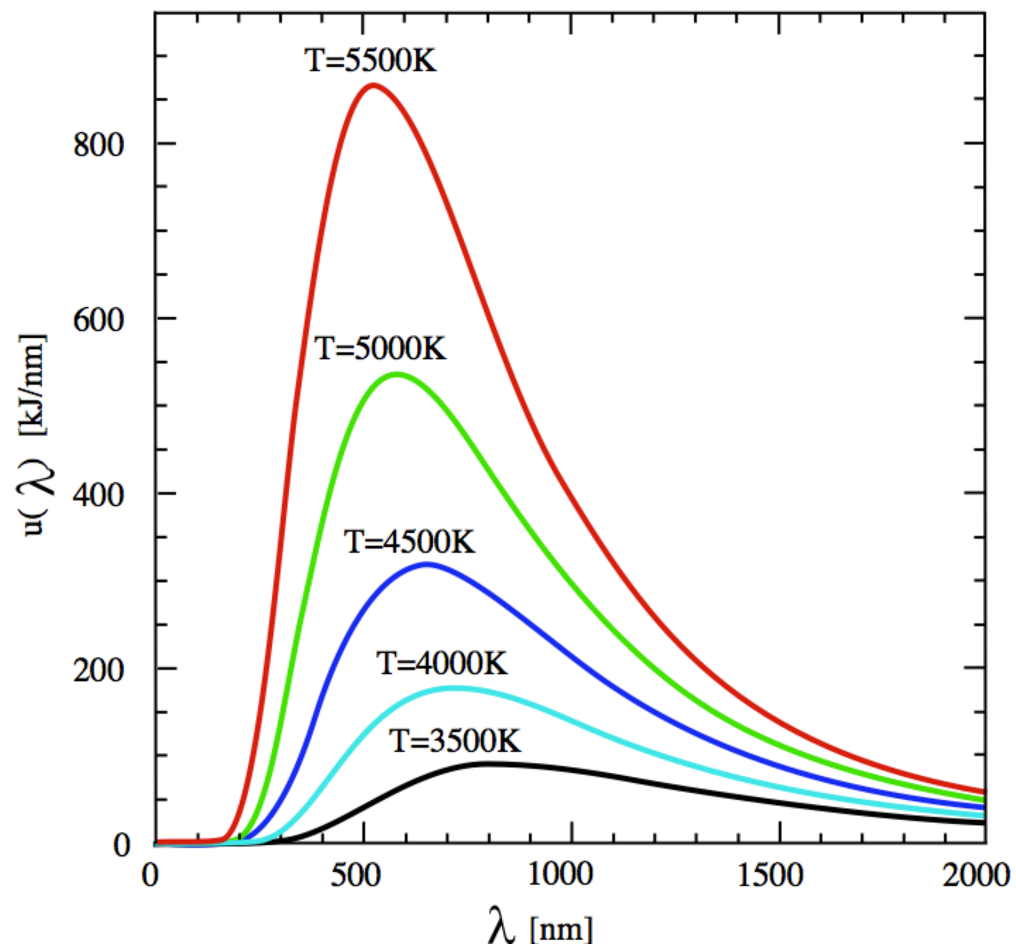
1. Some motivation... and history

2. Entanglement and statistical physics

3. Purity in dynamical state spaces

4. Decoupling and black-hole thermodynamics

1. Some motivation... and history



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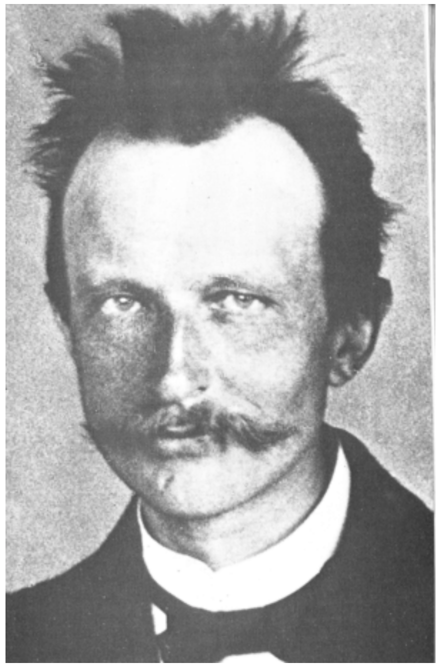
Max Planck derived correct law $\rho(\nu,T) = \frac{8\pi h\nu^3}{c^3} / \left(\exp[h\nu/(kT)] - 1 \right)$



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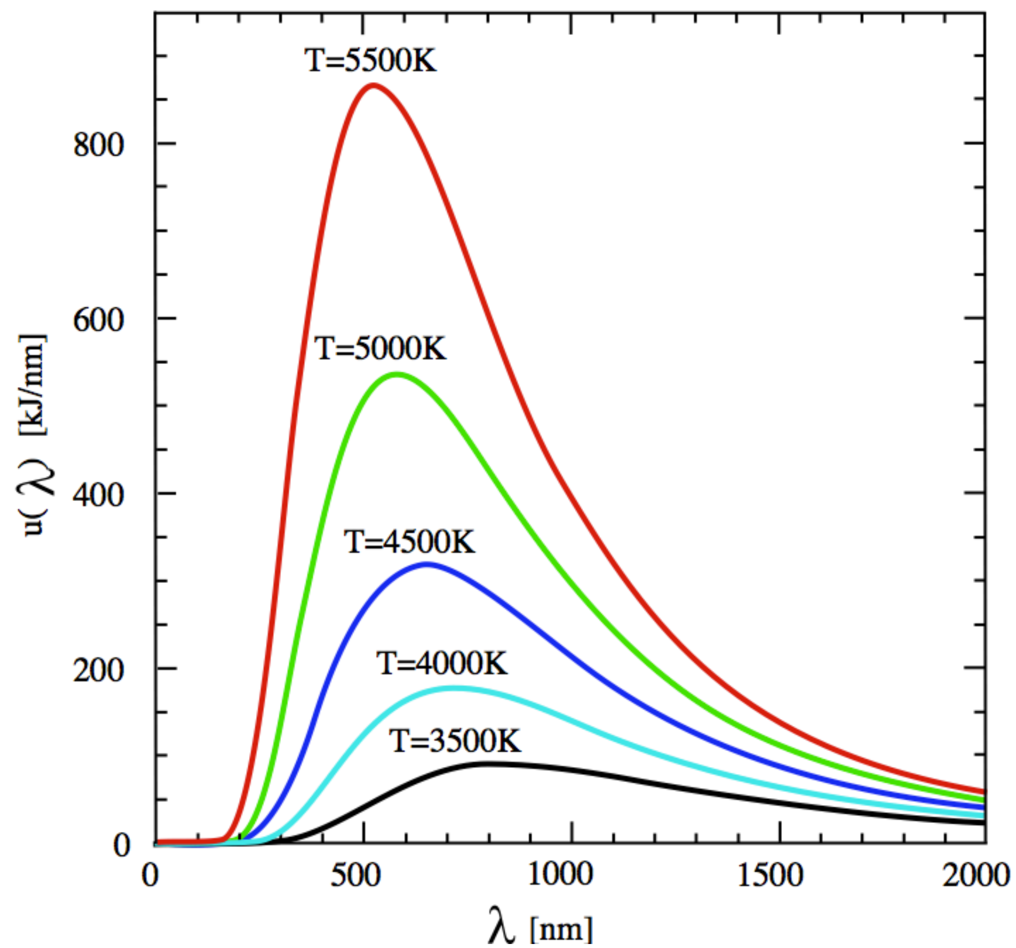

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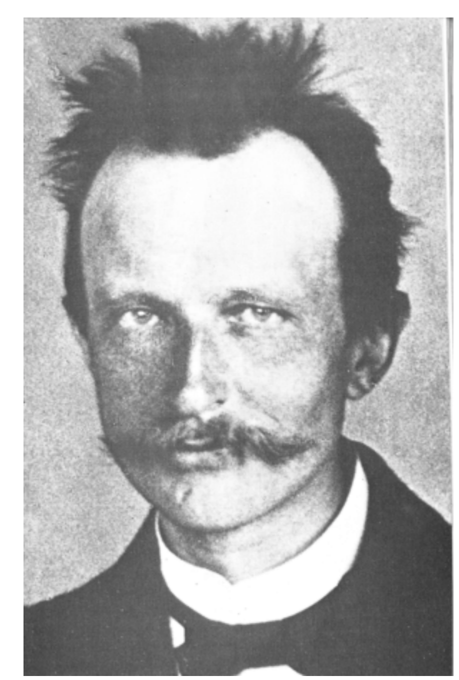

Main assumption: Thermodynamics remains valid in the realm of new physics.

This + quantization of energy assumption produce correct formula.

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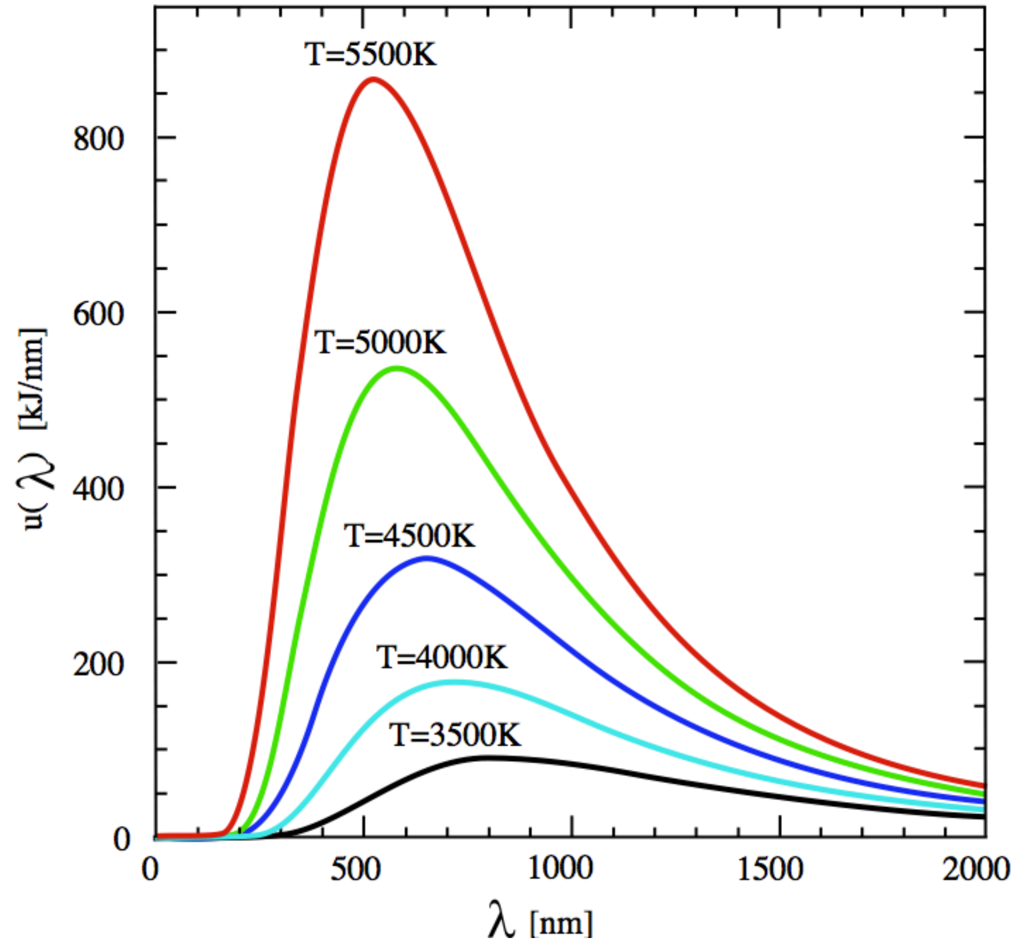

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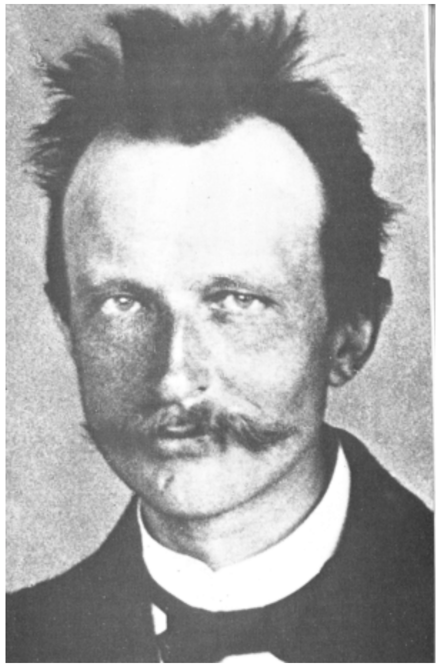

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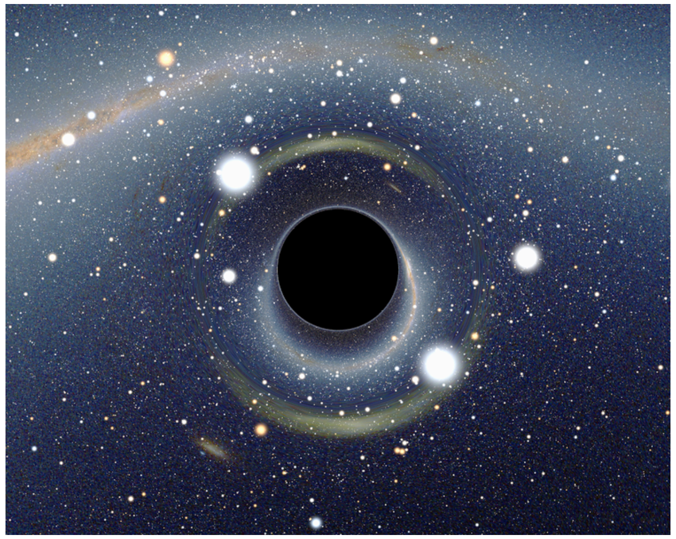
Nowadays: quantum gravity.
Black hole thermodynamics is used to get information on new physics.



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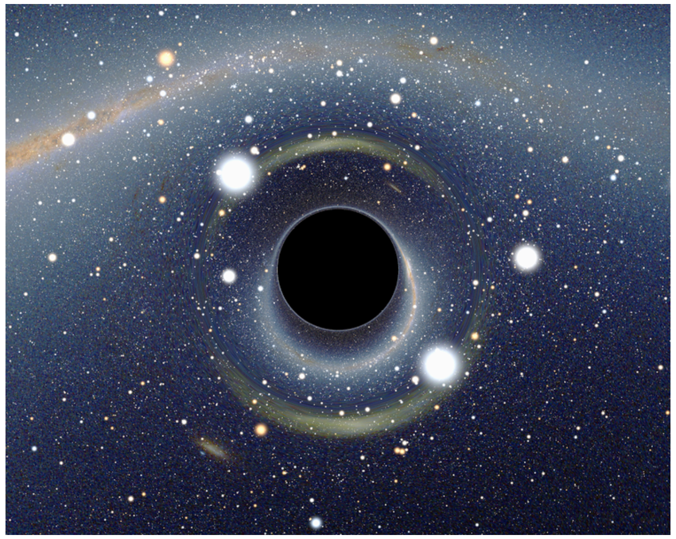

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How does statistical physics change?

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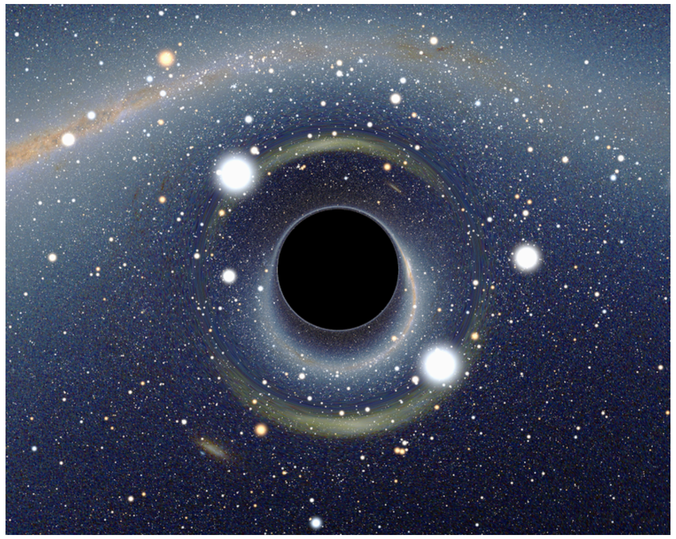
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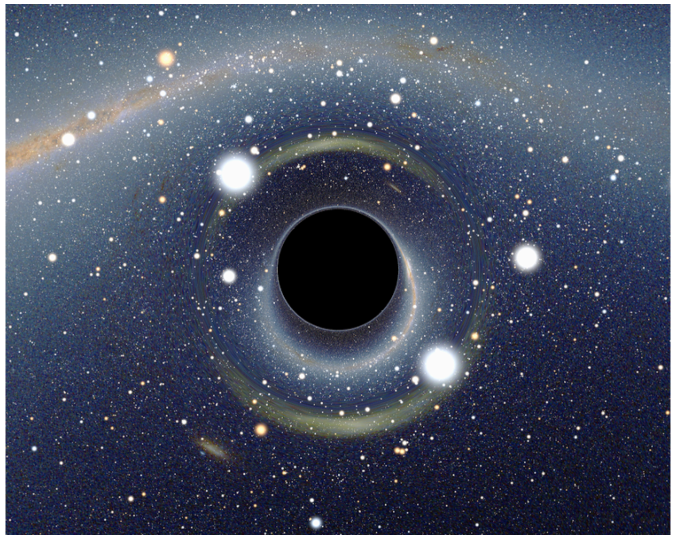

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How does statistical physics change?

In the meantime, we will also

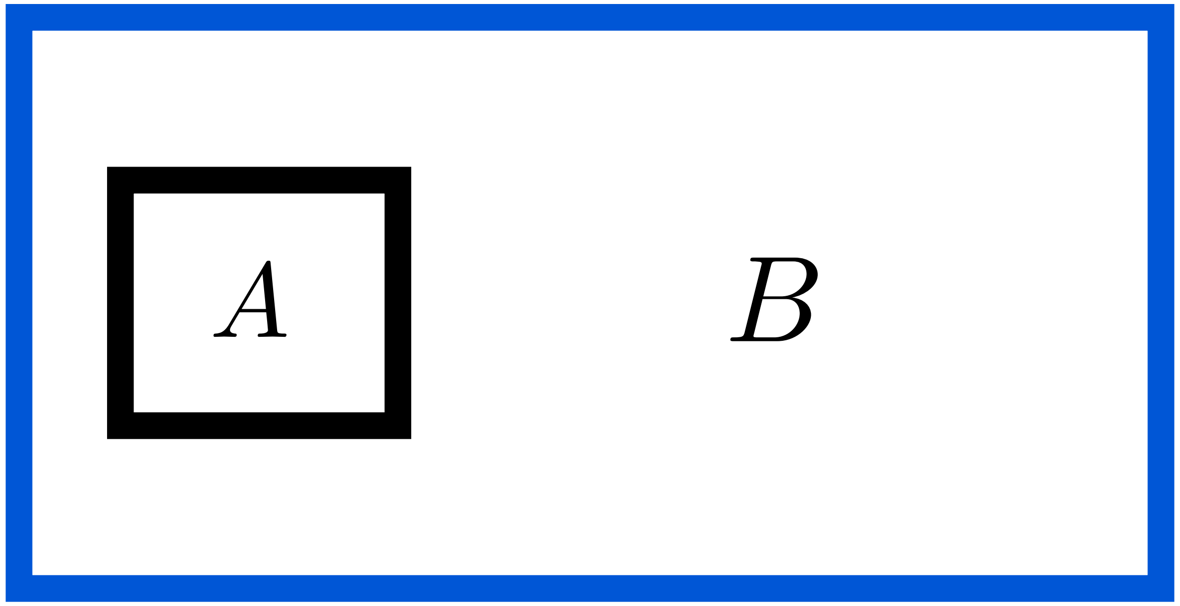
- get a unified view on typical entanglement and coin tossing,
- compute typ. ent. in (anti)symmetric subspaces,
- find a good entropy measure for general state spaces.

2. Entanglement and statistical physics

Why do systems thermalize despite global unitary evolution?

$$|\psi_{AB}(t)\rangle = e^{-iH_{AB}t}|\psi_{AB}(0)\rangle$$

$$\psi_A(t) := \mathrm{Tr}_B|\psi_{AB}(t)\rangle\langle\psi_{AB}(t)|$$



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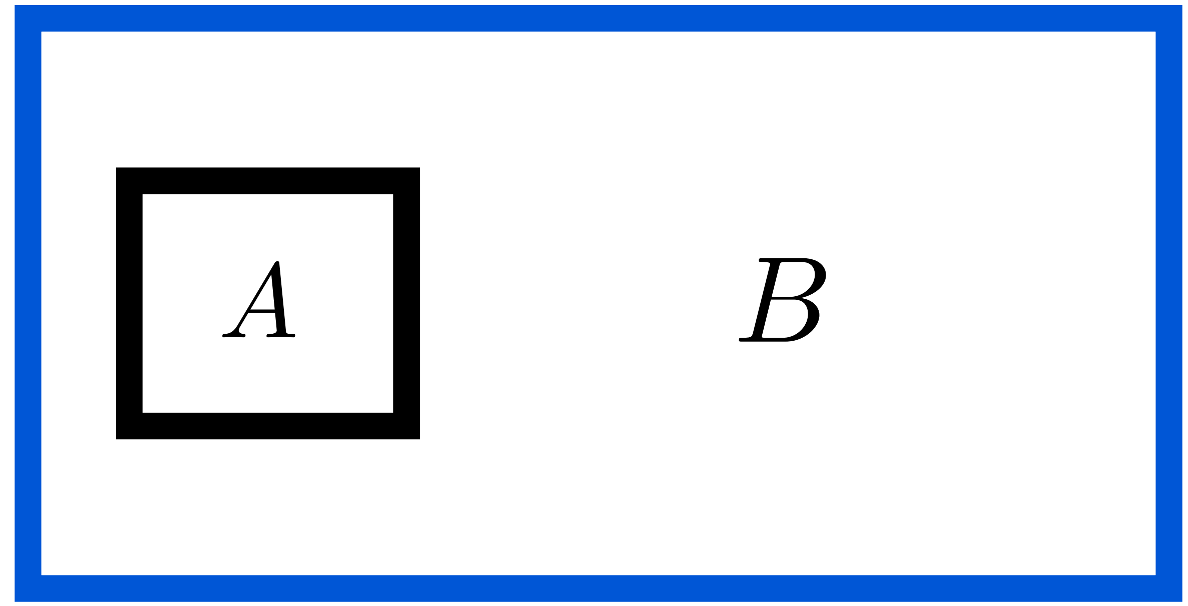

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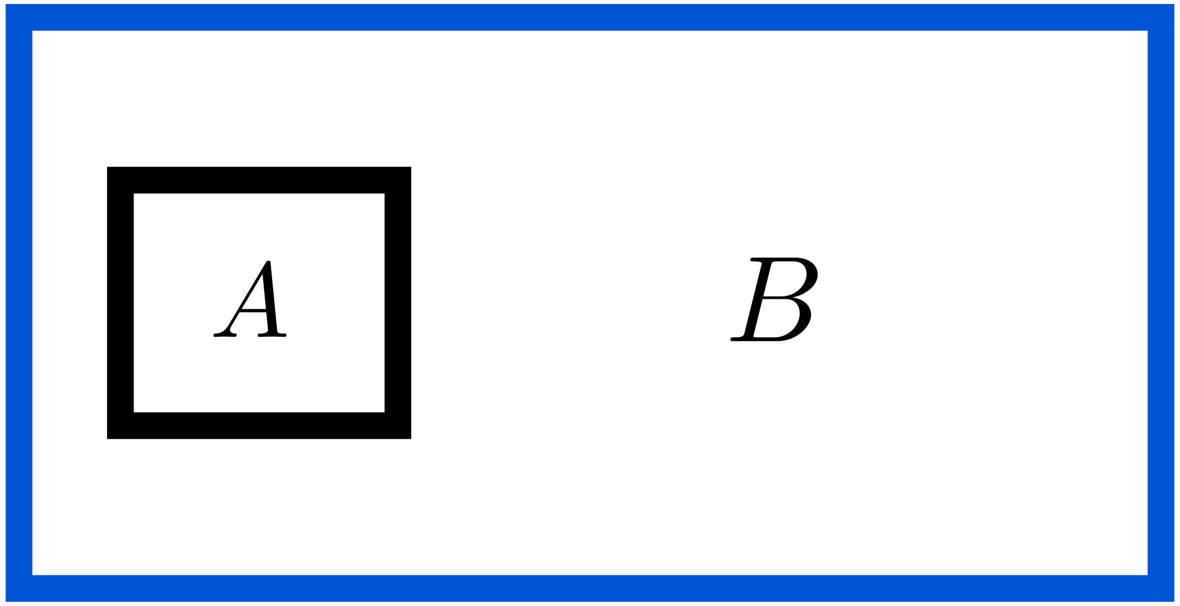

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Random states $|\psi\rangle \in AB$ typically

- look locally like statistical ensembles,
- are almost maximally entangled,
- have low local purity $\mathrm{Tr}(\psi_A^2)$.

2. Entanglement and statistical physics

Generalization: Constraint by subspace $|\psi\rangle \in R \subseteq AB.$

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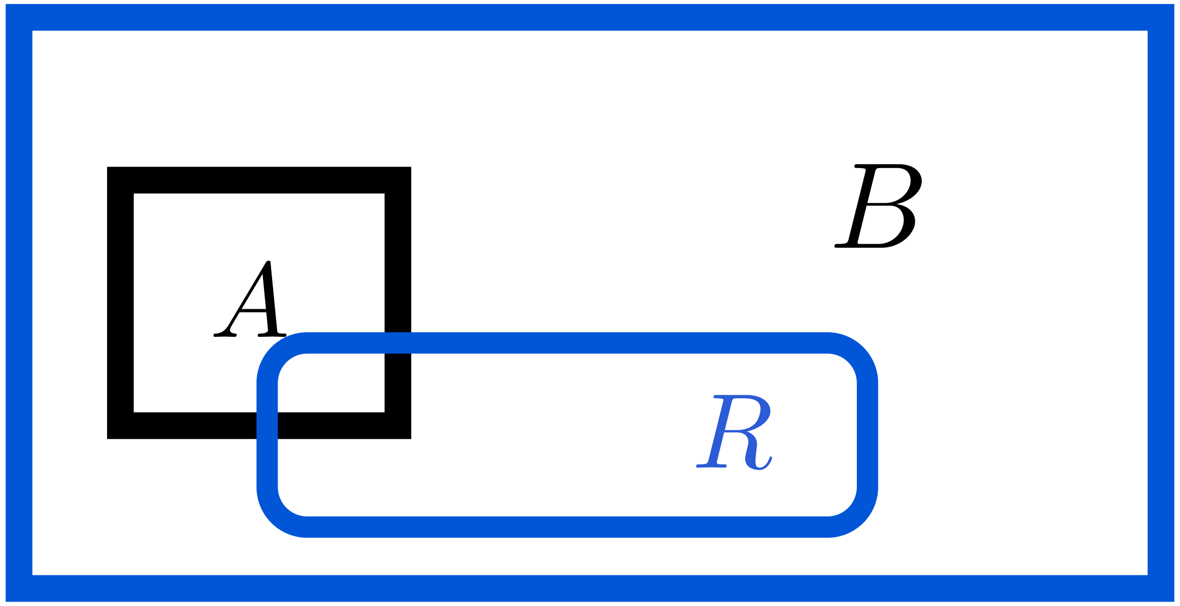

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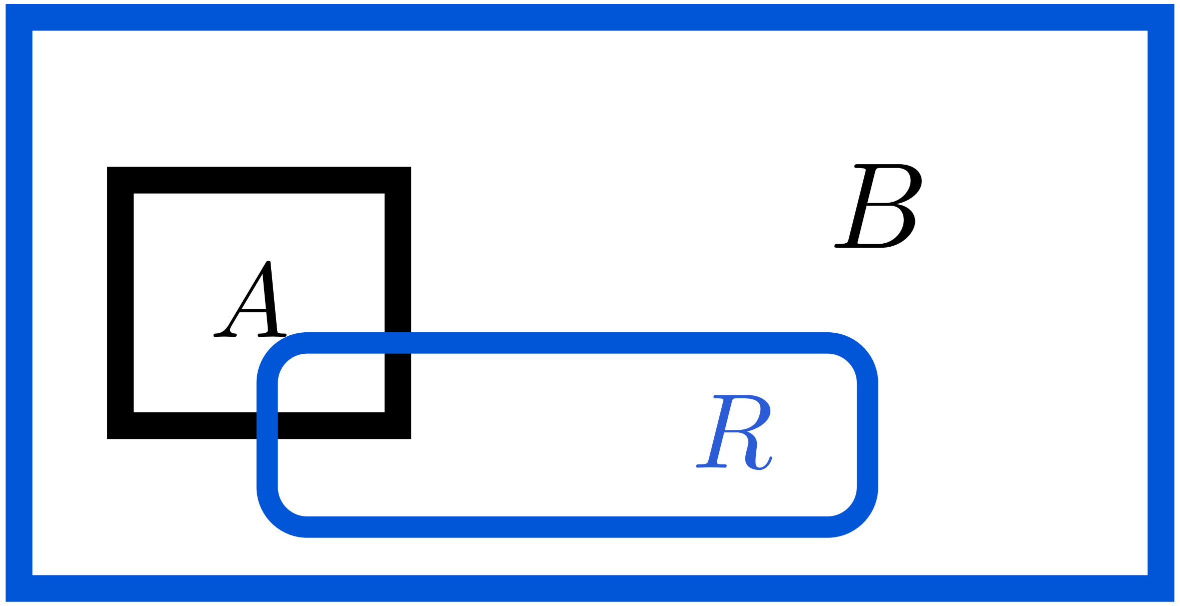

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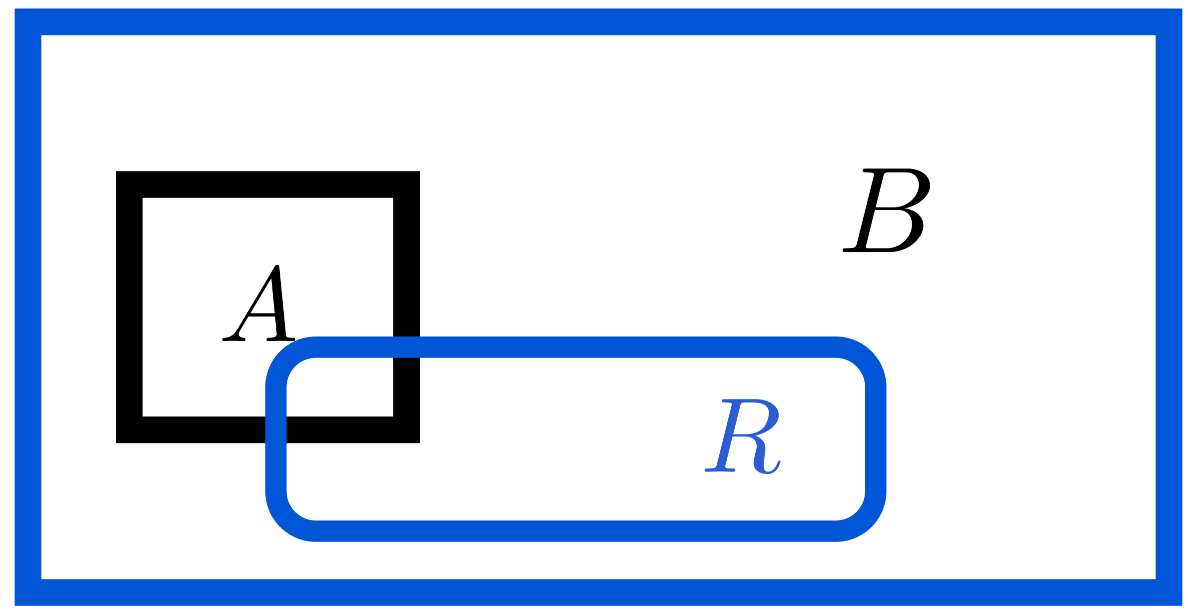

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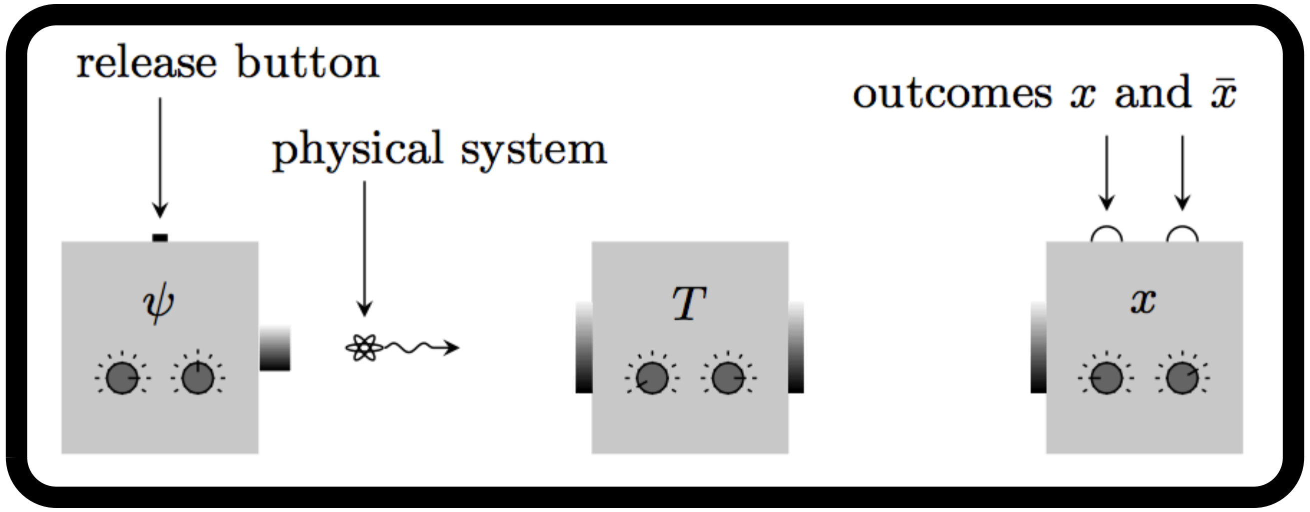
Now: What about typ. entanglement / local purity in general?

How is it in new probabilistic theories? Or old ones (classically)? ...

3. Purity in dynamical state spaces



(Unnormalized) state ω = list of all probabilities of „yes“-outcomes of all possible measurements.

$$\omega = (p_1, p_2, p_3, p_4, p_5, p_6, \ldots)$$

Sometimes, ω described by finitely many values. Example: Qubit

- What‘s the prob. of „spin up“ in X-direction?
- What‘s the prob. of „spin up“ in Y-direction?
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- Is the particle there at all?

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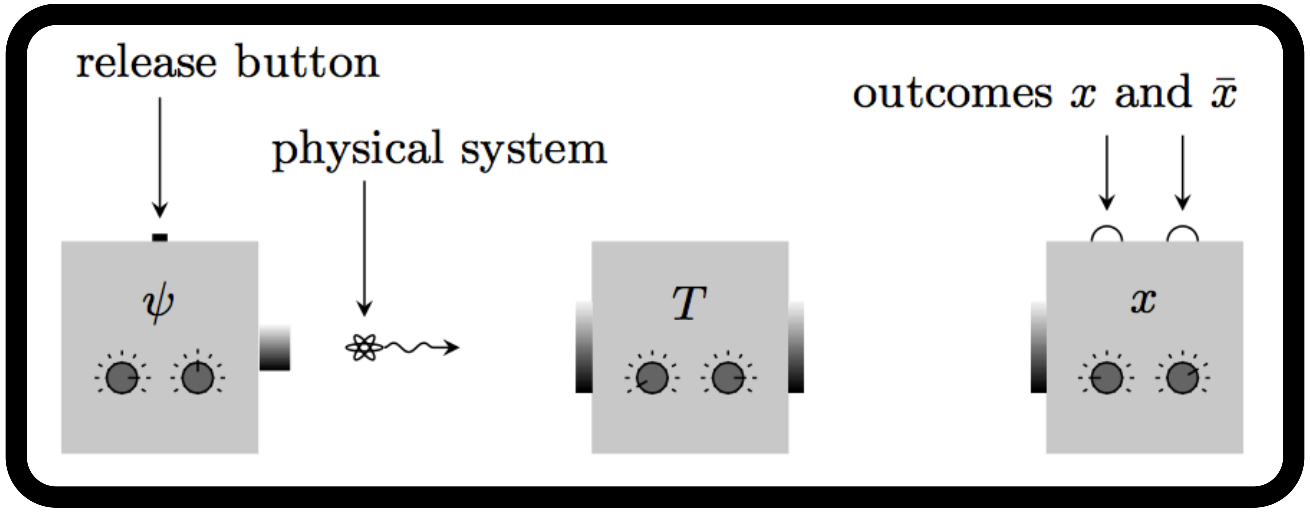

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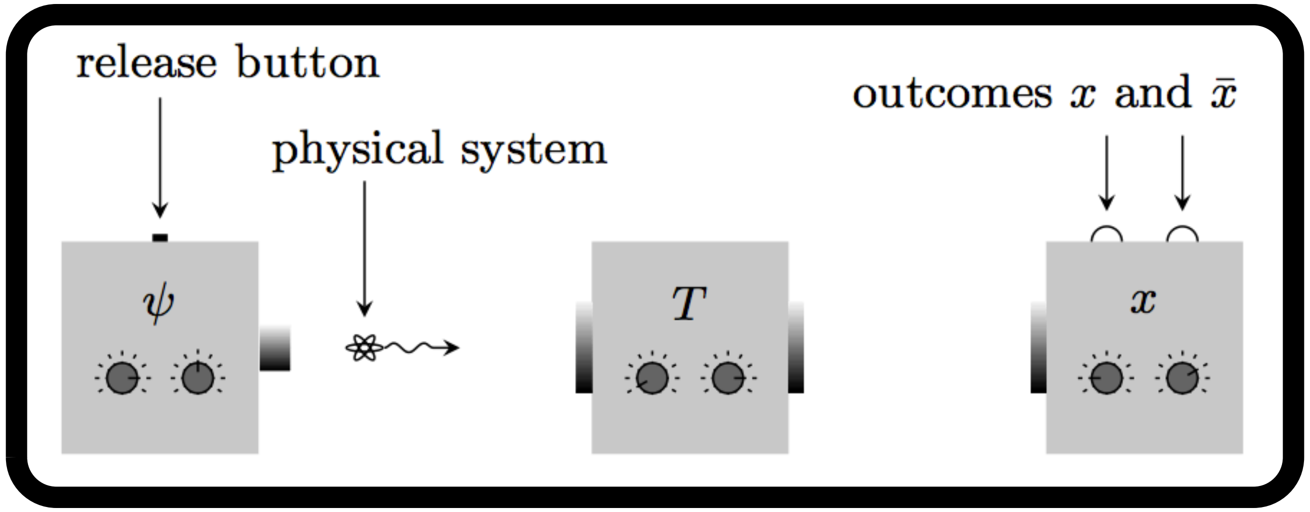

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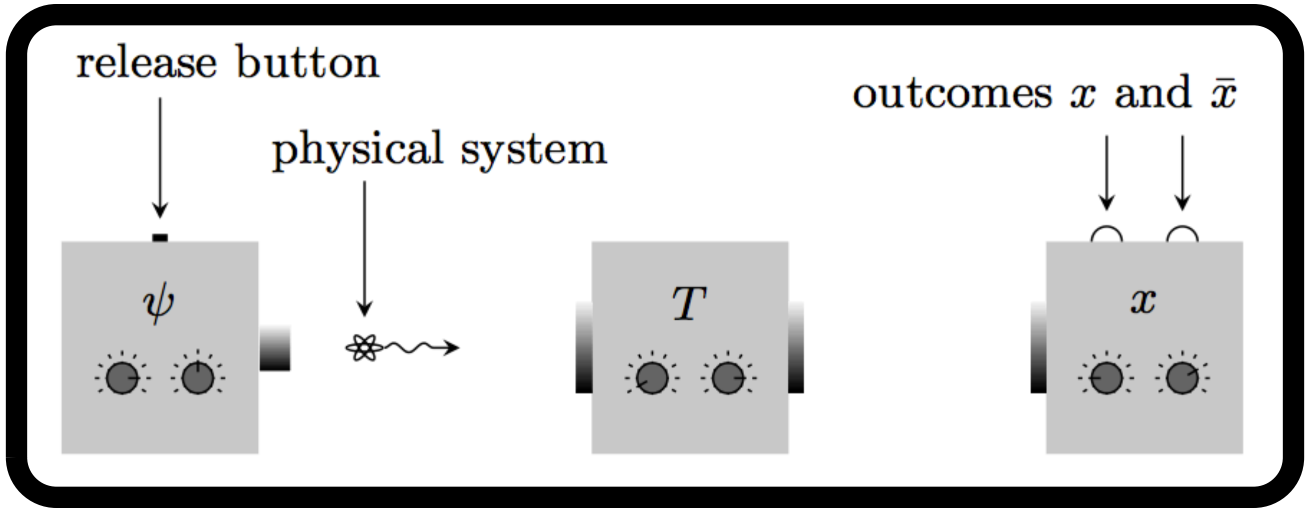

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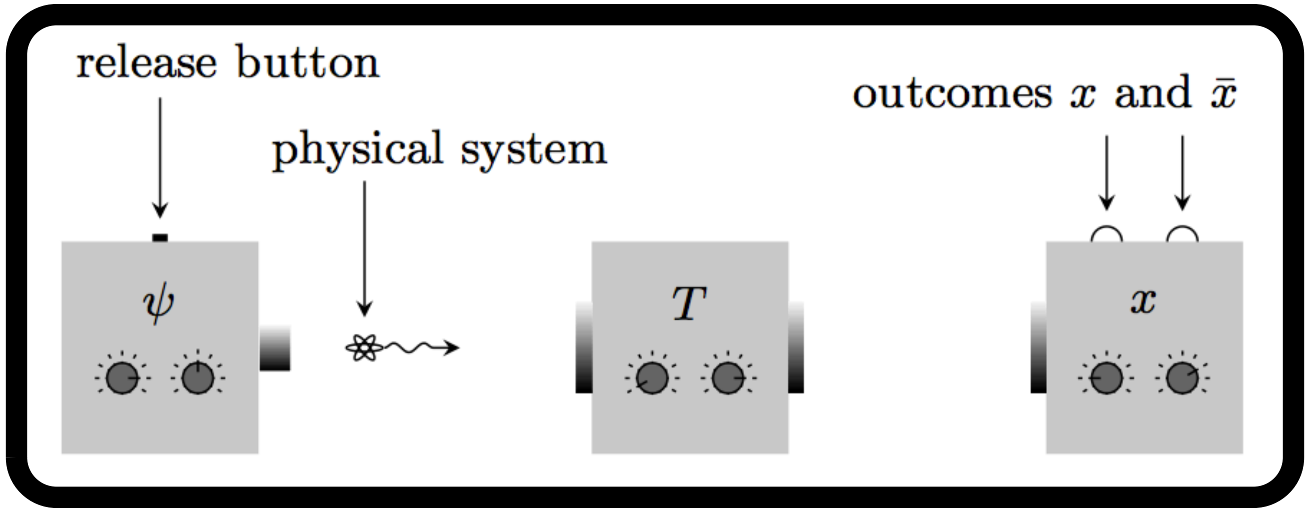

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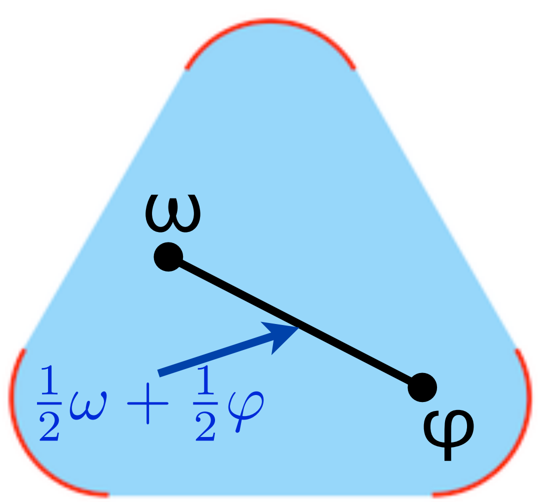

Normalized states are points in convex state spaces.
Extremal points are pure, others mixed.

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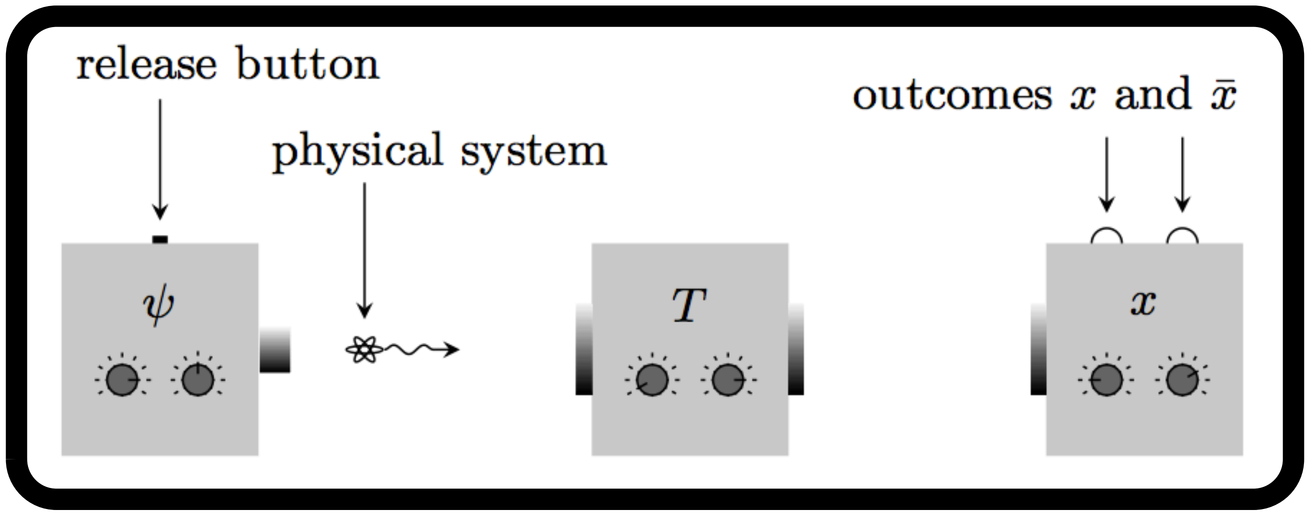

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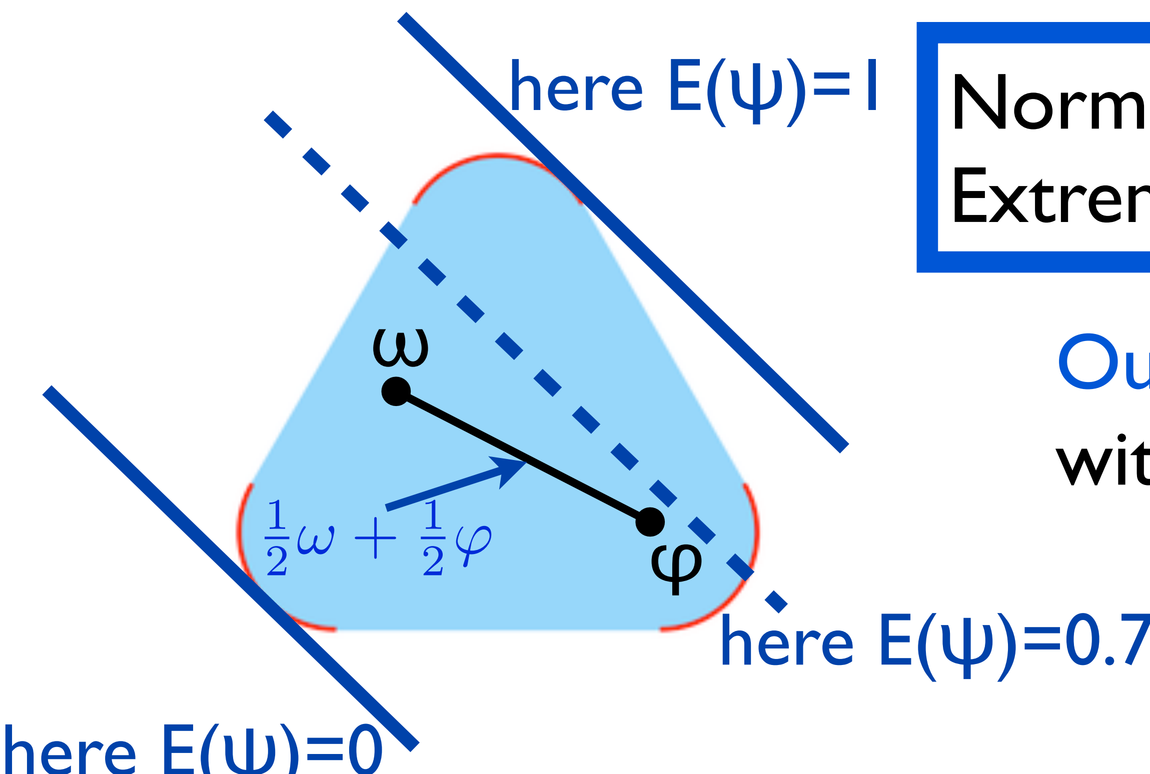

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Outcome probabilities are linear functionals *E*
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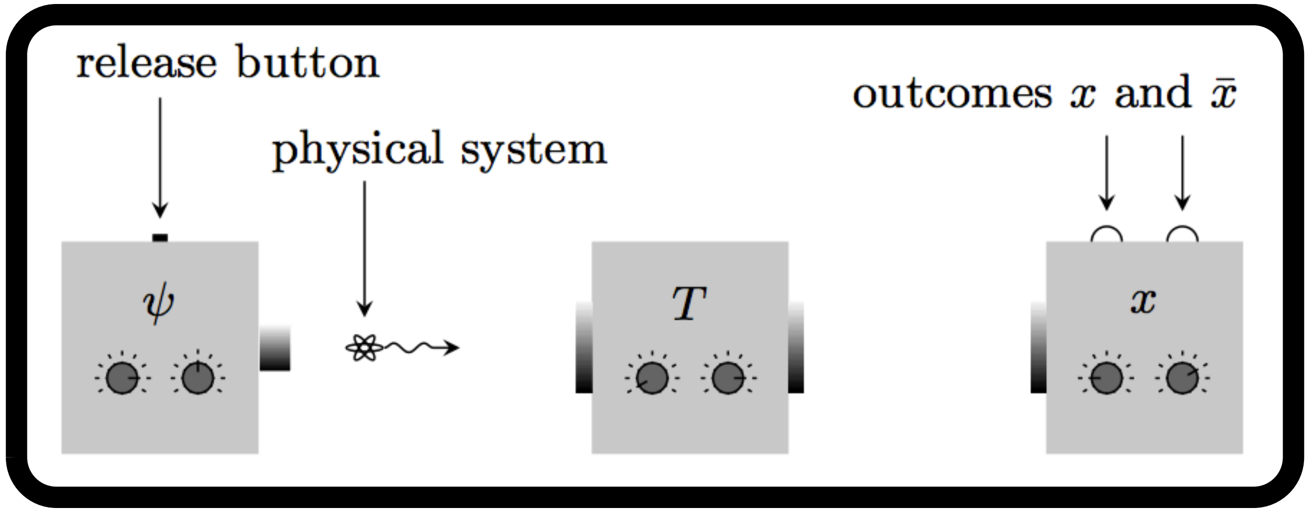

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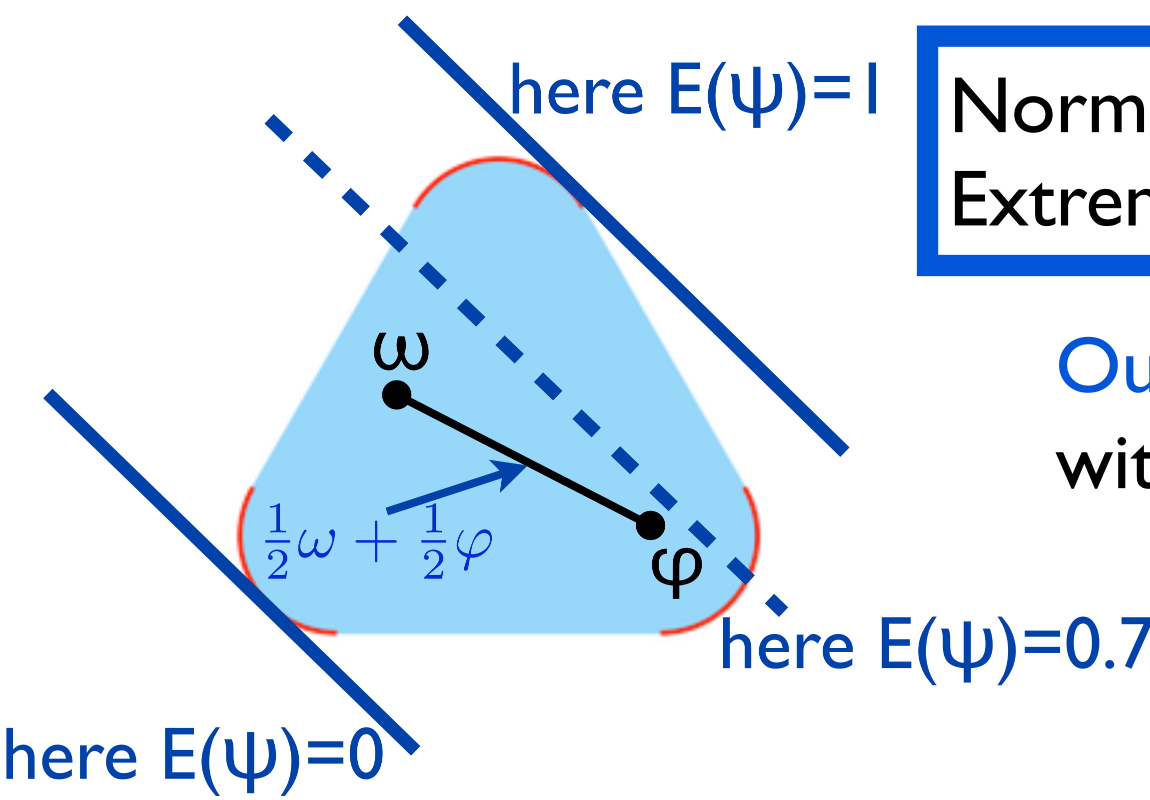

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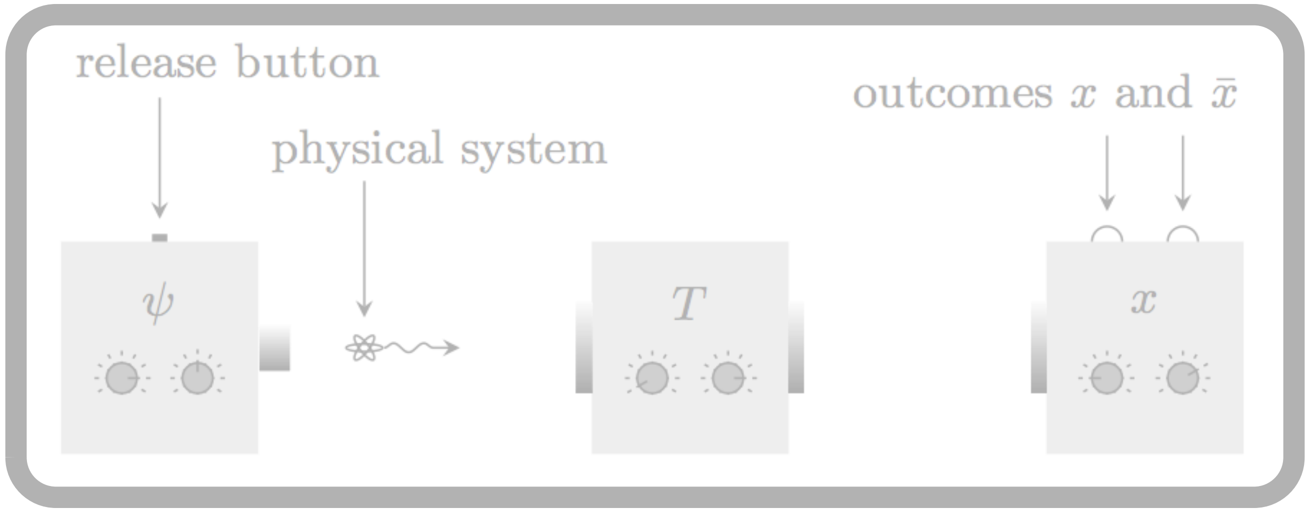

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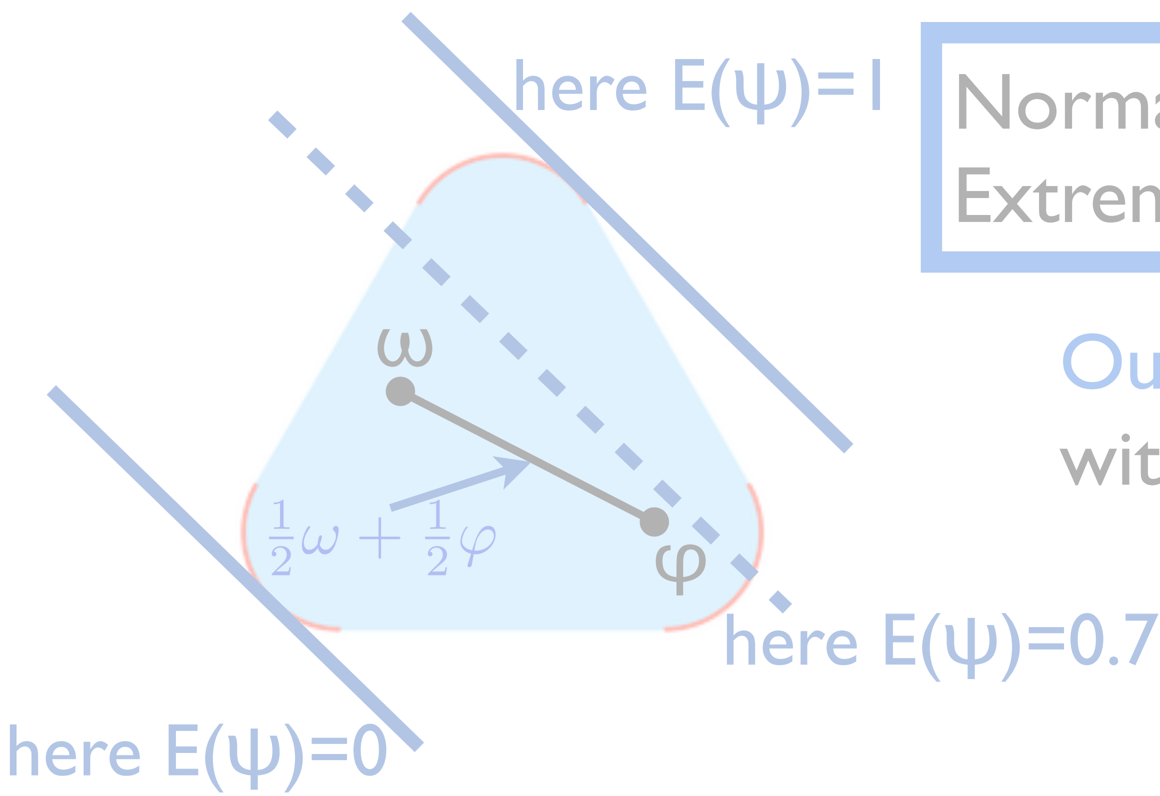

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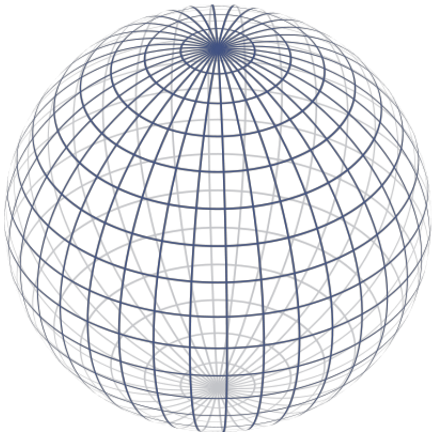

Qubit: ***N***=2, ***K***=4

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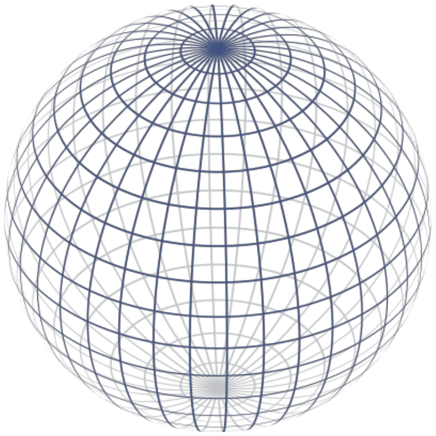

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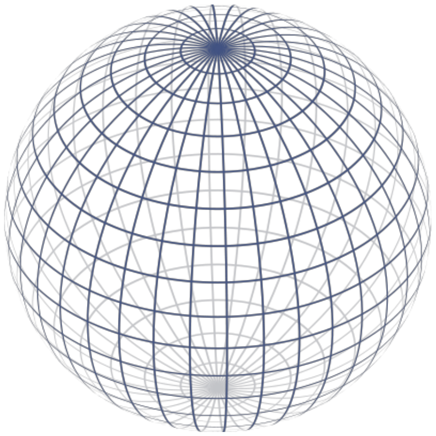

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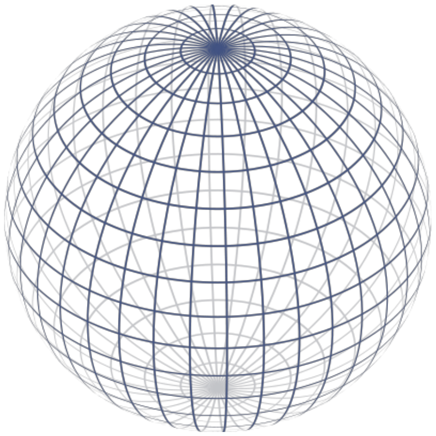

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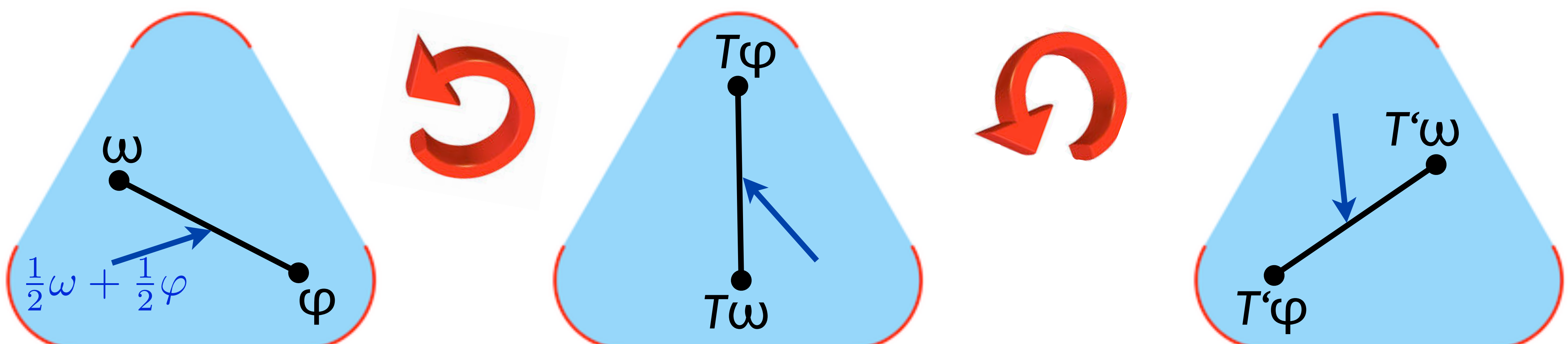
Transformations *T* map states to states, and are linear.

Reversible transformations form a group $\mathcal{G}_A$. In quantum theory: $\rho \mapsto U\rho U^\dagger$

They are symmetries of state space: $T(\Omega_A) = \Omega_A$



Normalized state space Ω_A

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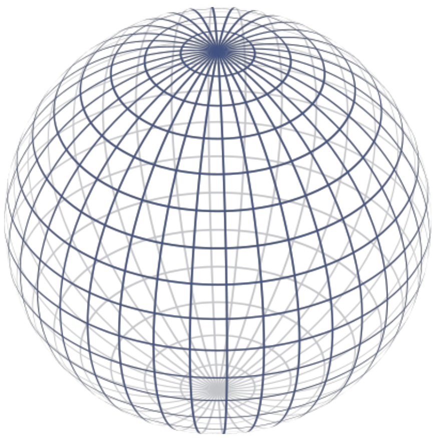

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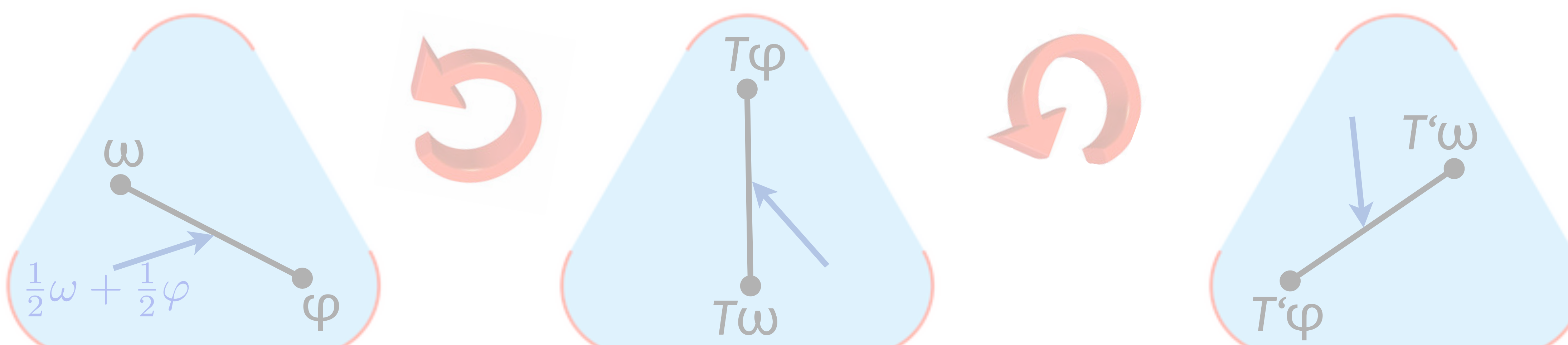

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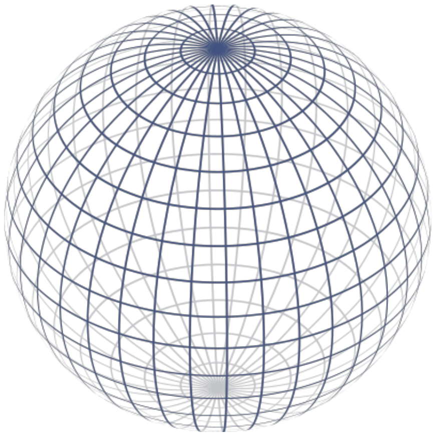

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⇒ A dynamical state space is a pair $(\Omega_A, \mathcal{G}_A)$.

state space

reversible transf.

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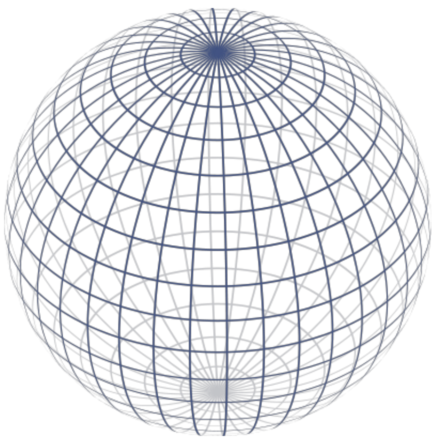

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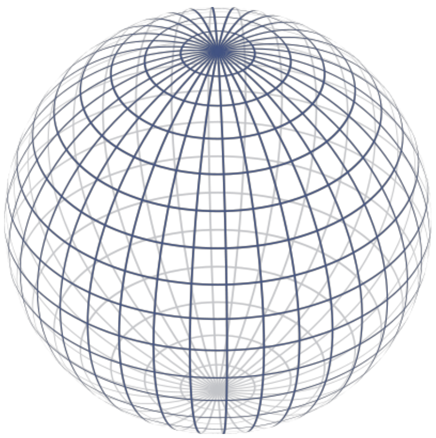

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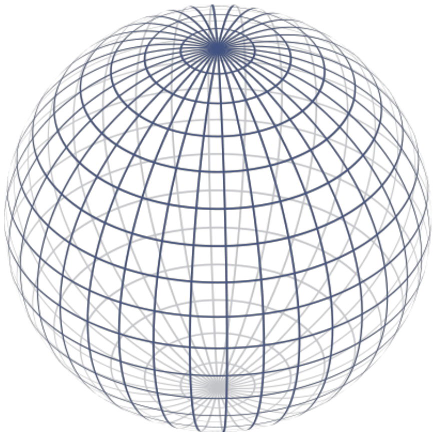
To do: define purity.

3. Purity in dynamical state spaces



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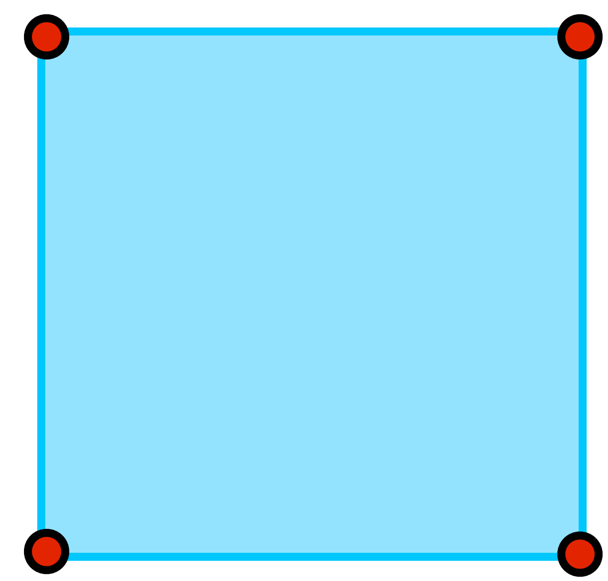
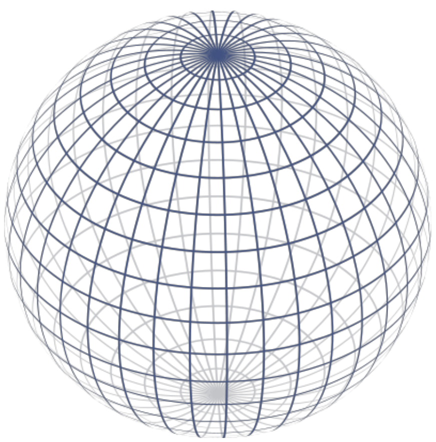
Assumptions:

- Transitivity: All pure states connected by reversible transf.
- Irreducibility: No invariant subspaces.



N=2, ***K***=3

3. Purity in dynamical state spaces

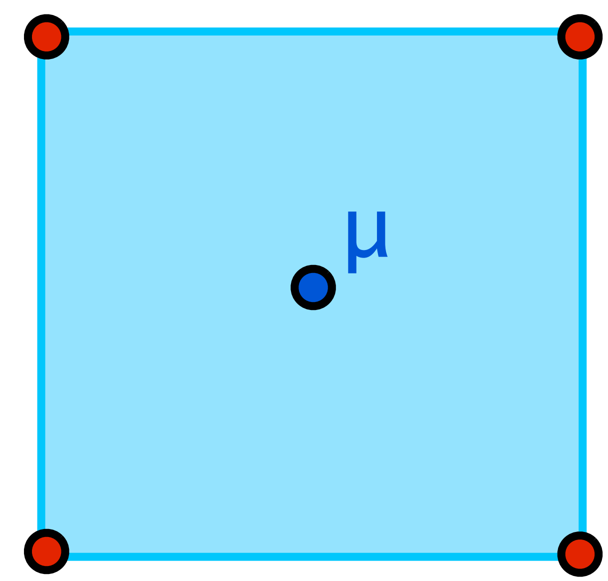


Qubit: ***N***=2, ***K***=4

Assumptions:

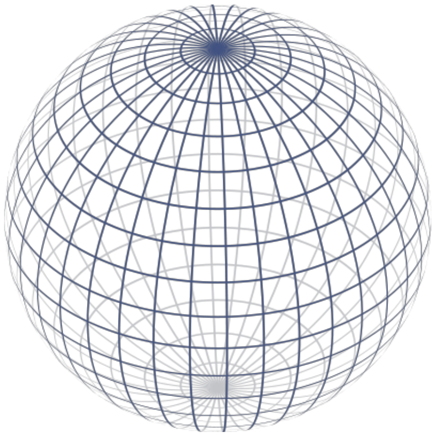
- Transitivity: All pure states connected by reversible transf.
- Irreducibility: No invariant subspaces.

⇒ unique invariant maximally mixed state μ.



N=2, ***K***=3

3. Purity in dynamical state spaces



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$\Rightarrow$ unique invariant maximally mixed state μ.

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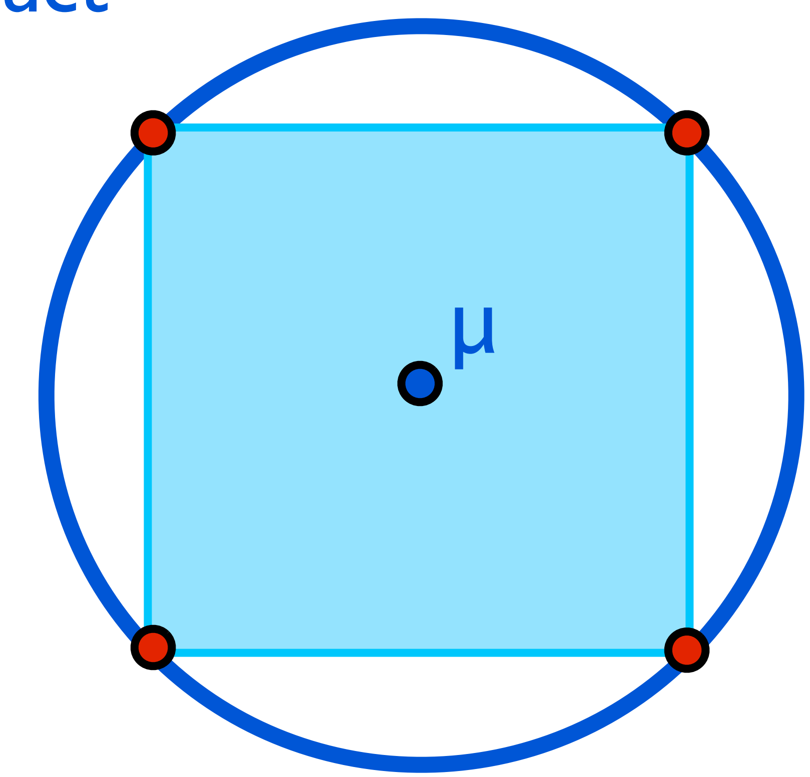

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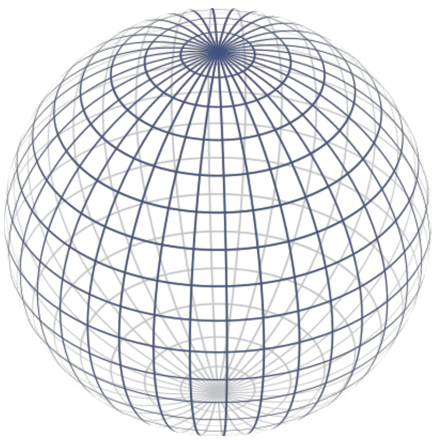

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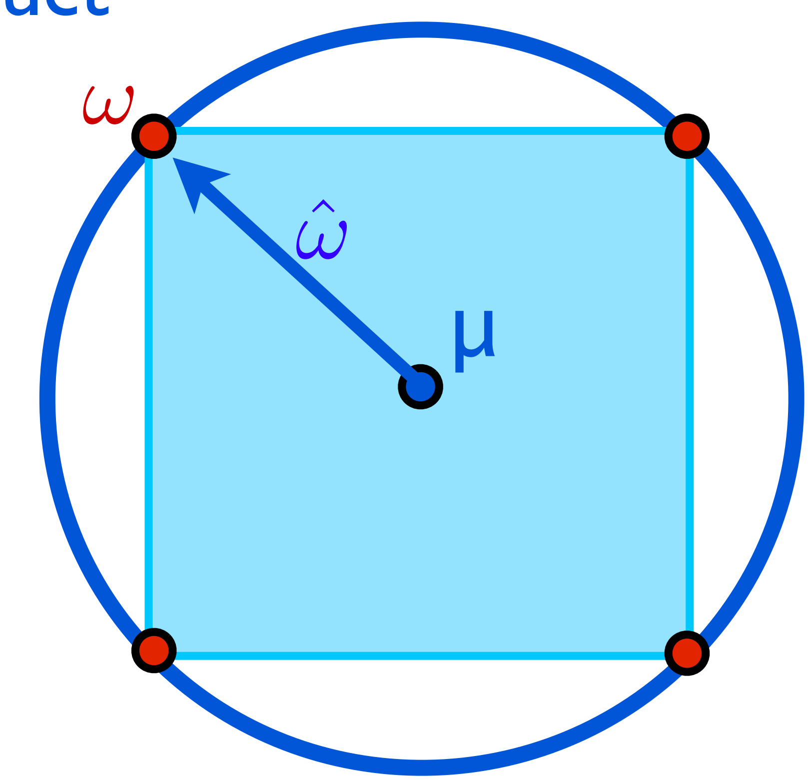

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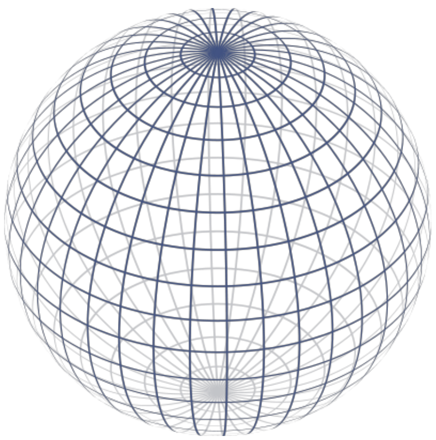

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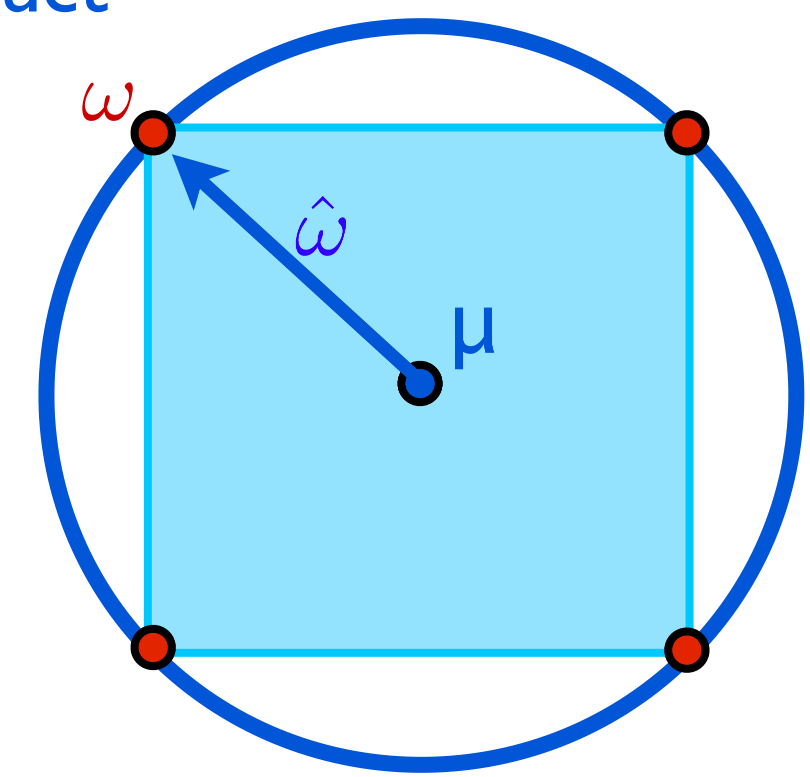

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scaled such that pure states have purity 1.

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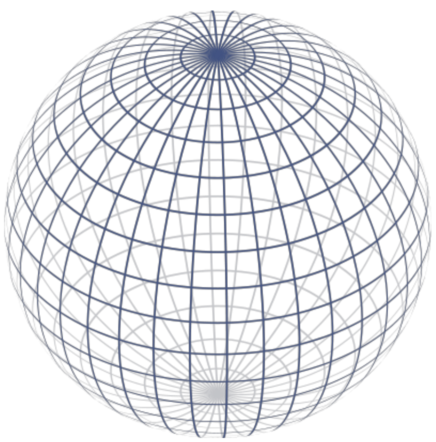

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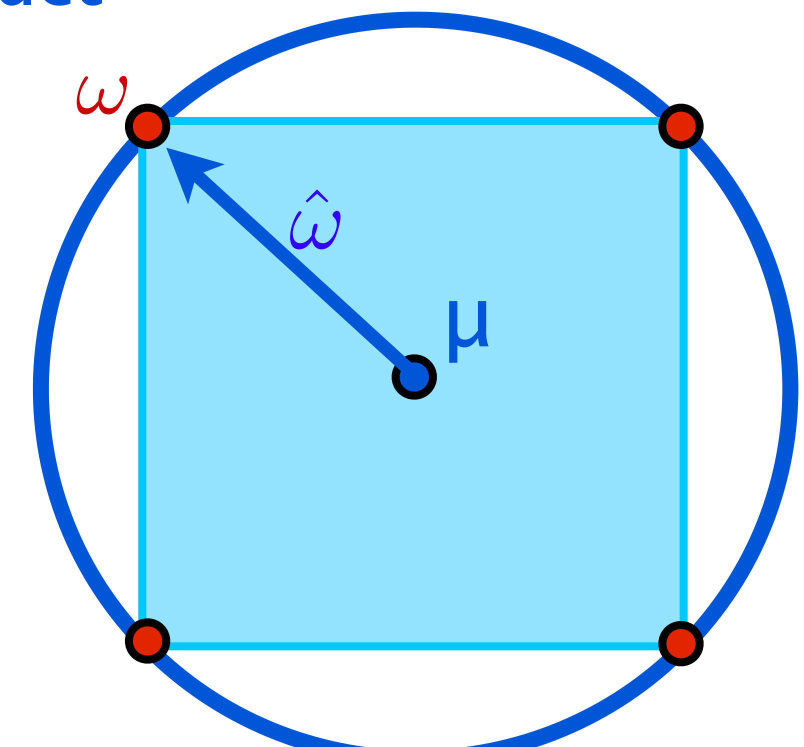

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- $0 \leq \mathcal{P}(\omega) \leq 1,$ and $\sqrt{\mathcal{P}}$ is convex,
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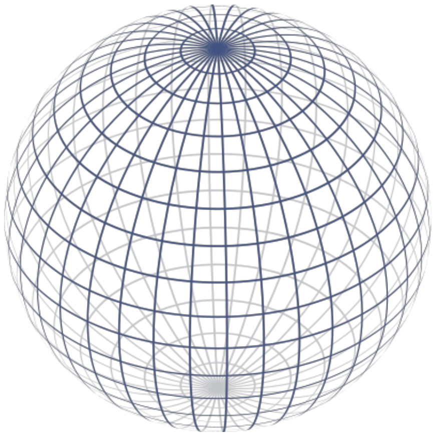

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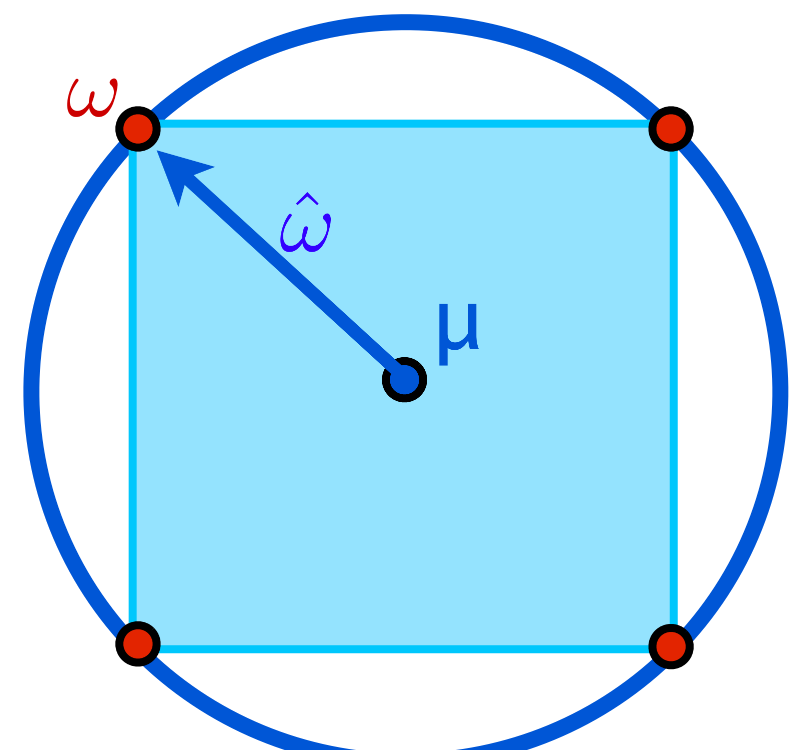

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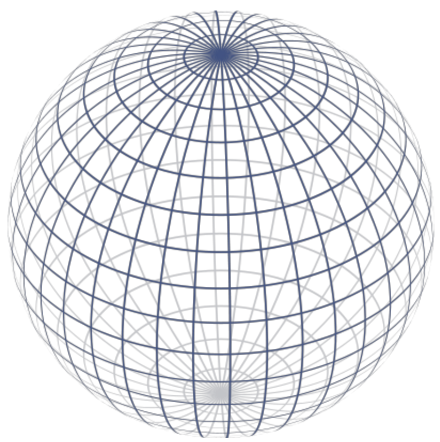

Quantum states on $\mathbb{C}^n$: $\mathcal{P}(\rho) = \frac{n}{n-1}\mathrm{Tr}(\rho^2) - \frac{1}{n-1}$

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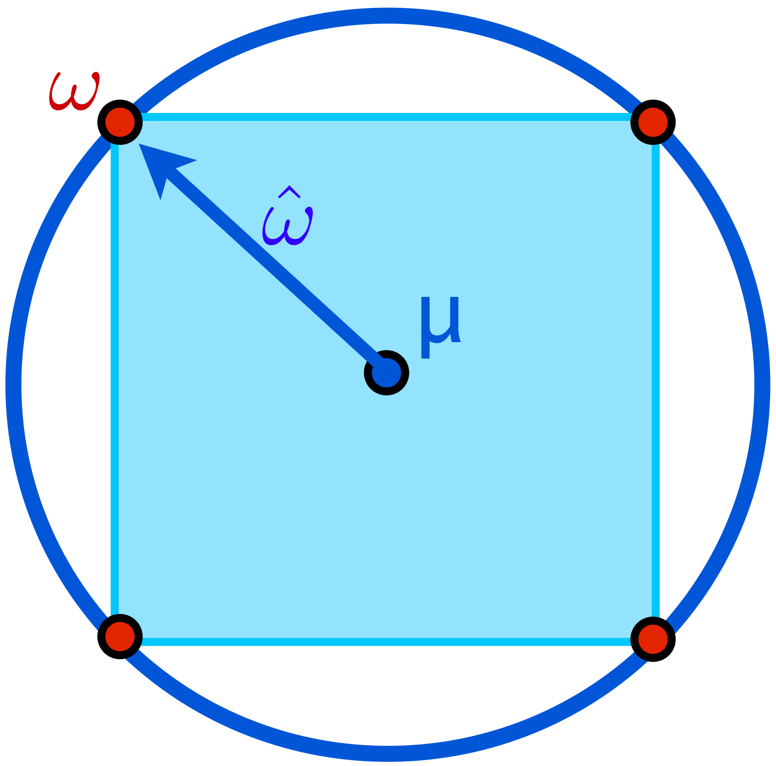

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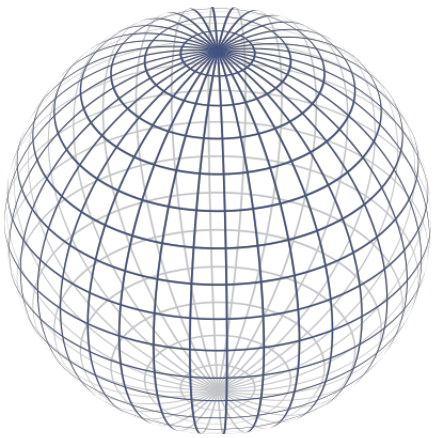

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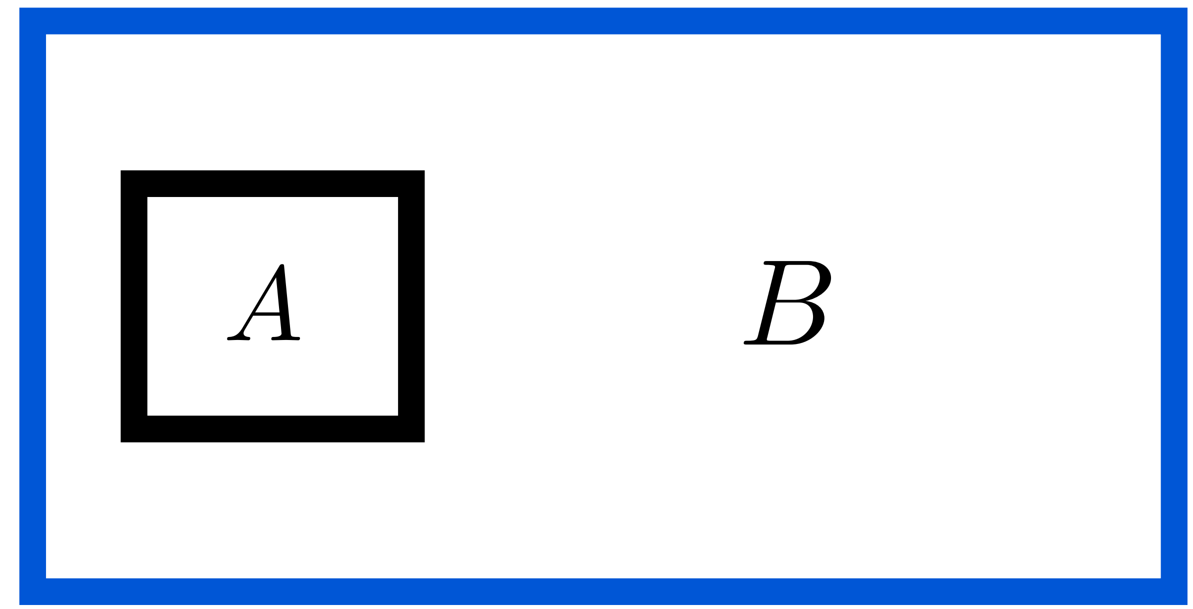

How to draw a random state ω^{AB} with fixed purity $\mathcal{P}(\omega^{AB}) = \mathcal{P}_0$:

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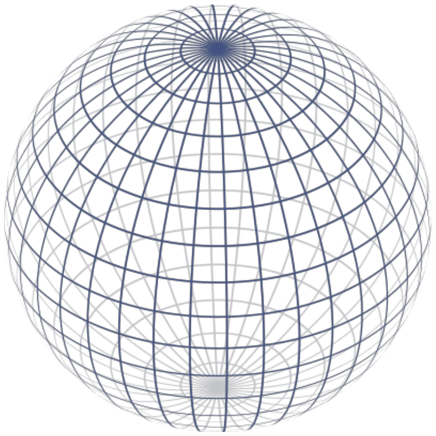

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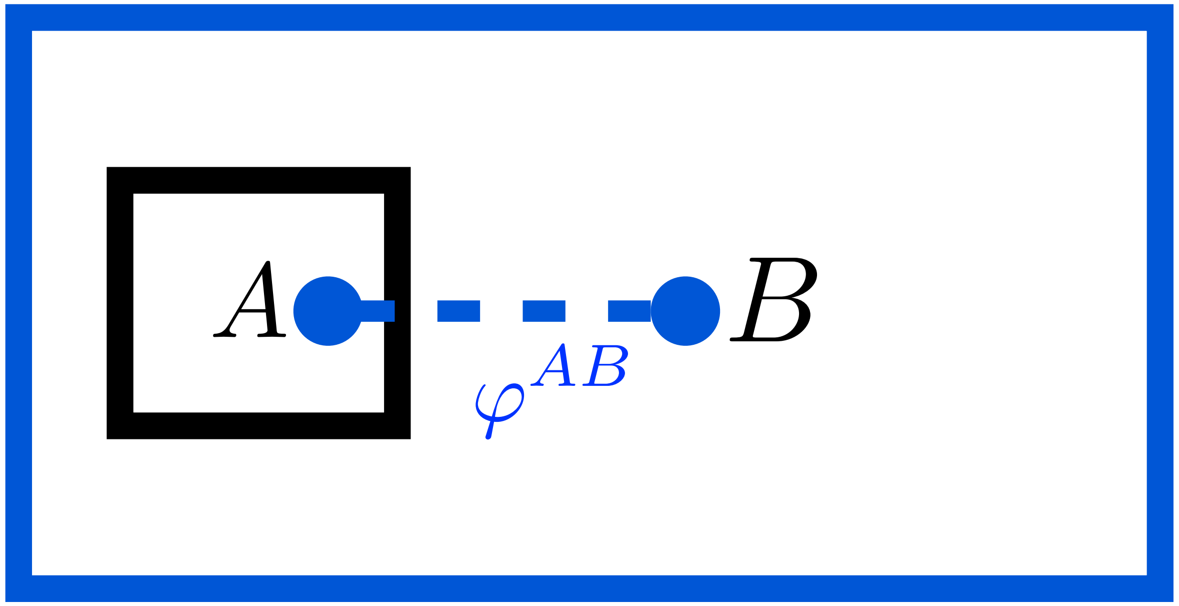

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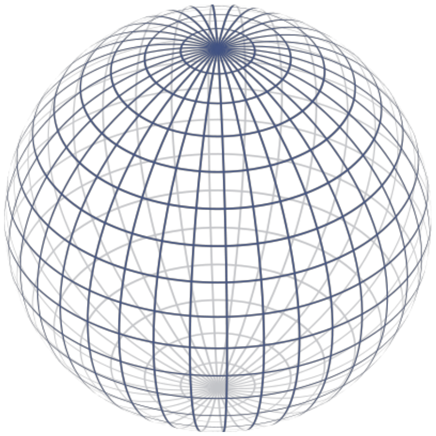

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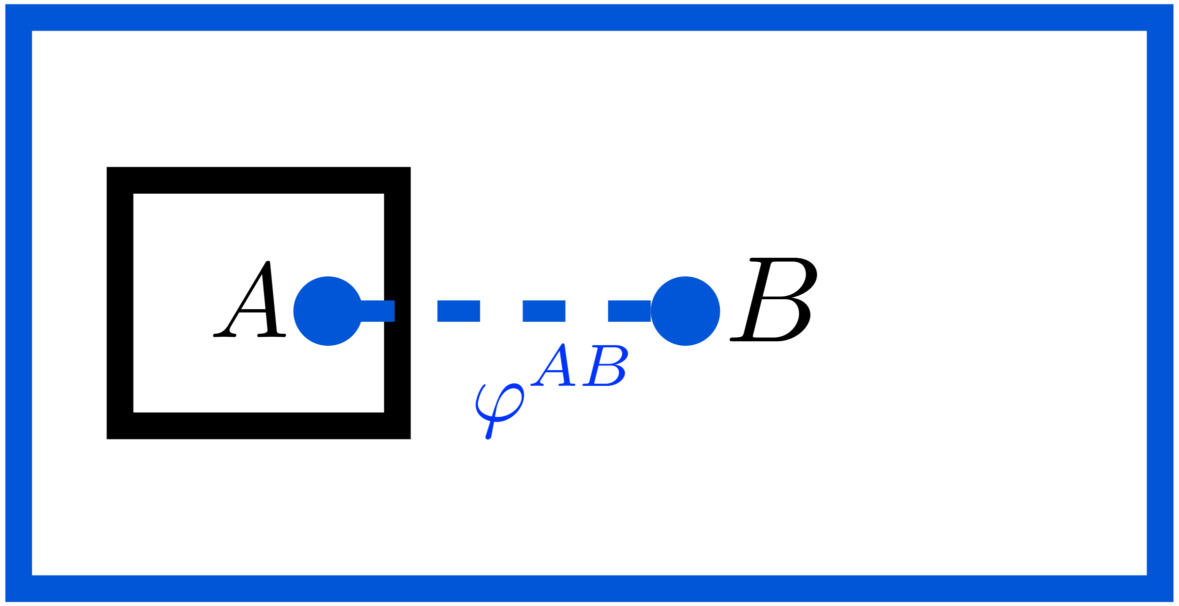

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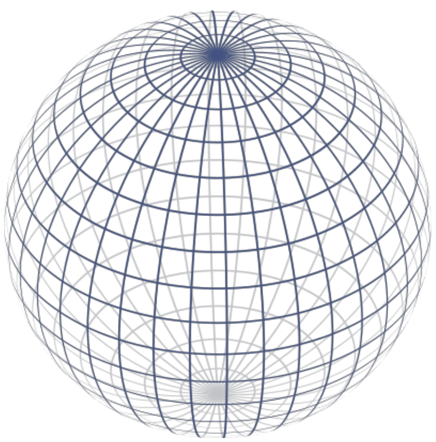

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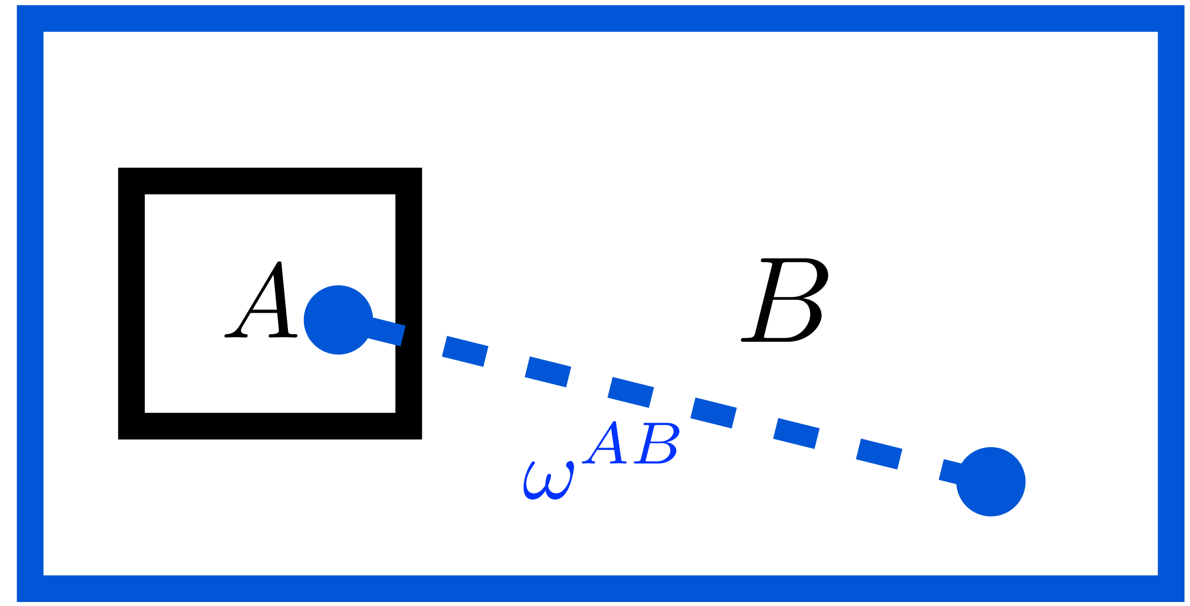

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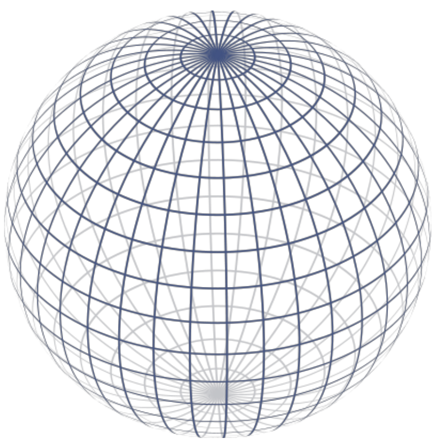
- Fix an arbitrary initial state φ^{AB} with $\mathcal{P}(\varphi^{AB}) = \mathcal{P}_0$.
- apply a Haar-random reversible transformation *T* to it.
- the result is $\omega^{AB} := T\varphi^{AB}$.

3. Purity in dynamical state spaces



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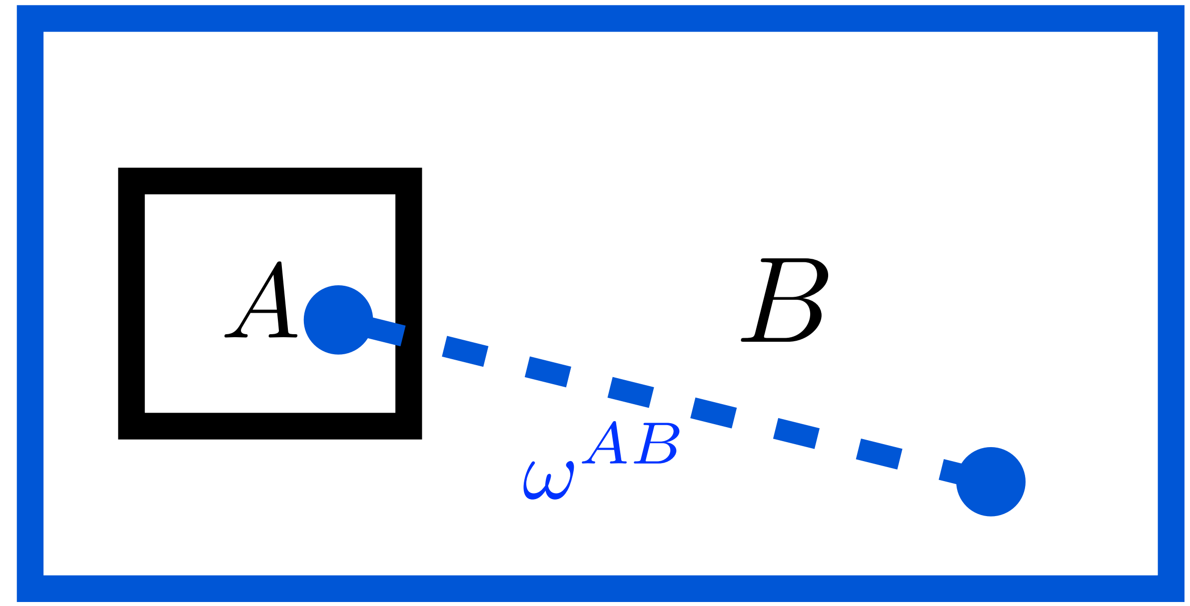

Theorem: If *AB* is locally tomographic and contains a composite classical subsystem, then

$$\mathbb{E}_\omega \mathcal{P}(\omega^A) = \frac{K_A - 1}{K_A K_B - 1} \cdot \frac{N_A N_B - 1}{N_A - 1} \cdot \mathcal{P}(\omega^{AB}).$$

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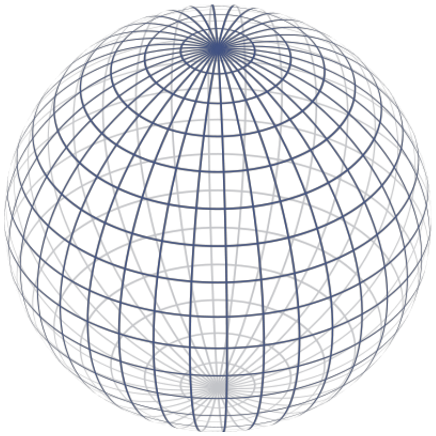

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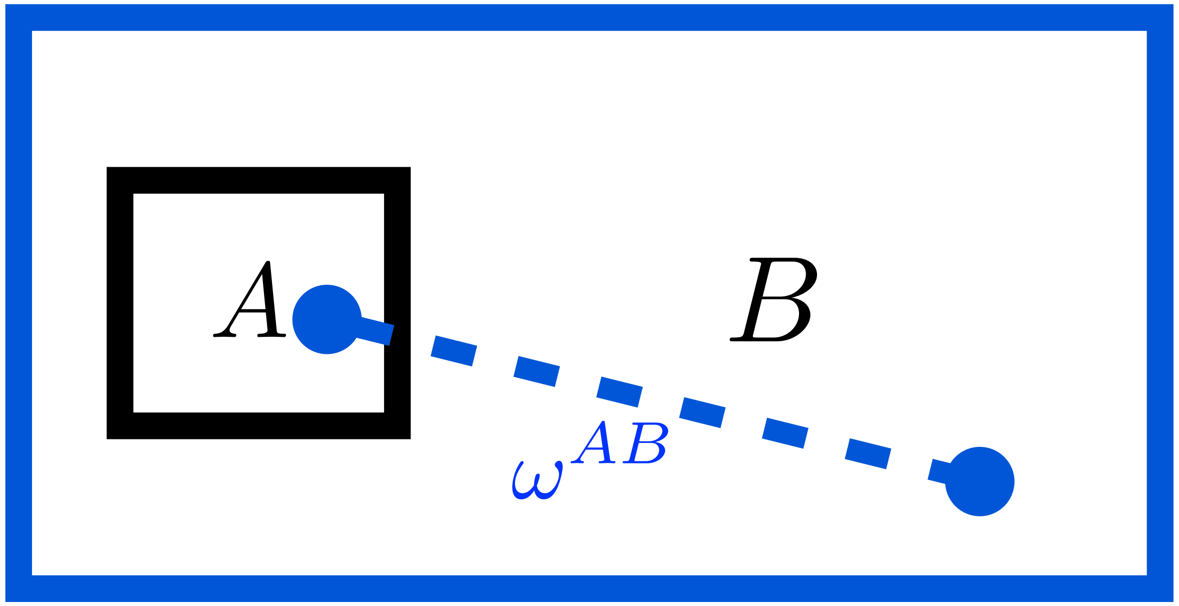

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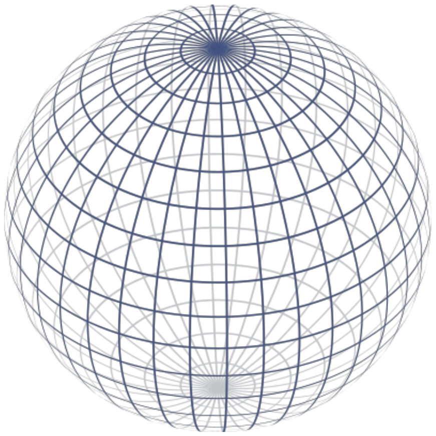
Random pure quantum states: $K = N^2, \ \mathcal{P}(\omega^{AB}) = 1.$

$$\mathbb{E}_\omega \mathcal{P}(\omega^A) = \frac{N_A + 1}{N_A N_B + 1} \approx \frac{1}{N_B}$$

almost maximally entangled!

Due to Markov's inequality, this is the typical behaviour.

3. Purity in dynamical state spaces



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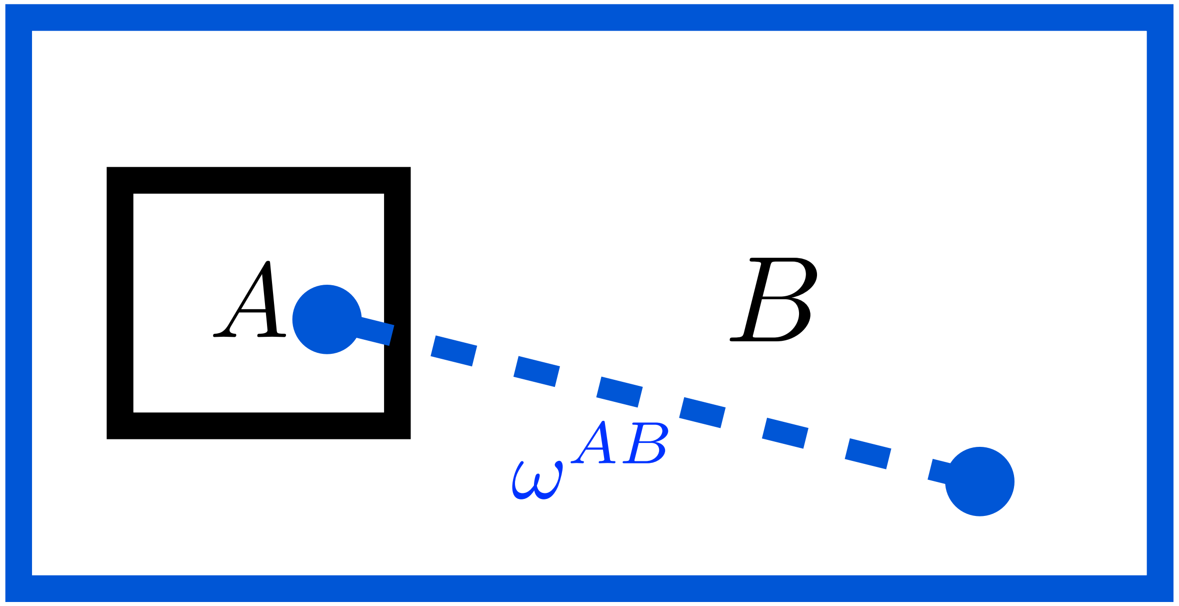

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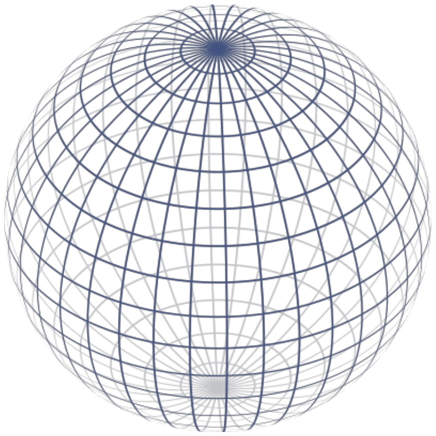

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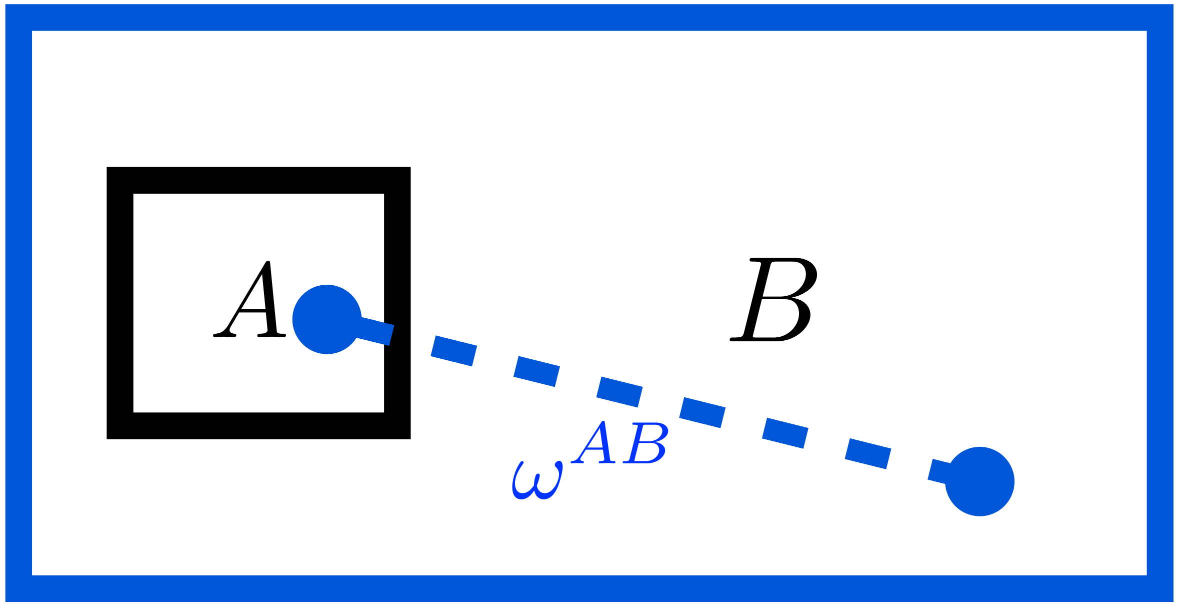

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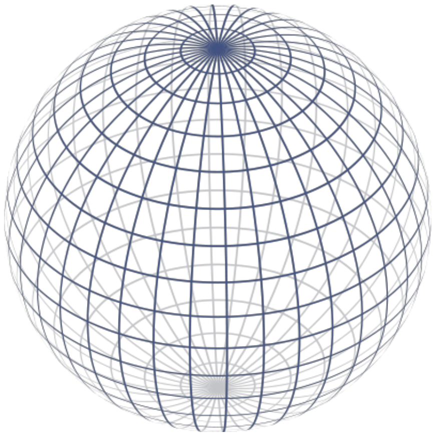
Random pure classical states: $K = N,\ \mathcal{P}(\omega^{AB}) = 1.$

$\mathbb{E}_\omega \mathcal{P}(\omega^A) = 1.$ There are no entangled classical states.

3. Purity in dynamical state spaces



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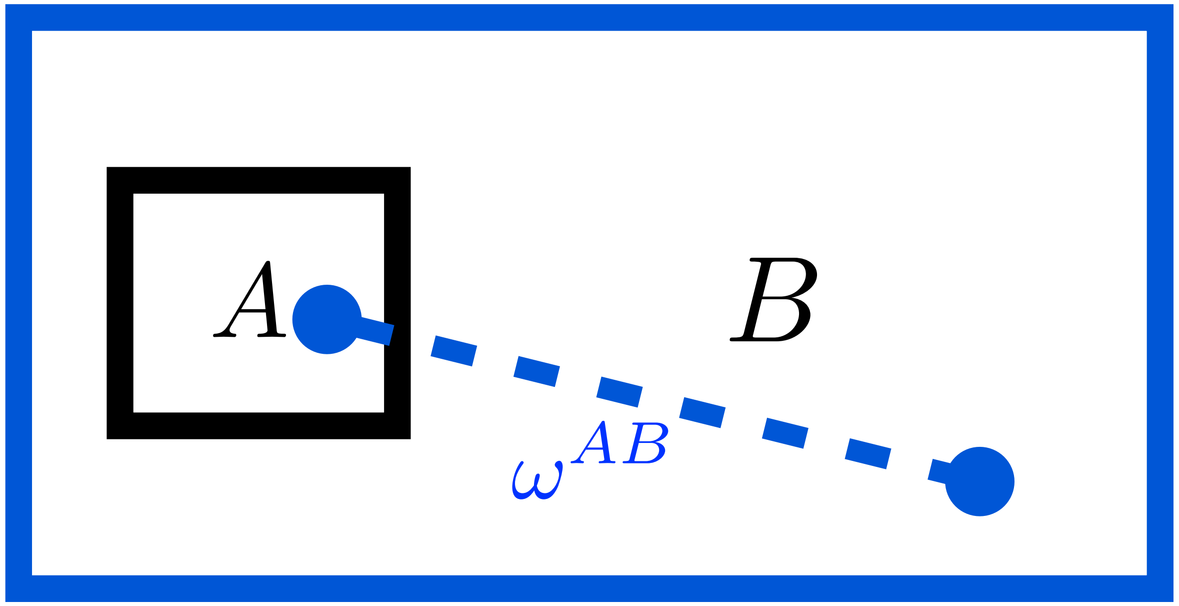

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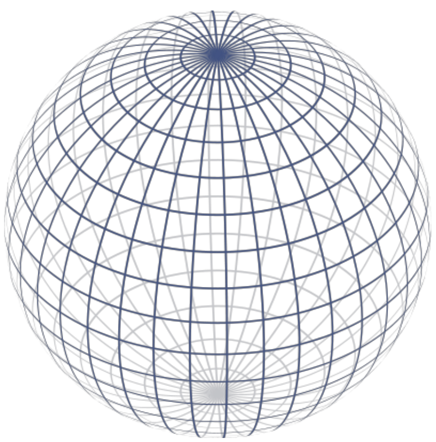

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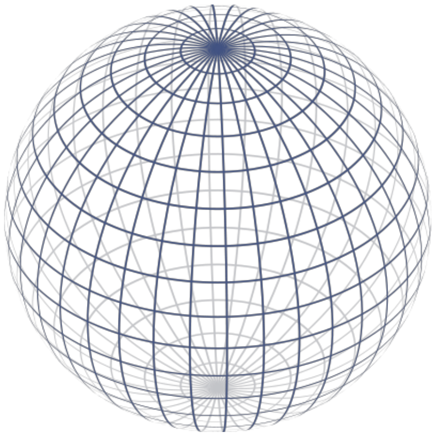
Classical coin tossing: $N = K$; initially, $\varphi^{AB} = \varphi^A \otimes \varphi^B$.

Coin's state φ^A pure (fully known), environment state φ^B mixed.

$\omega^{AB} = T\varphi^{AB}$, random permutation *T*. $\mathbb{E}_\omega \mathcal{P}(\omega^A) = \mathcal{P}(\omega^{AB}) = \mathcal{P}(\varphi^{AB})$.

3. Purity in dynamical state spaces



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Ignorance about environment gets transferred to coin.

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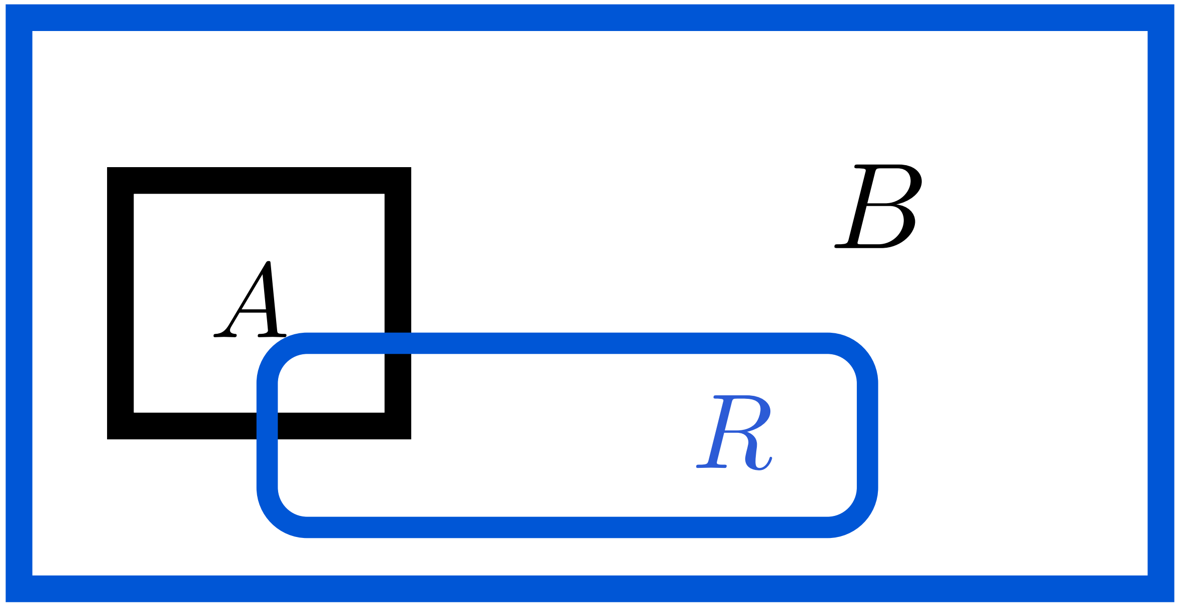
3. Purity in dynamical state spaces

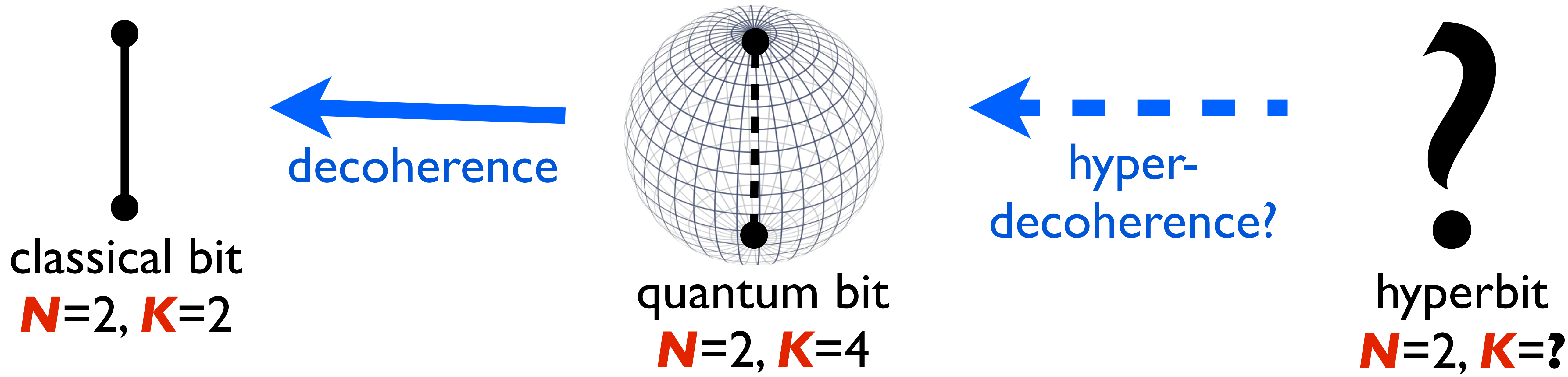
Analogous calculations with constraint $R \subseteq AB$ give new quantum results as by-products, e.g.:



Theorem. If $\omega_\pm$ is a random pure state on the symmetric $R_+ = \mathbb{C}^n \vee \mathbb{C}^n$ or antisymmetric subspace $R_- = \mathbb{C}^n \wedge \mathbb{C}^n$ on $AB = \mathbb{C}^n \otimes \mathbb{C}^n$, then

$$\mathbb{E}_{\omega_\pm} \mathrm{Tr}\left[\left(\omega_\pm^A\right)^2\right] = \frac{2(n \pm 1)}{n^2 \pm n + 2}.$$

4. Decoupling and black-hole thermodynamics



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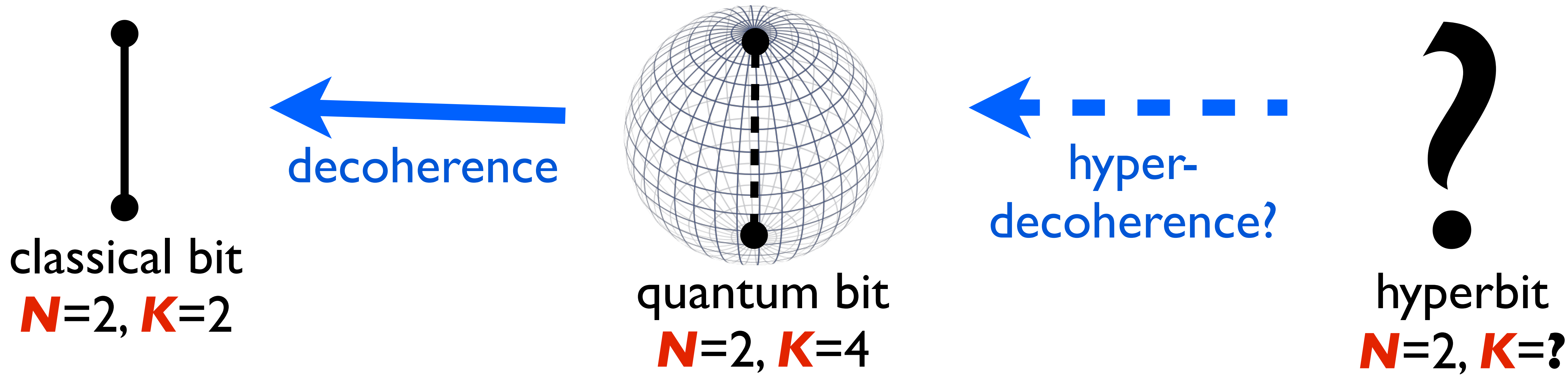
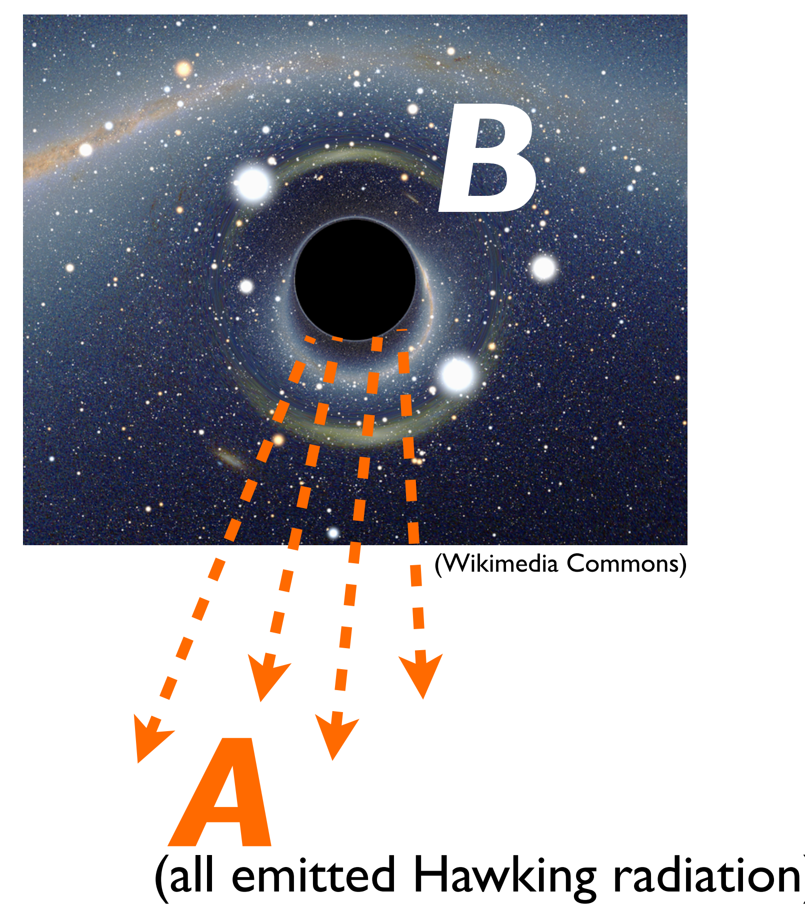
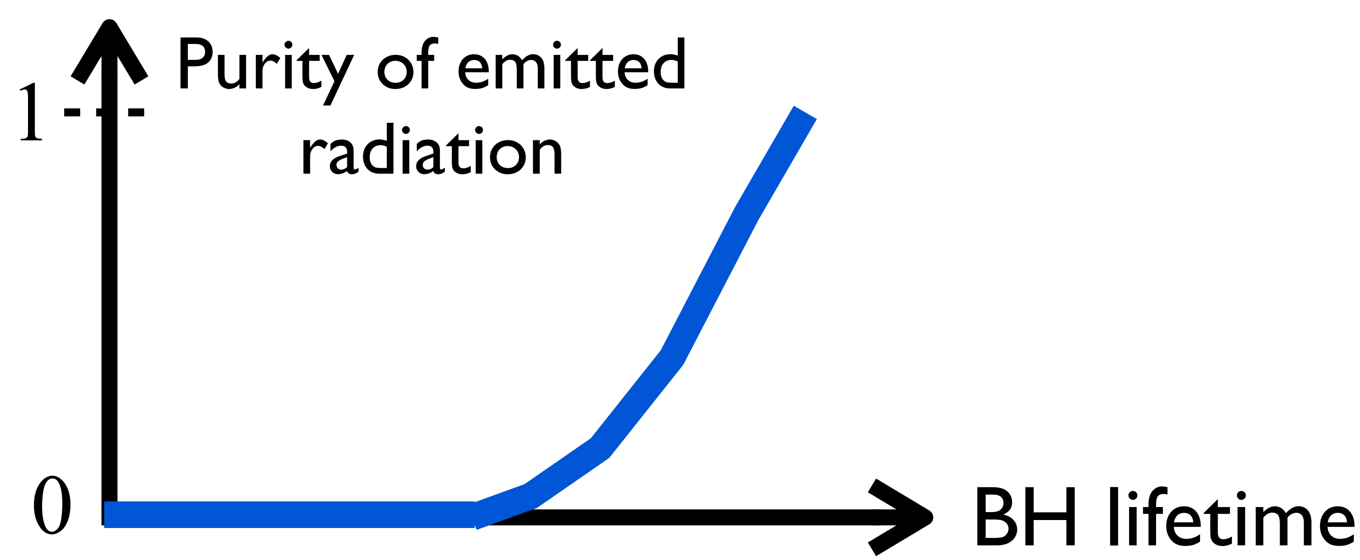


If all is quantum, and B.H. formed from pure state:
N qubits in Hawking radiation A can only be maximally mixed if B keeps N qubits for purification.





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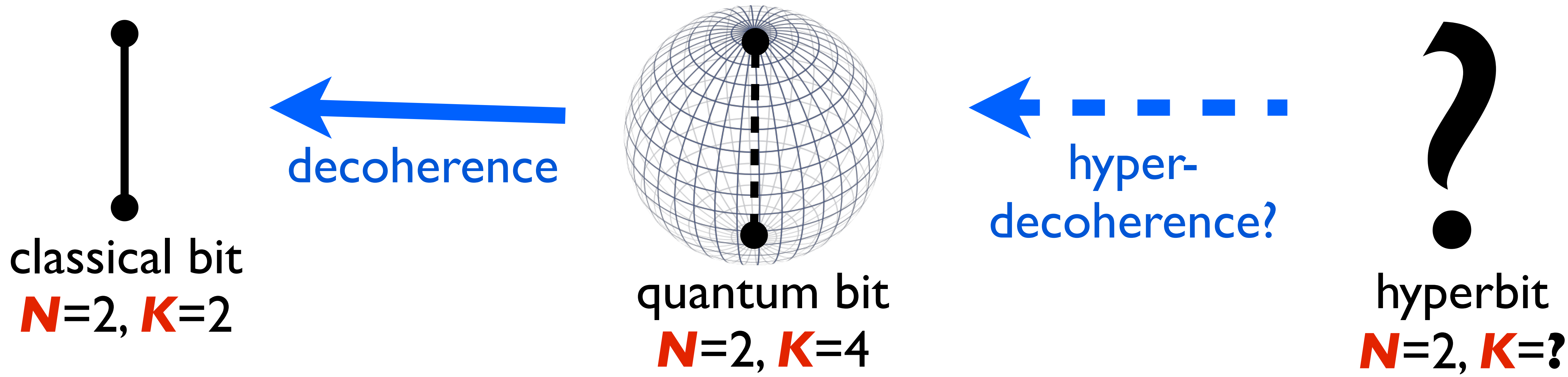
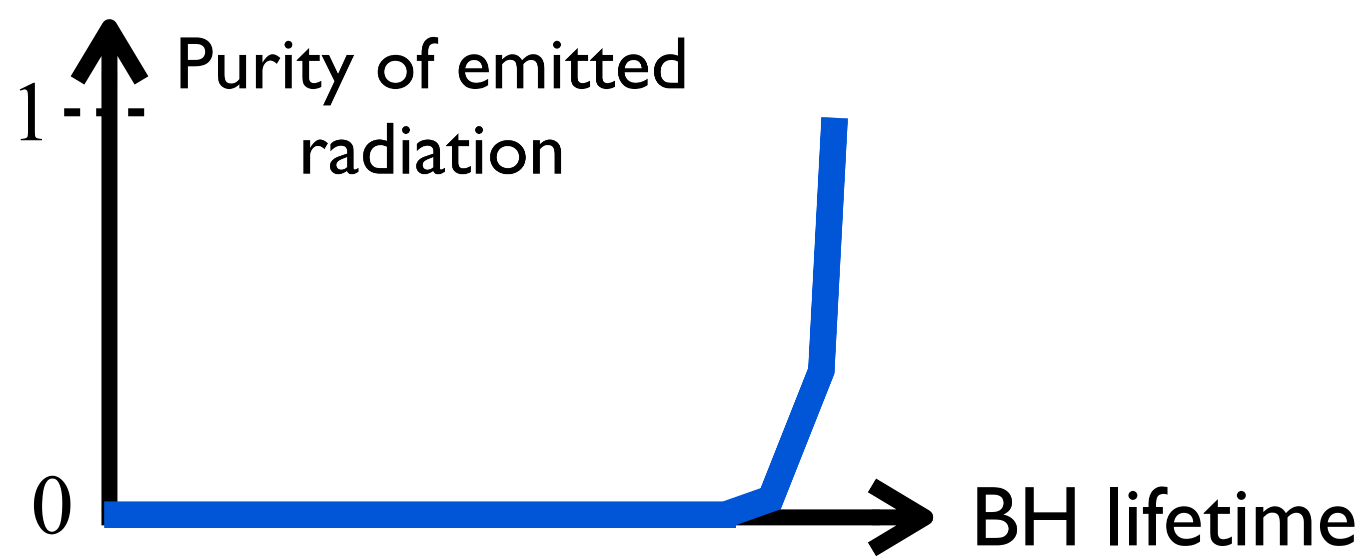
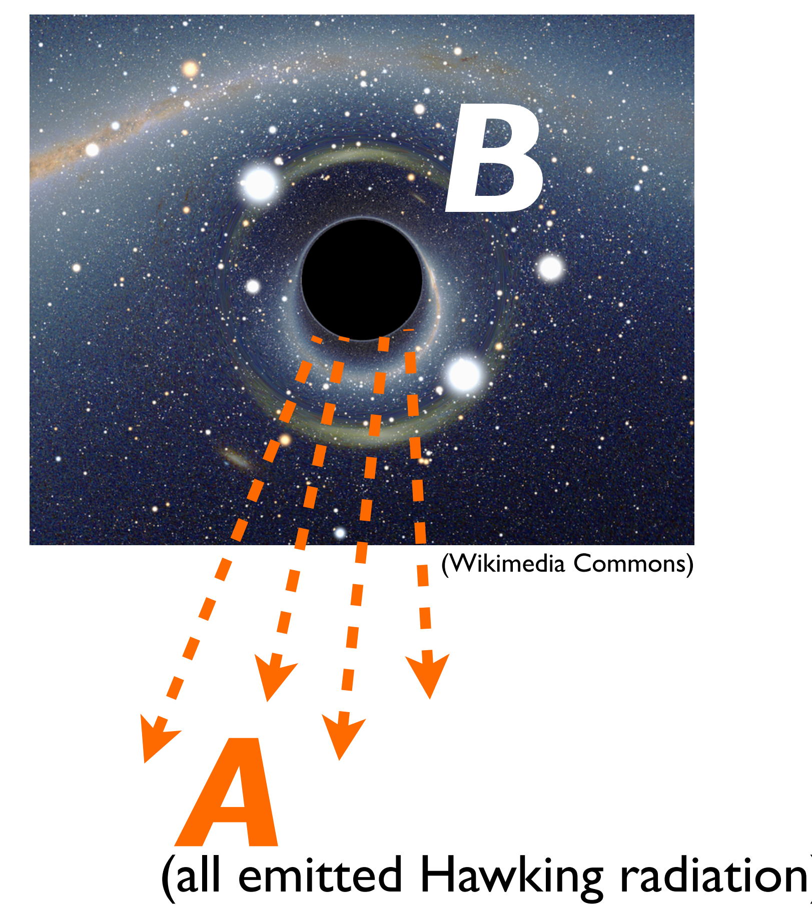


If B is post-quantum with $K_B \gg N_B^2$, typical pure states ω^{AB} have $\mathcal{P}(\omega^A) \approx N_B/K_B \ll 1/N_B$. Small B can purify large A.





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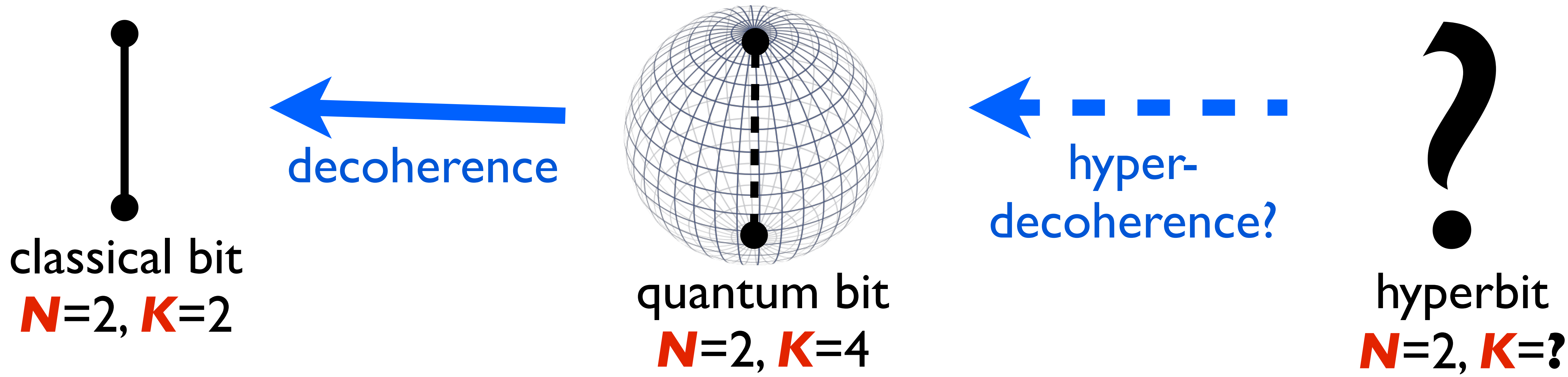

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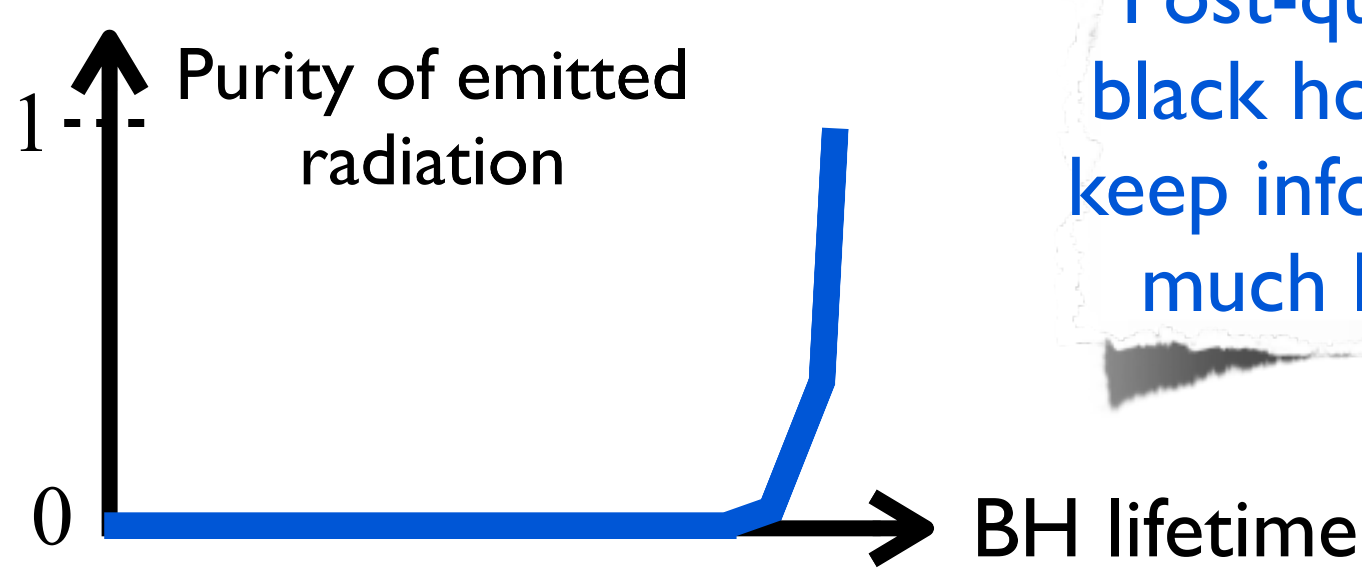
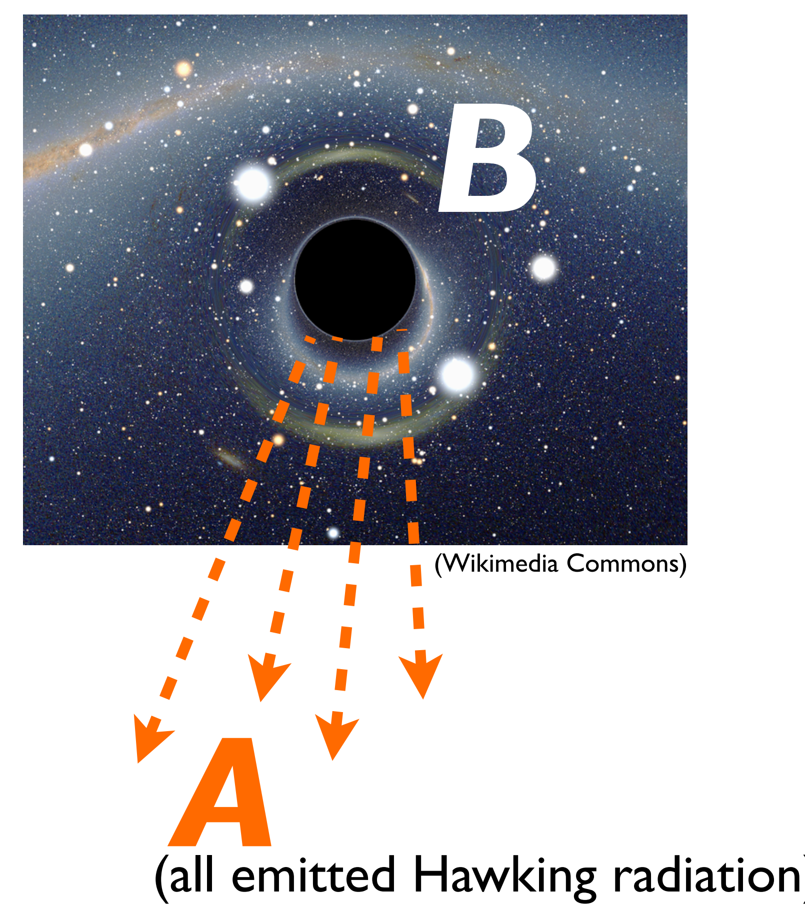
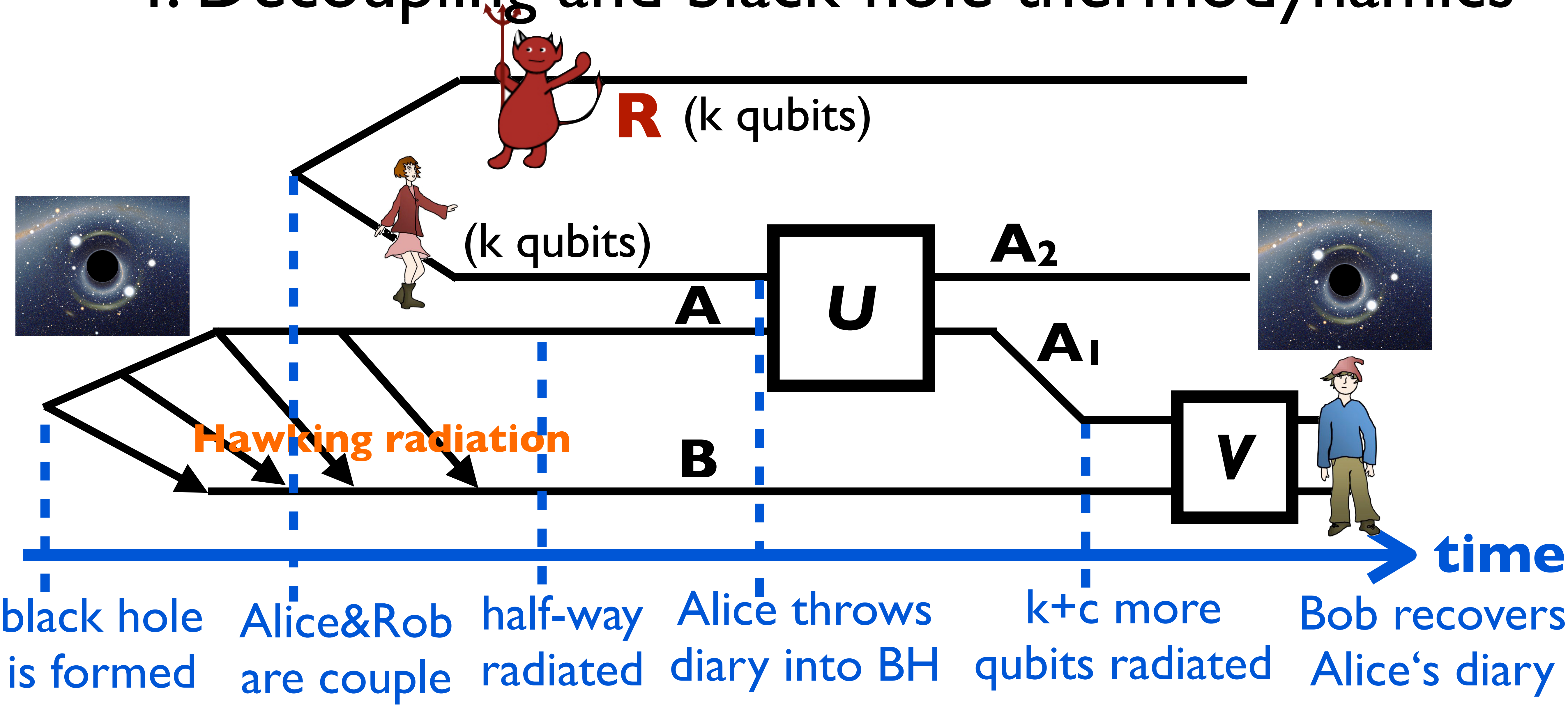


Post-quantum black holes may keep information much longer.

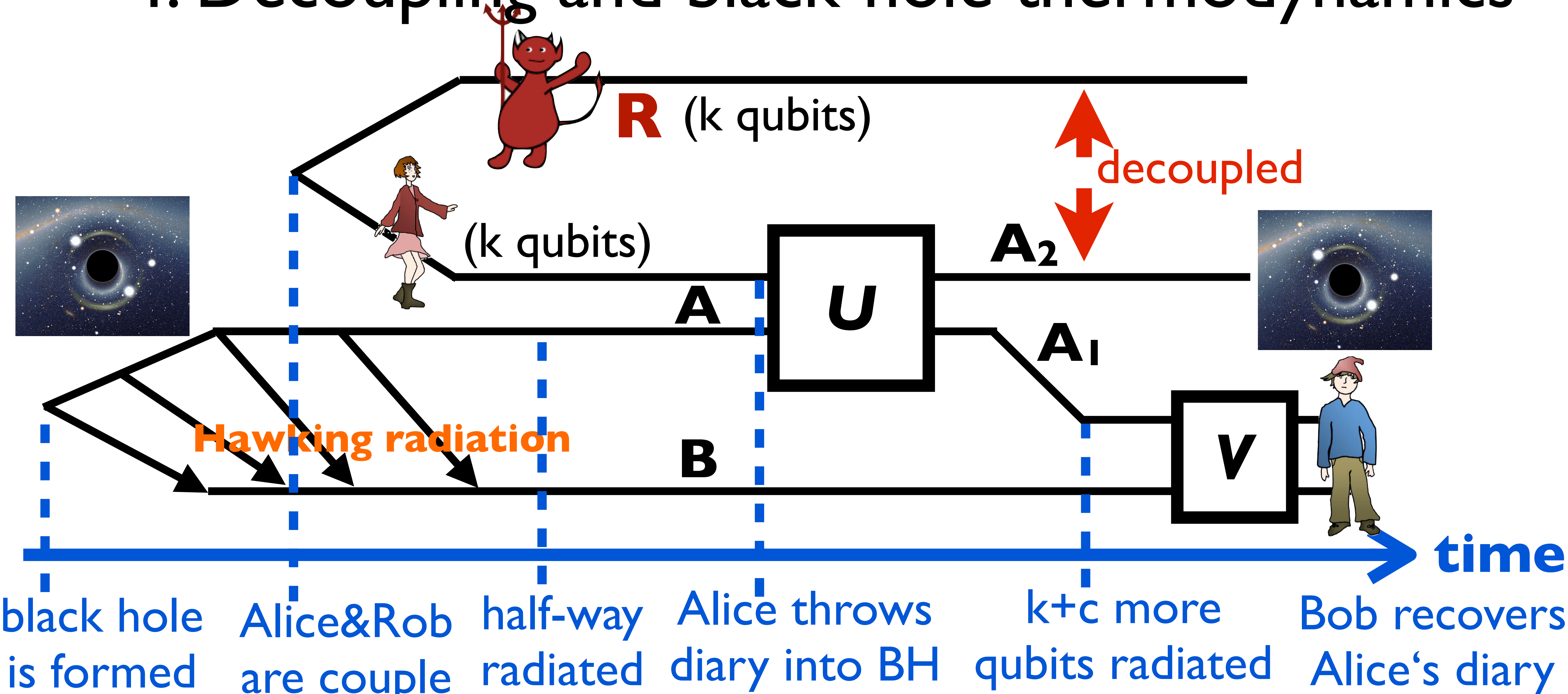


4. Decoupling and black-hole thermodynamics



P. Hayden and J. Preskill, *Black holes as mirrors*, J. HEP **09**(120), 2007.

4. Decoupling and black-hole thermodynamics



$$|\omega(U)^{RAB}\rangle = U^A \otimes \mathbf{1}^{RB} |\psi^{RAB}\rangle$$

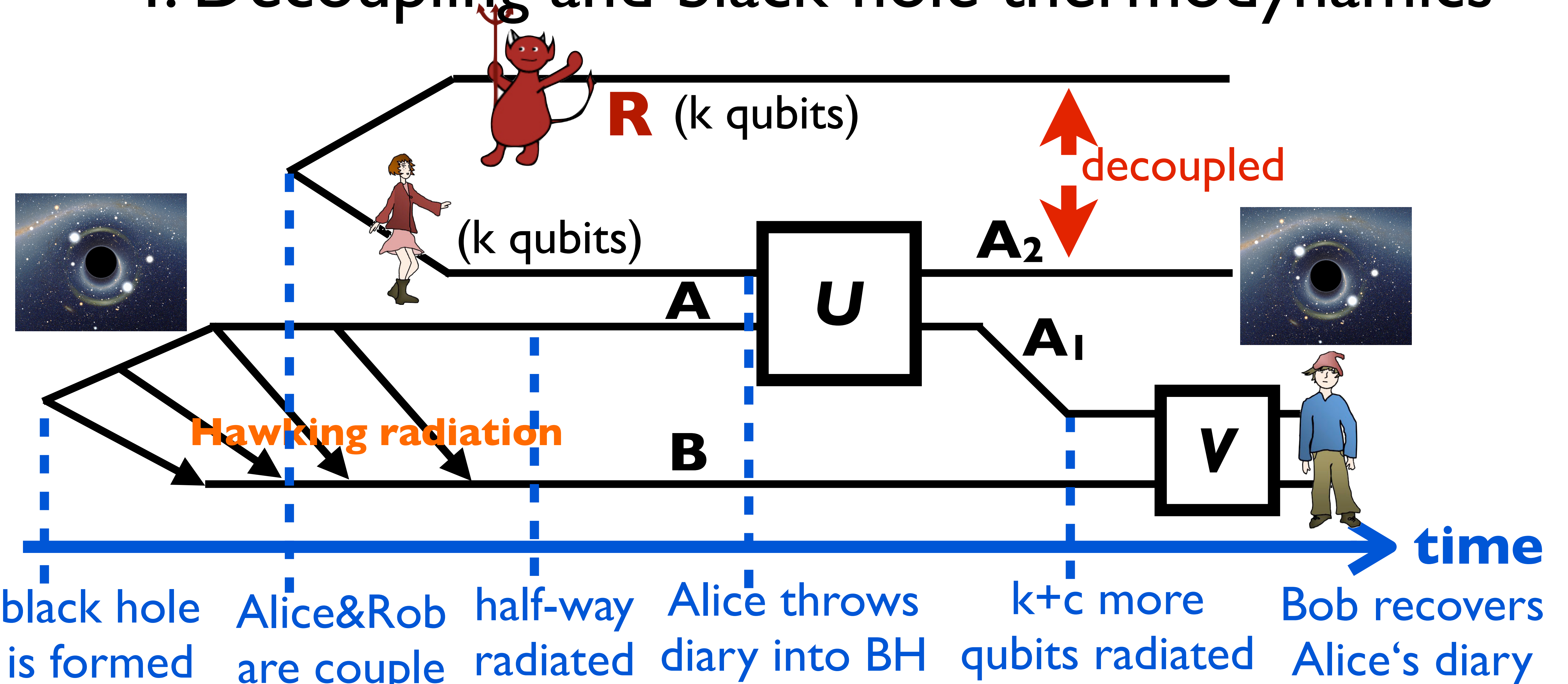
Decoupling:

$$\int dU \, \|\omega(U)^{RA_2} - \psi^R \otimes \mu^{A_2}\|_1^2 \leq \frac{|R|\,|A|}{|A_1|^2} \mathrm{Tr}\left[\left(\psi^{RA}\right)^2\right] \leq \frac{|R|^2}{|A_1|^2} = 2^{-2c}.$$

max. mix.

P. Hayden and J. Preskill, *Black holes as mirrors*, J. HEP **09**(120), 2007.

4. Decoupling and black-hole thermodynamics



$$\omega(U)^{RAB} = (U^A \otimes \mathbf{1}^{RB})\psi^{RAB}$$

Post-quantum decoupling:

$$\int_{\mathcal{G}_A} dU \, \|\hat{\omega}(U)^{RA_2} - (\psi^R \otimes \mu^{A_2})^\wedge\|_2^2 = \mathcal{P}(\psi^{RA}) \cdot \frac{(K_{A_2}-1)(N_R N_A - 1)}{(N_R N_{A_2}-1)(K_A - 1)} - \mathcal{P}(\psi^R) \cdot \frac{(N_R - 1)(K_{A_2}-1)}{(N_R N_{A_2}-1)(K_A-1)}$$

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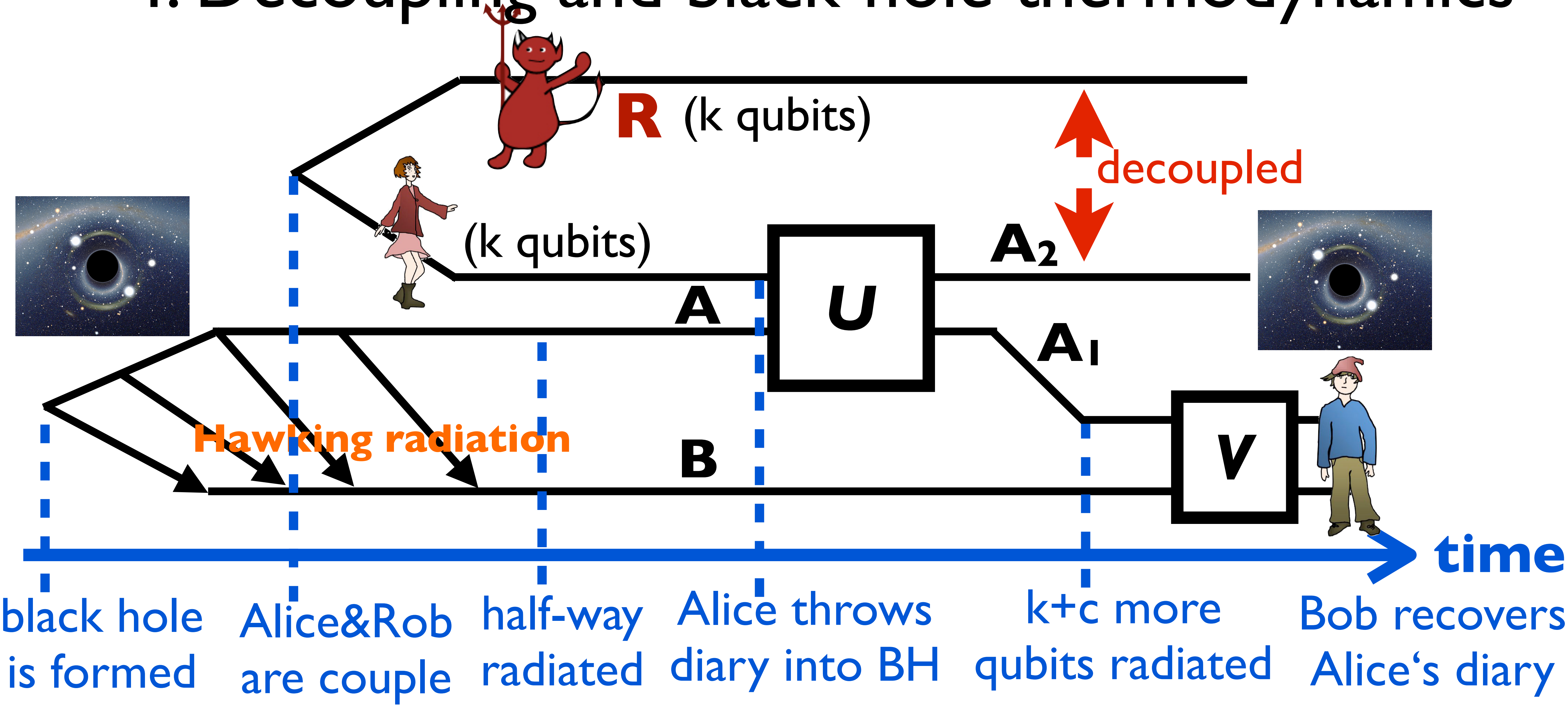

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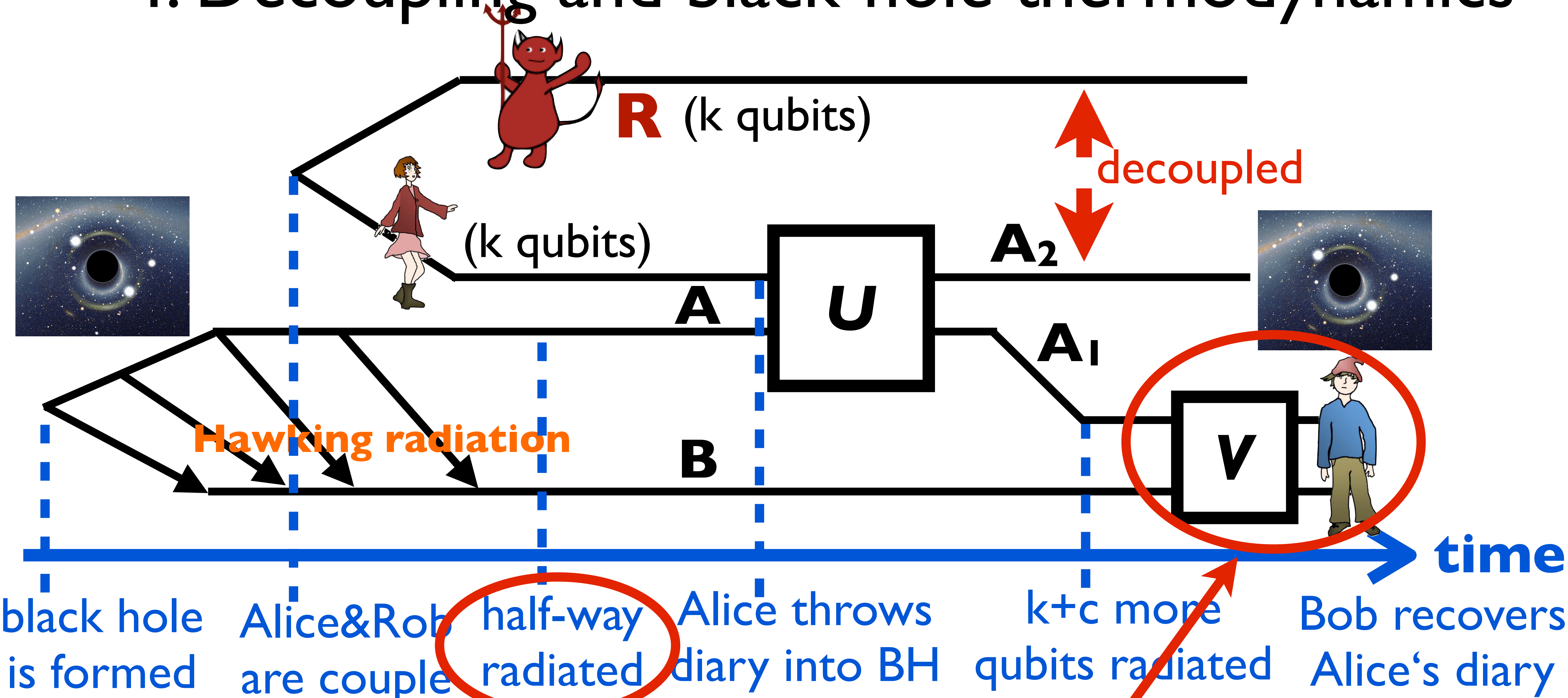

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But: "Half-way" much later? Bob unable to decode?

Conclusions

- Purity is a nice entropy measure in probabilistic theories.

- Unified view on typical entanglement and coin tossing: randomization depends basically on parameters $\boldsymbol{N}$ and $\boldsymbol{K}$.

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Thank you!

(arXiv:1107.6029)