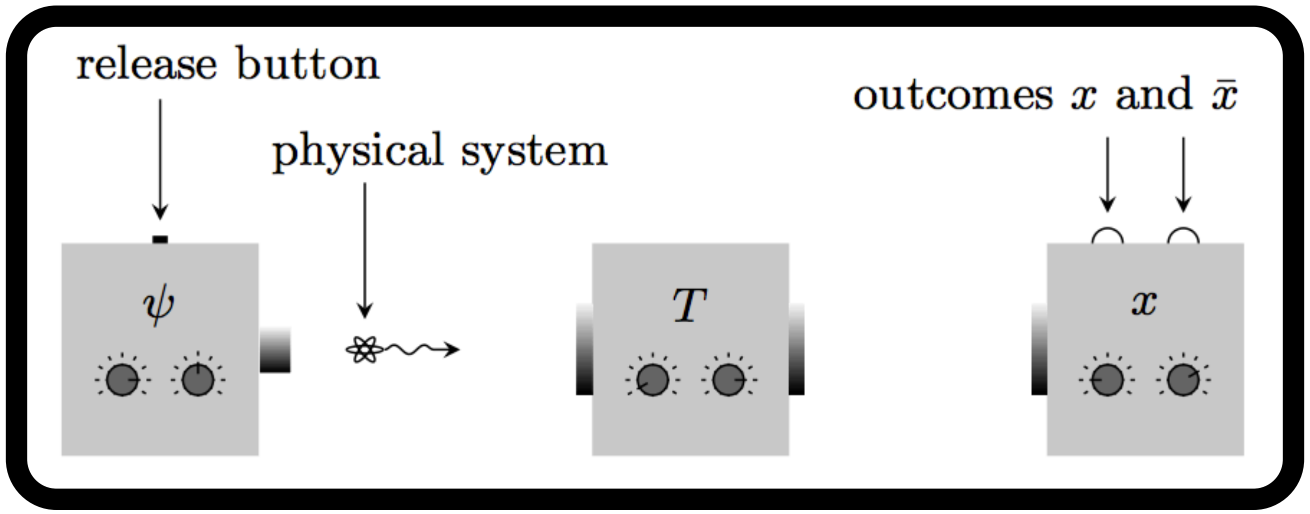
2. General Probabilistic Theories



Prepare state ω or φ with prob. ½. Result: $\frac{1}{2}\omega + \frac{1}{2}\varphi$

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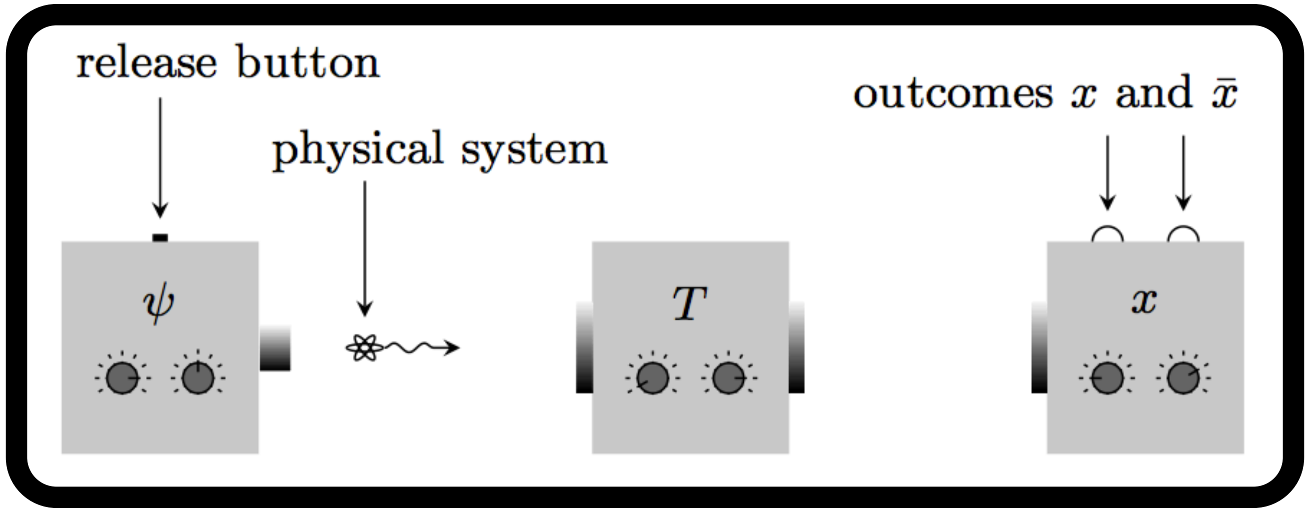

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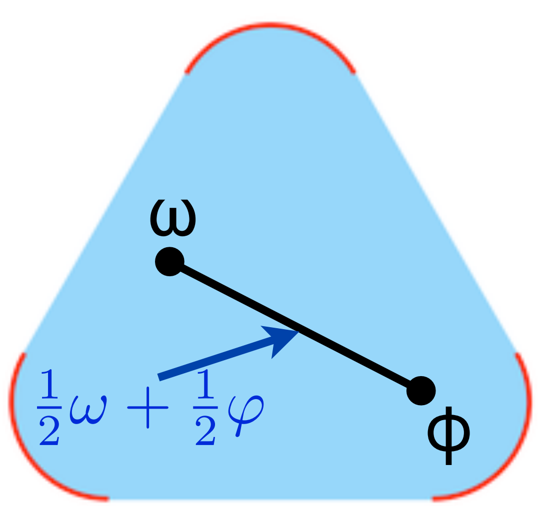
(Normalized) state spaces are convex sets.
Extremal points are pure states, others mixed.



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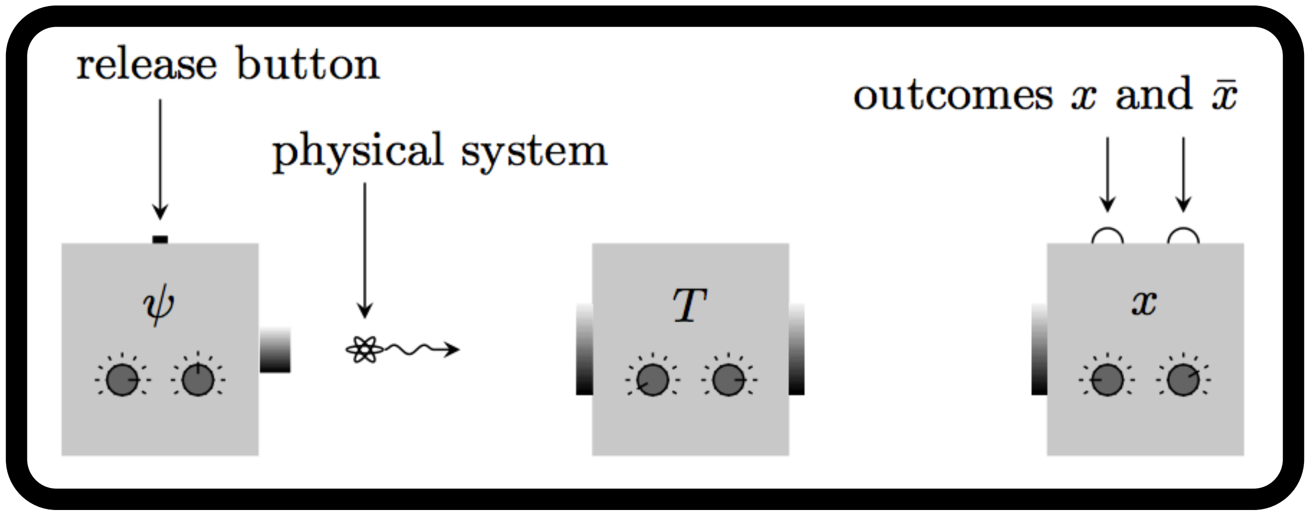

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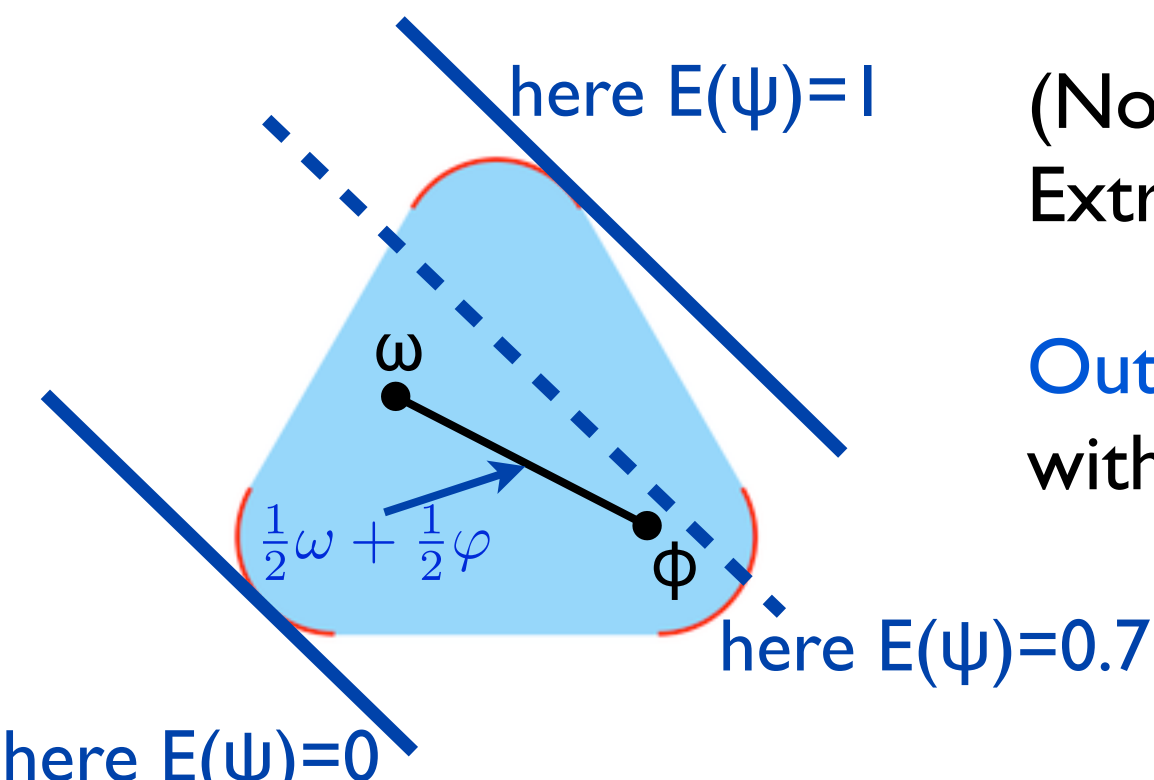

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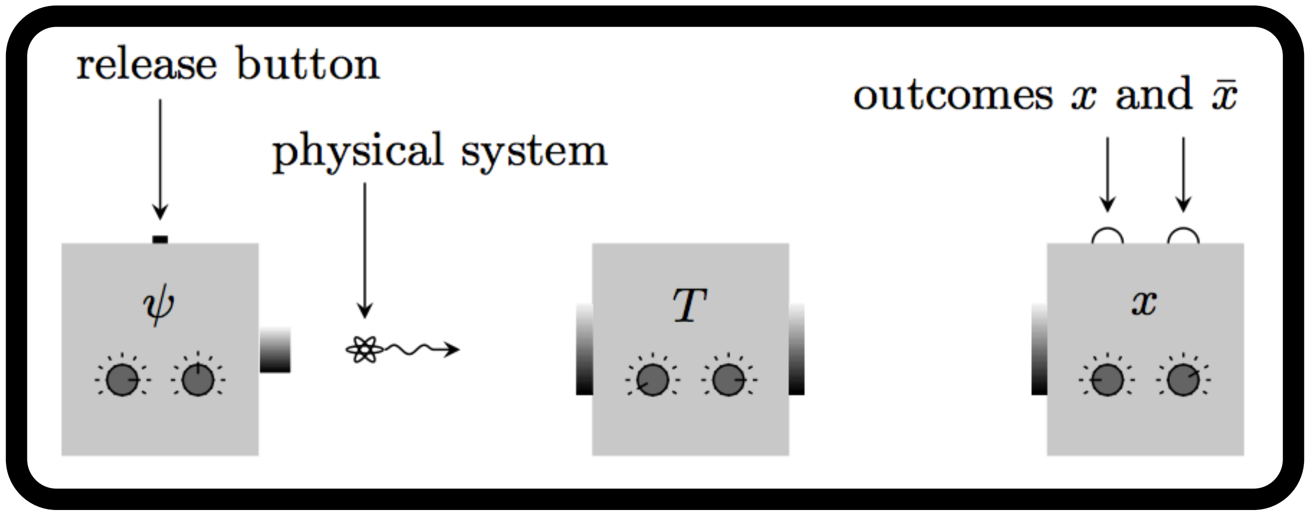

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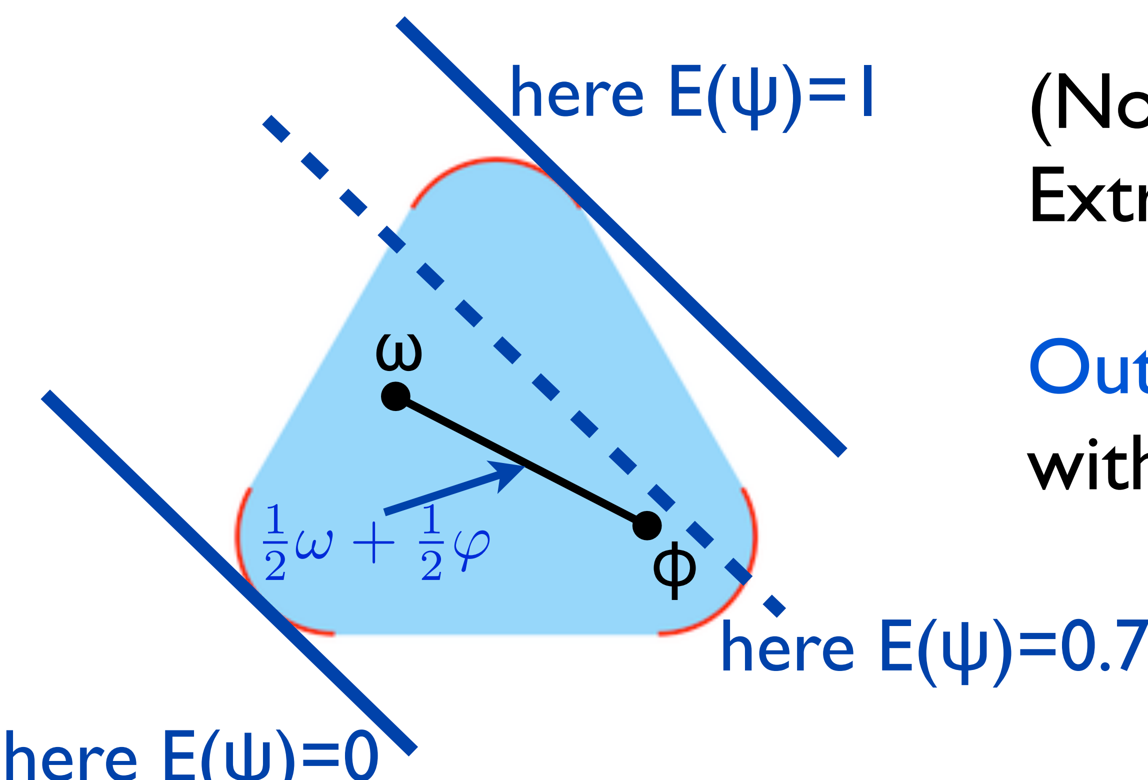

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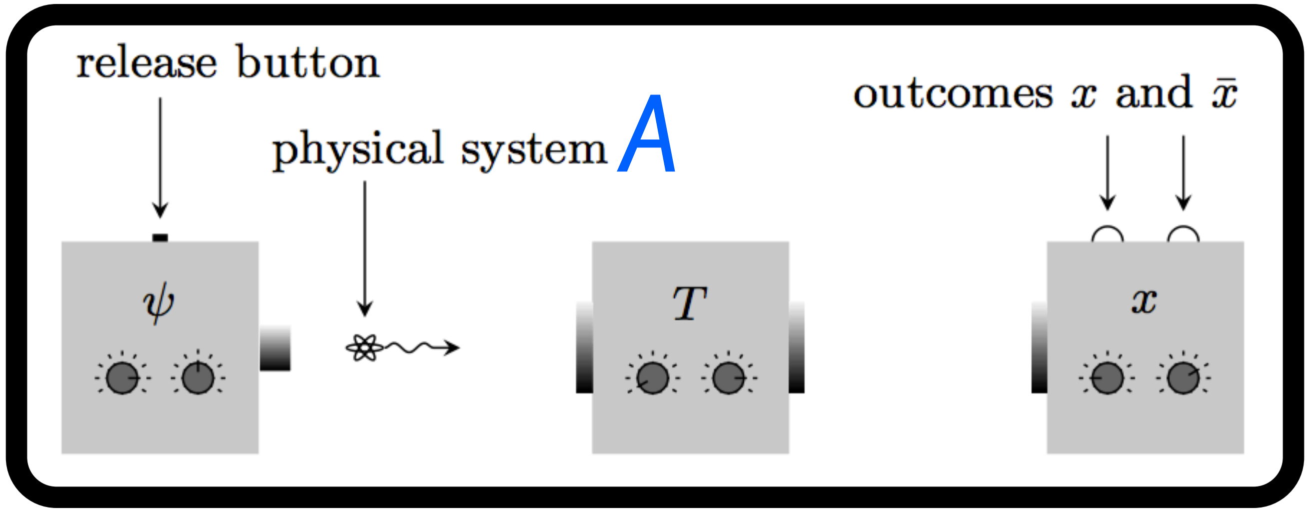
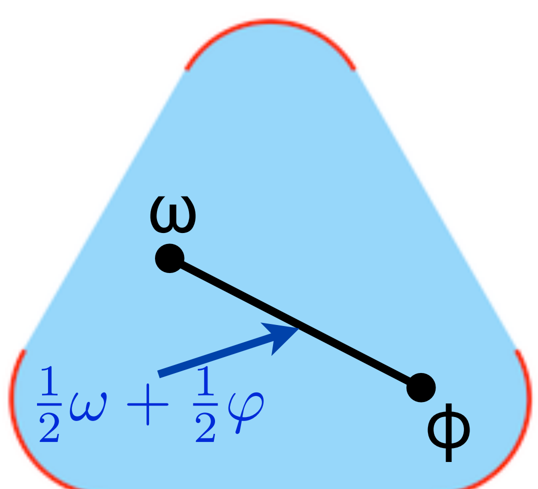
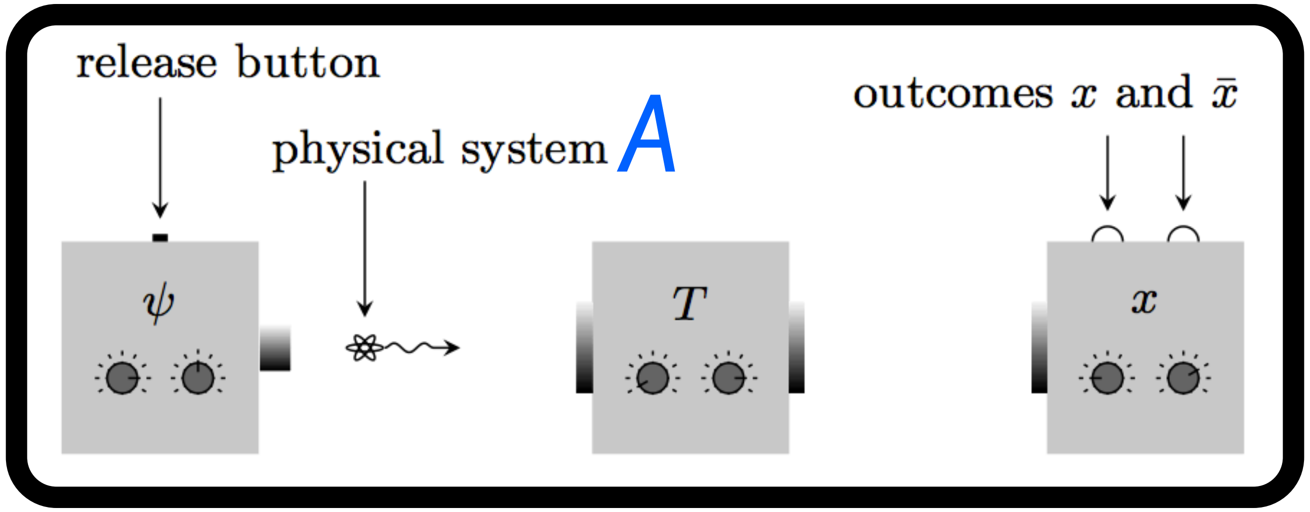


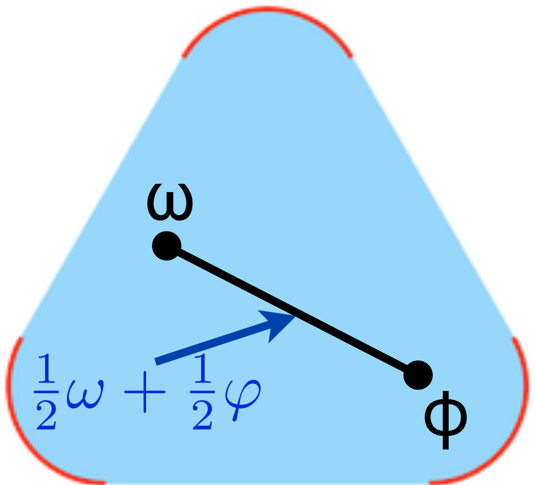


Normalized state space Ω_A

2. General Probabilistic Theories



Transformations T map (unnormalized) states to states, and are linear.



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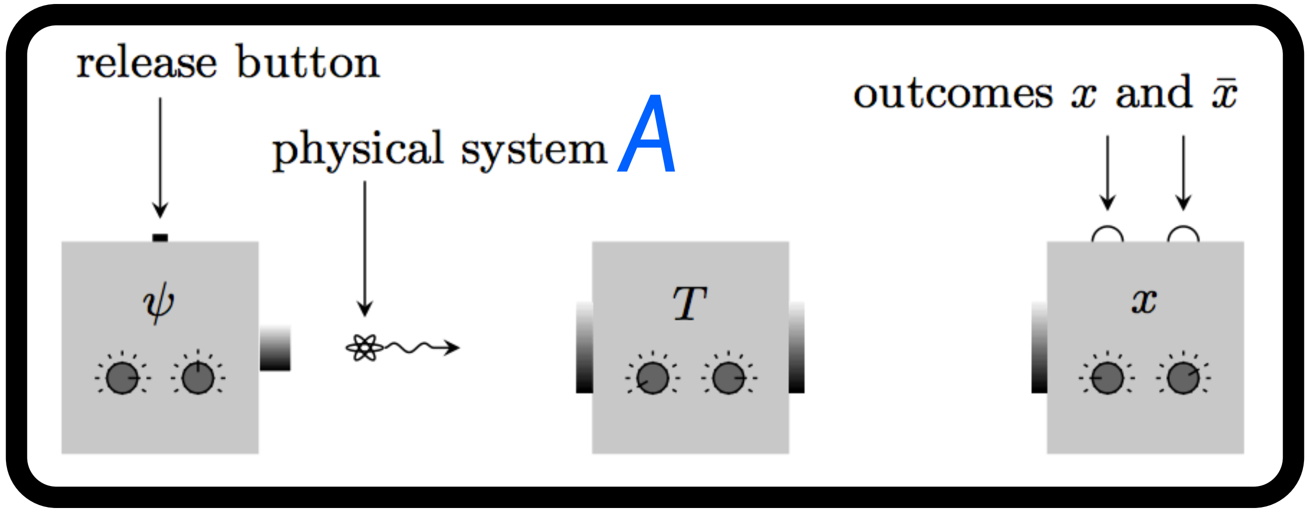

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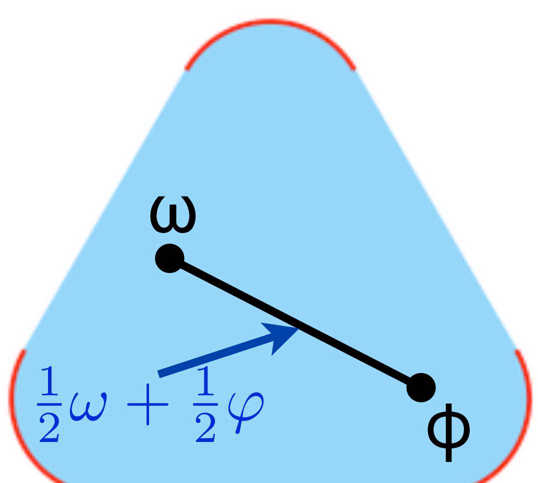

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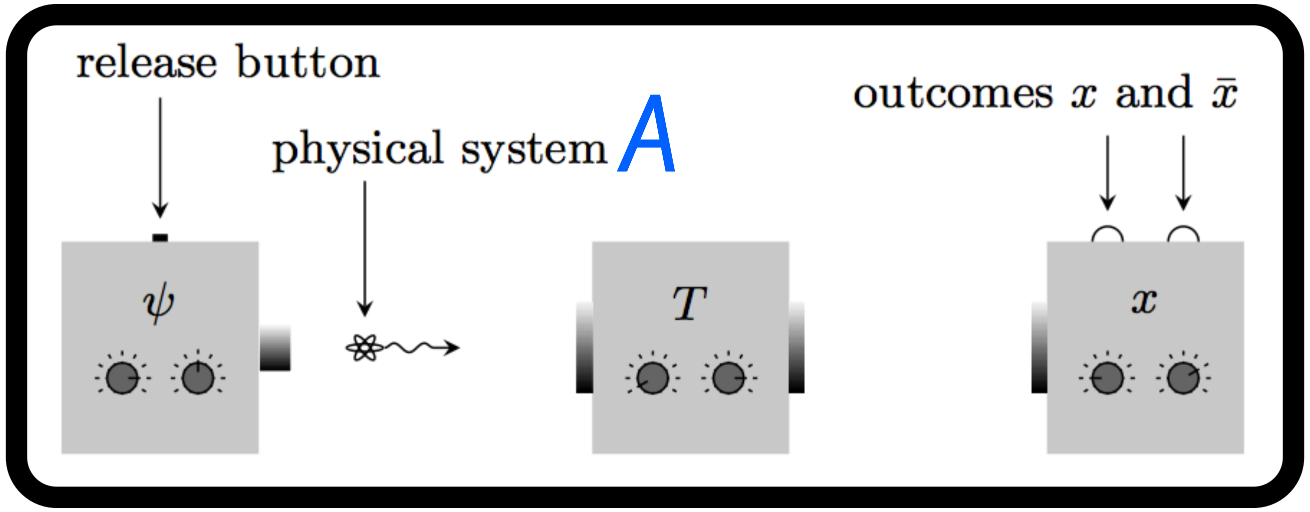

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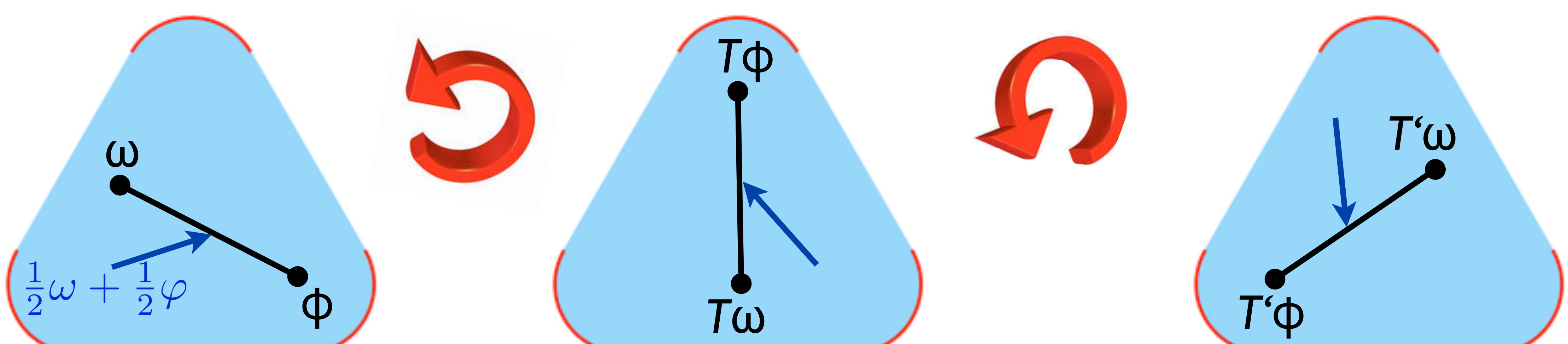

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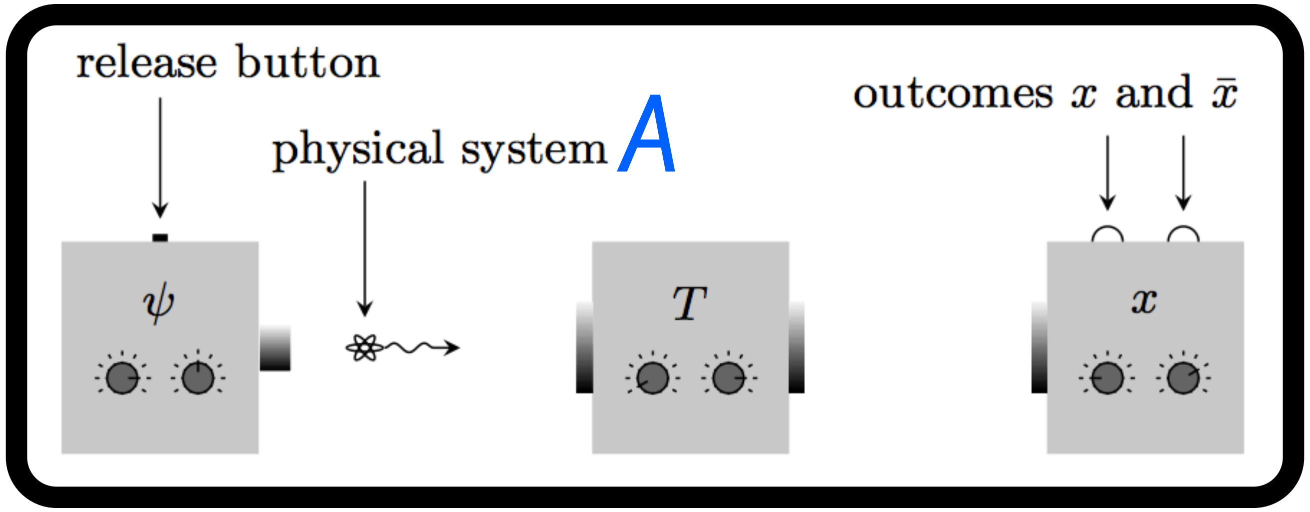
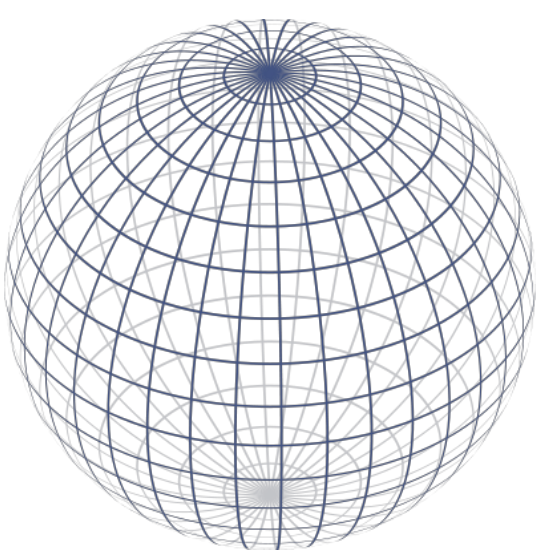


Not all symmetries have to be in $\mathcal{G}_A$.



Qubit: Ω_A is the 3D unit ball,

$\mathcal{G}_A = SO(3)$ (no reflections!)

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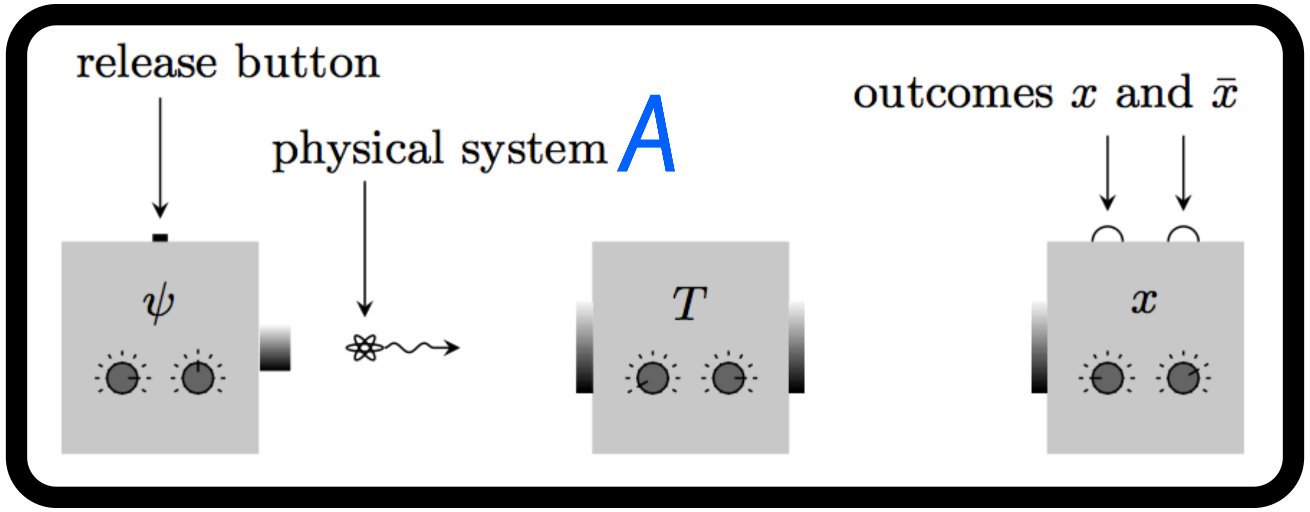

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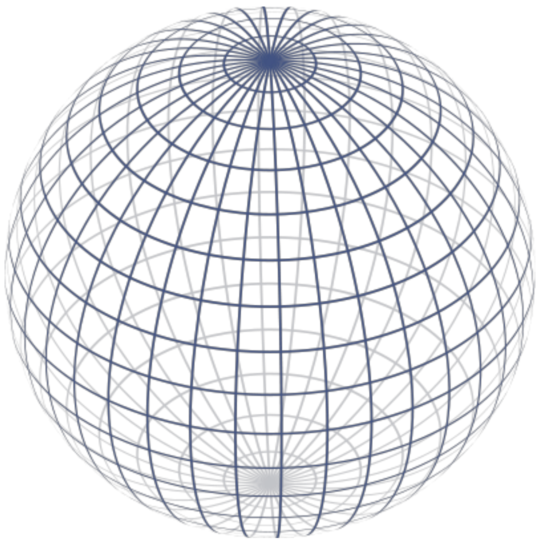

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$\Rightarrow$ A system is a pair $(\Omega_A, \mathcal{G}_A)$.

state space

reversible transformations

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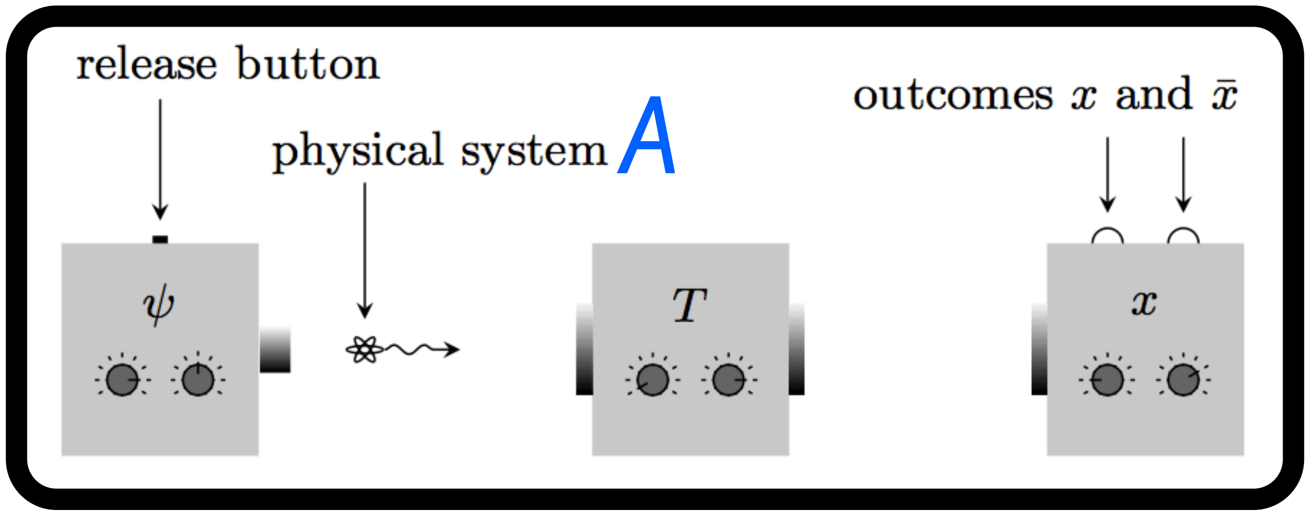

Axiom II (Reversibility):
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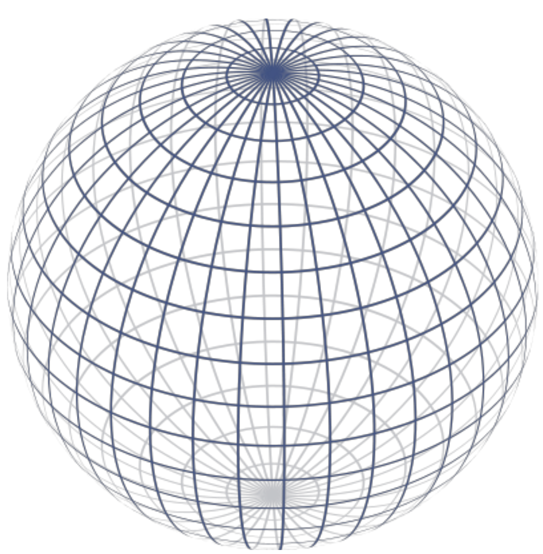

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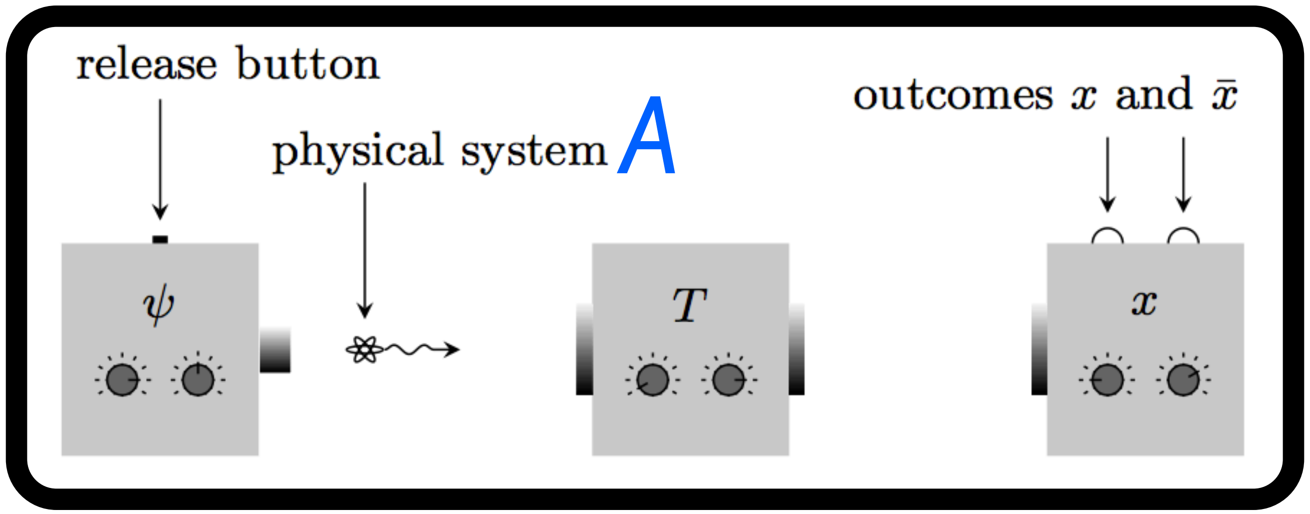

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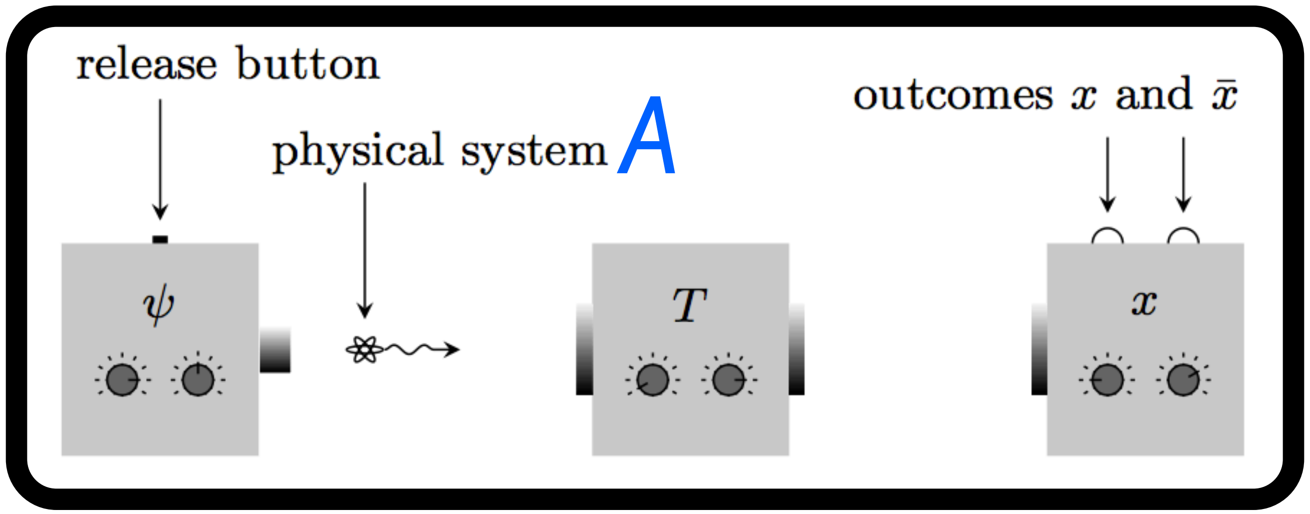

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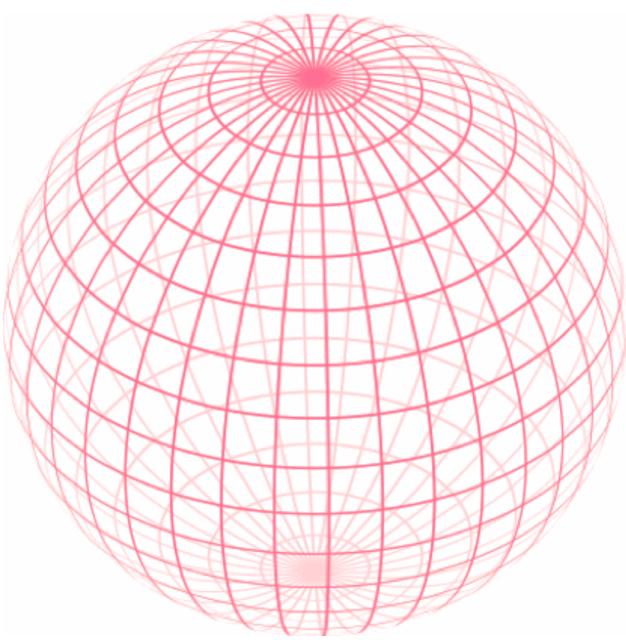

✓

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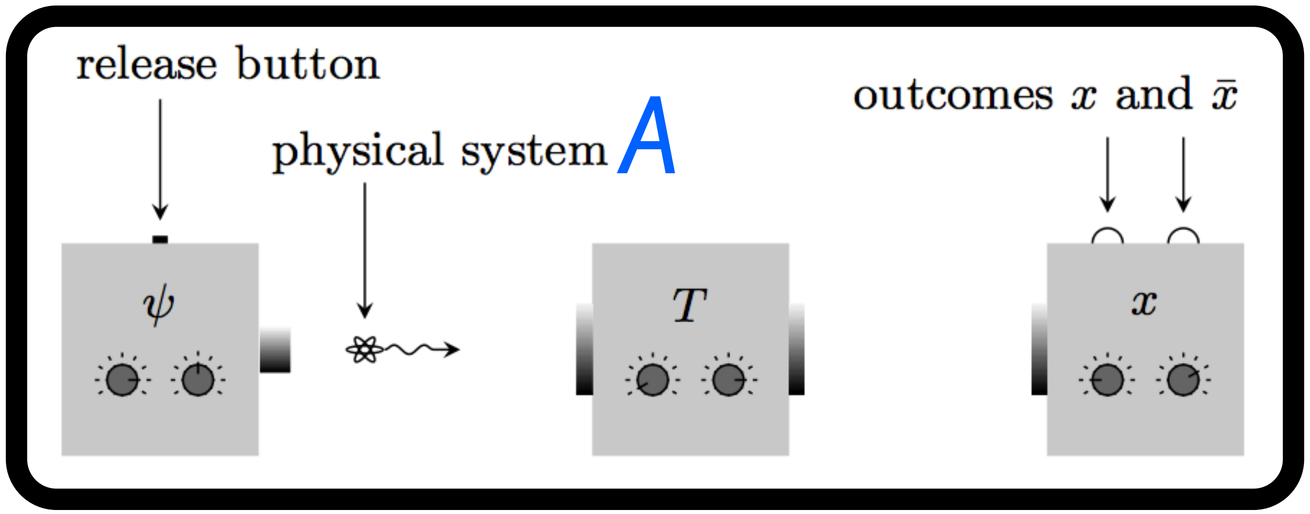

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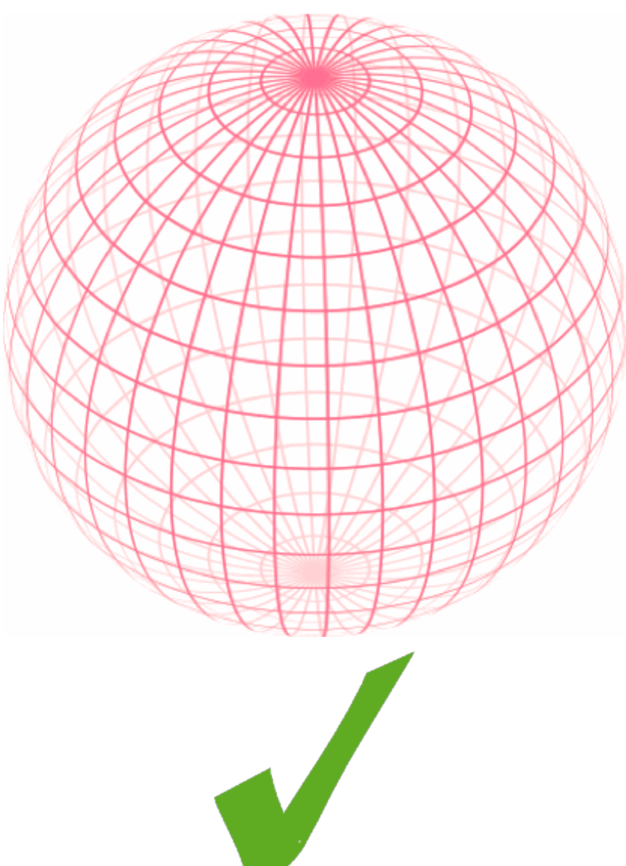

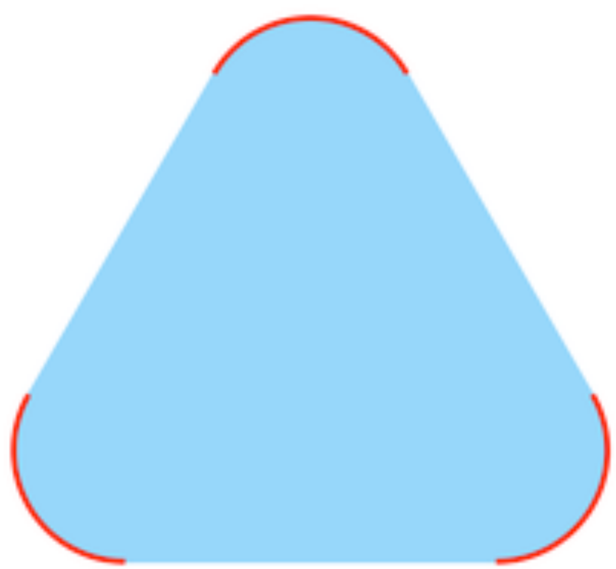

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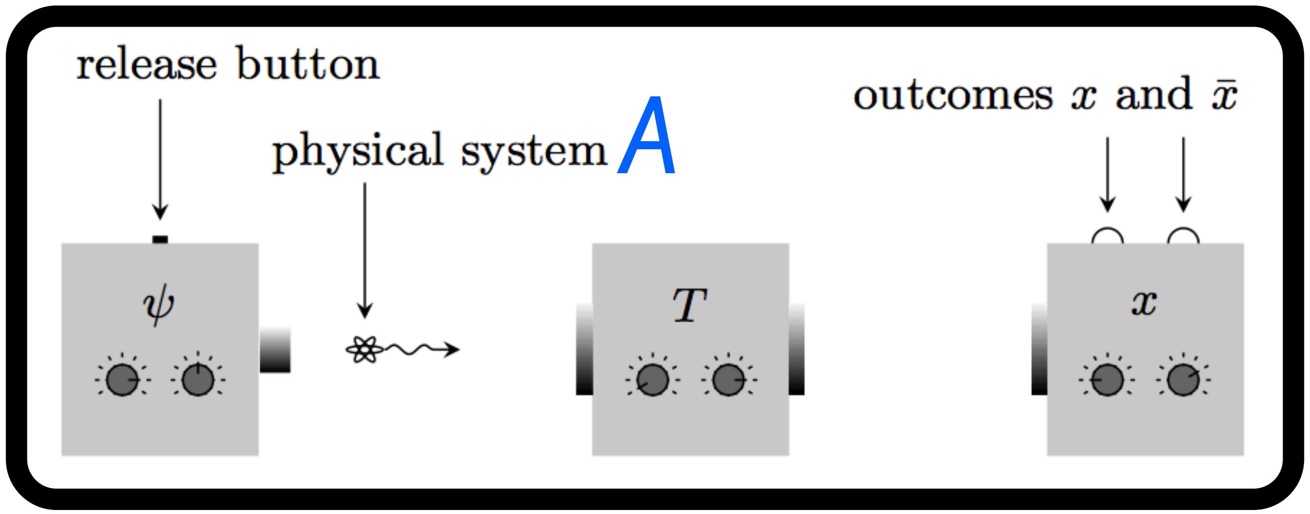

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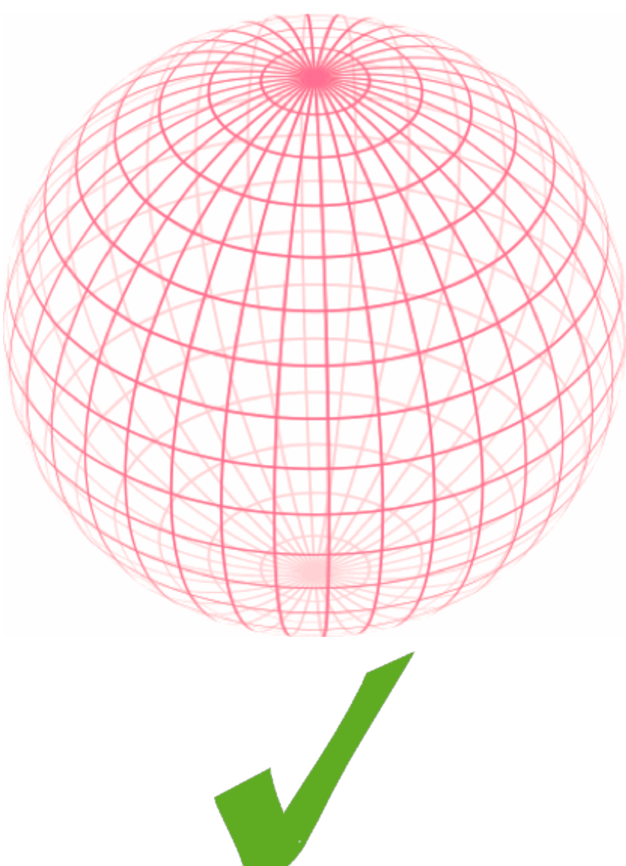

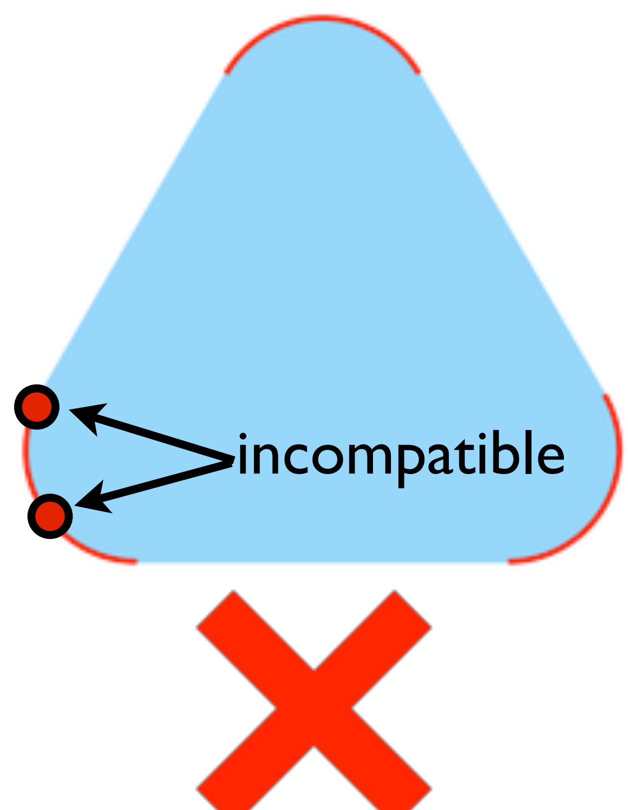

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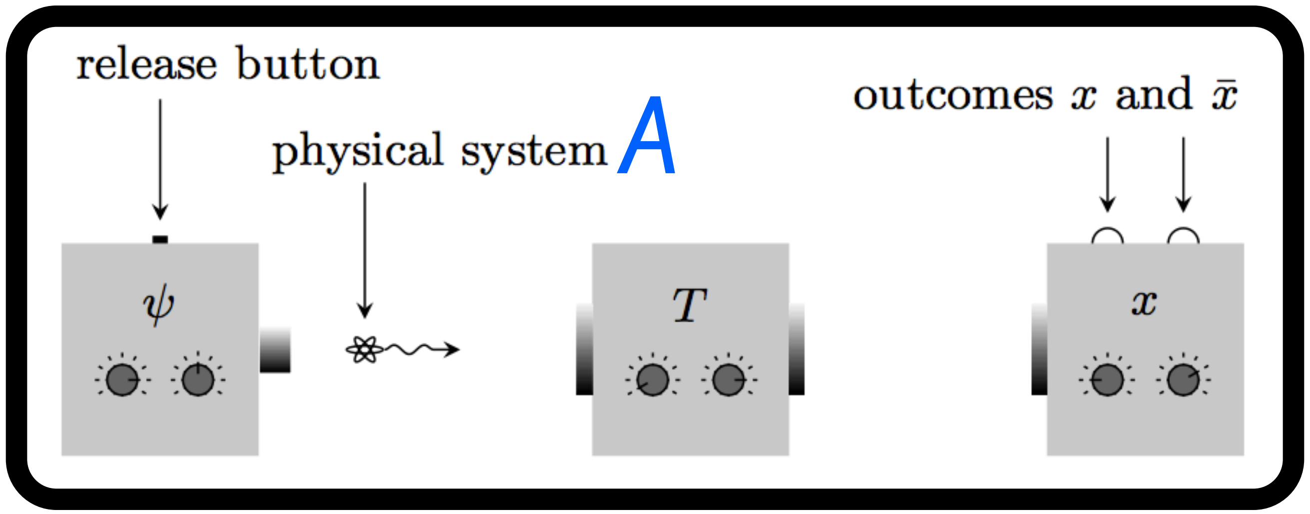

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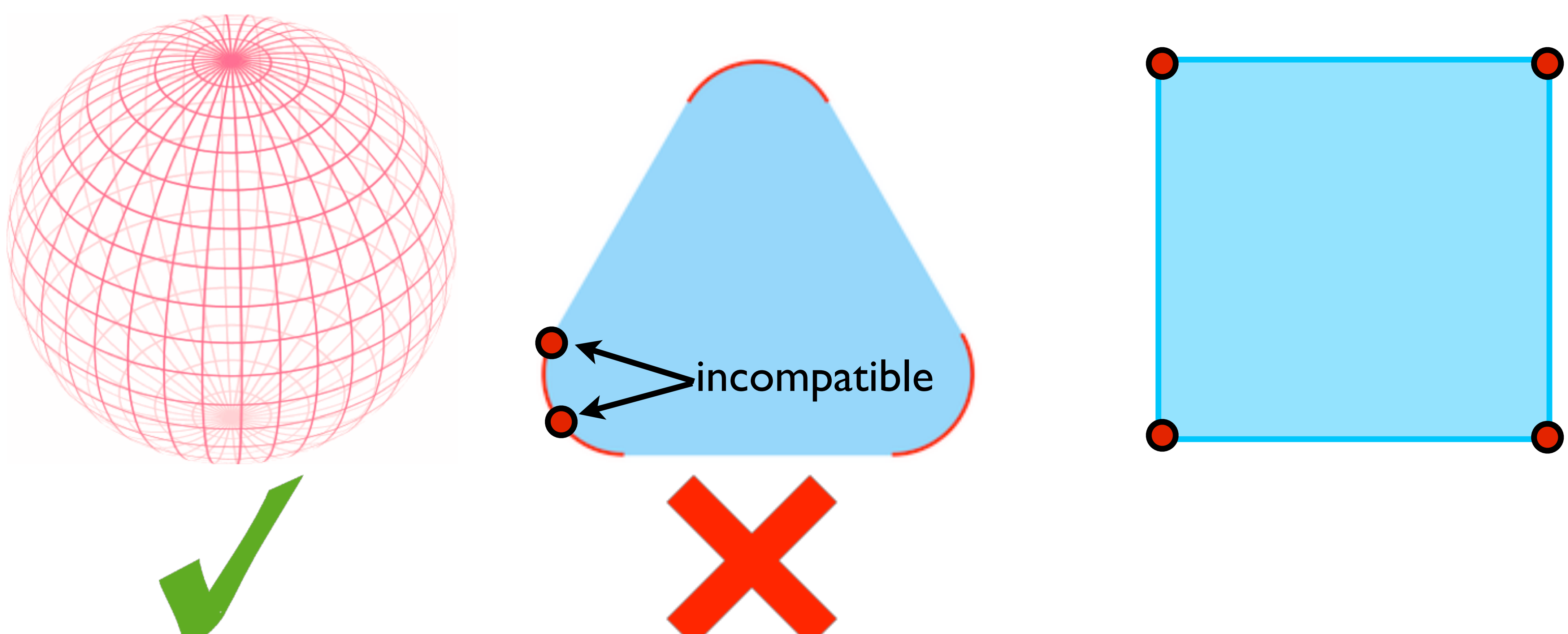

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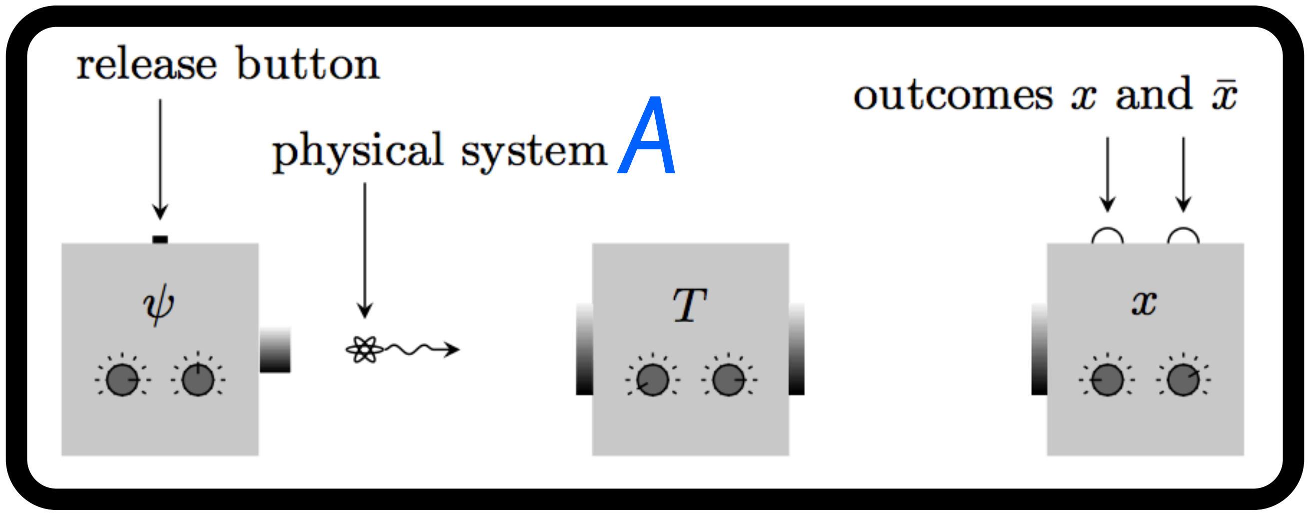

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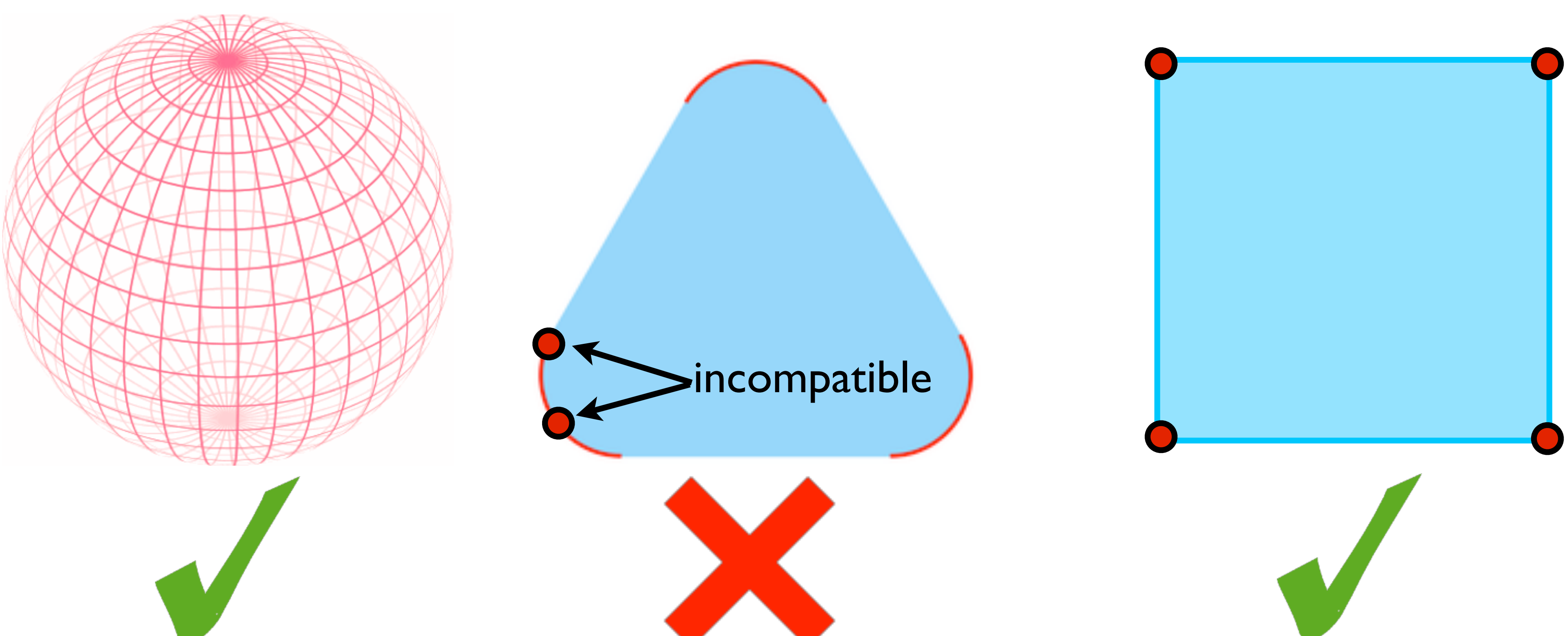

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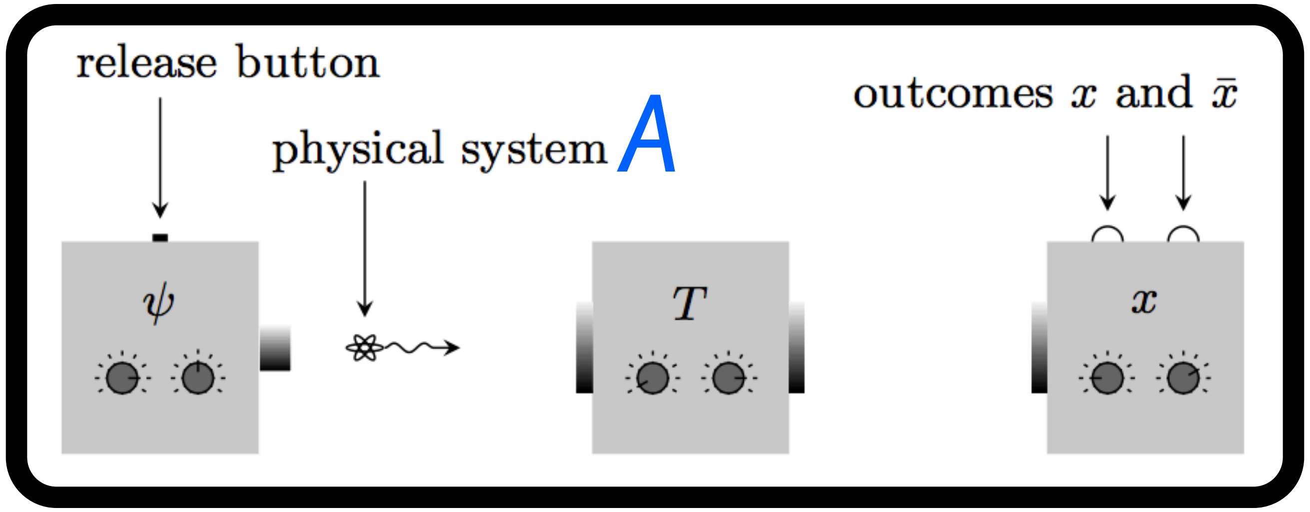

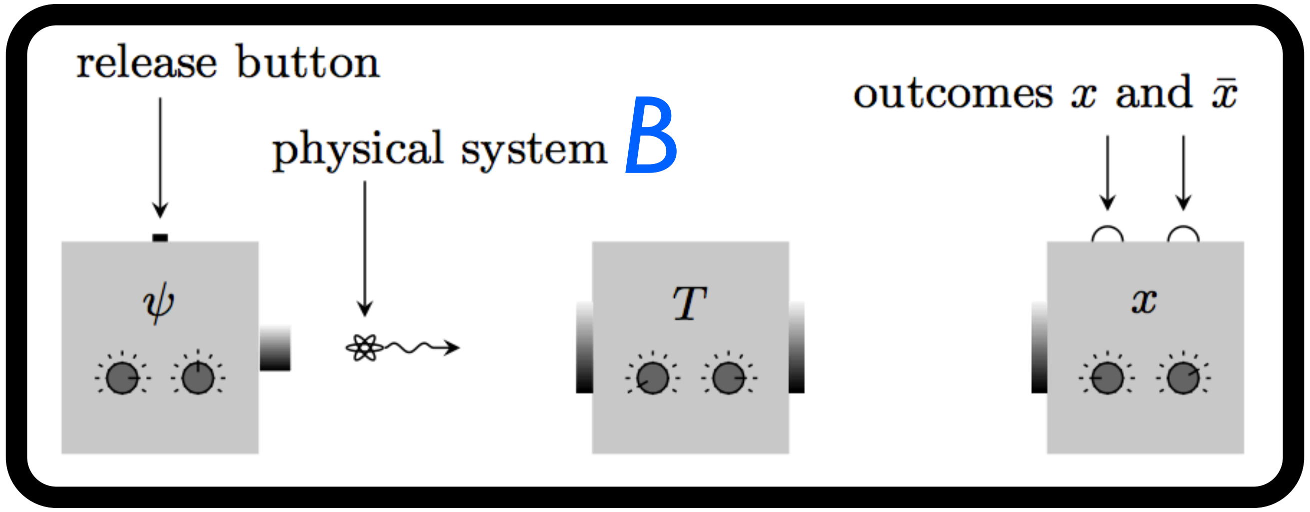

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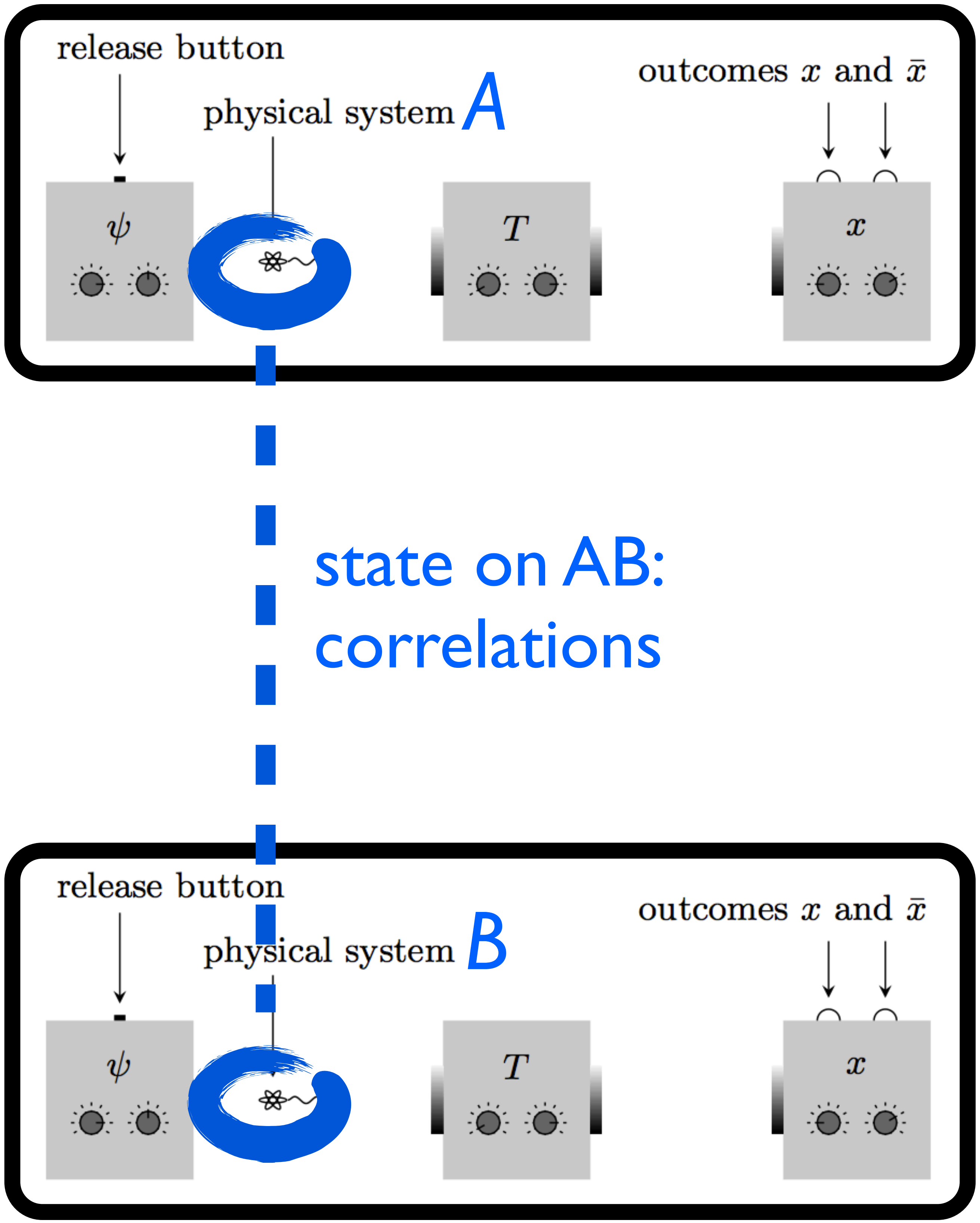

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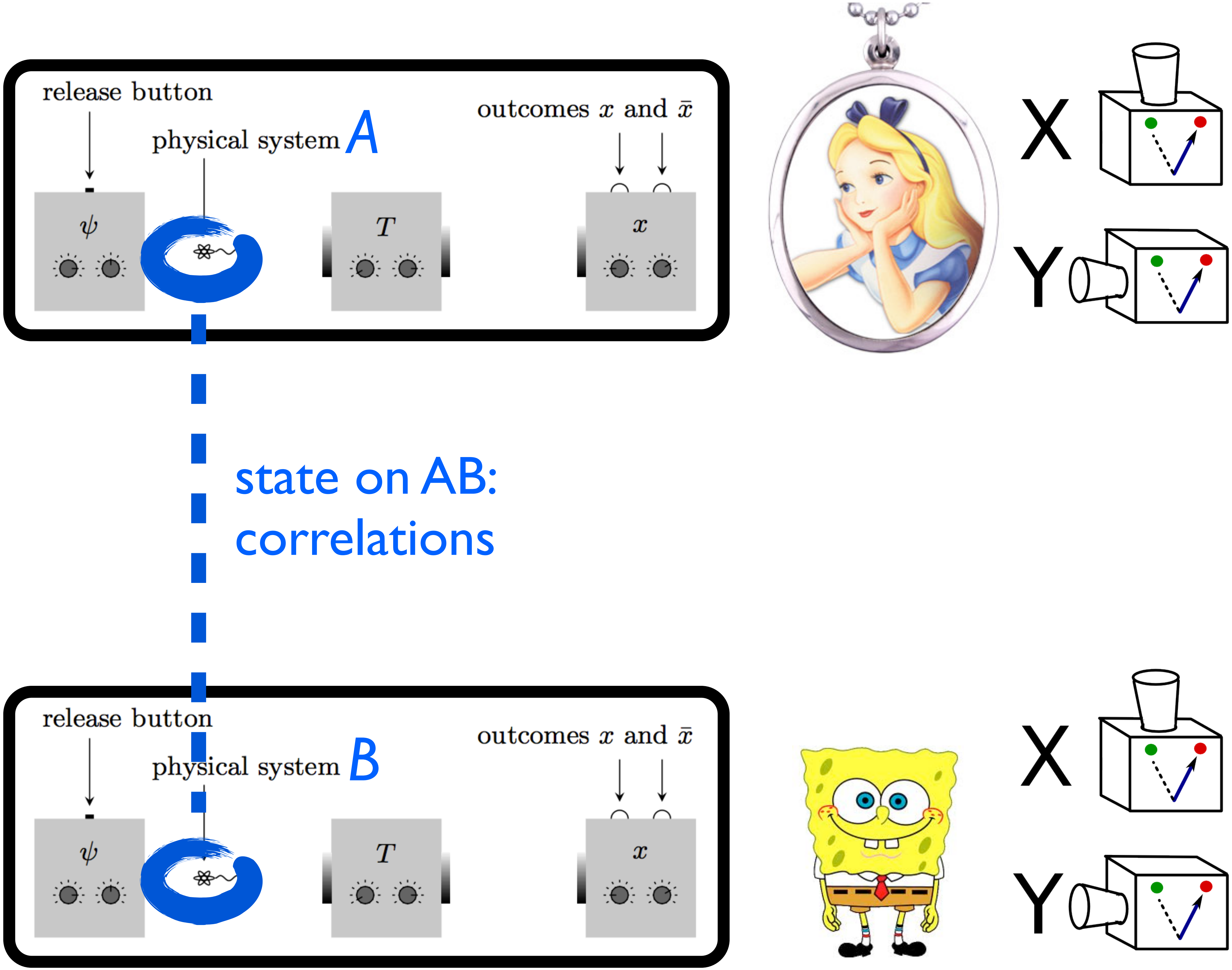

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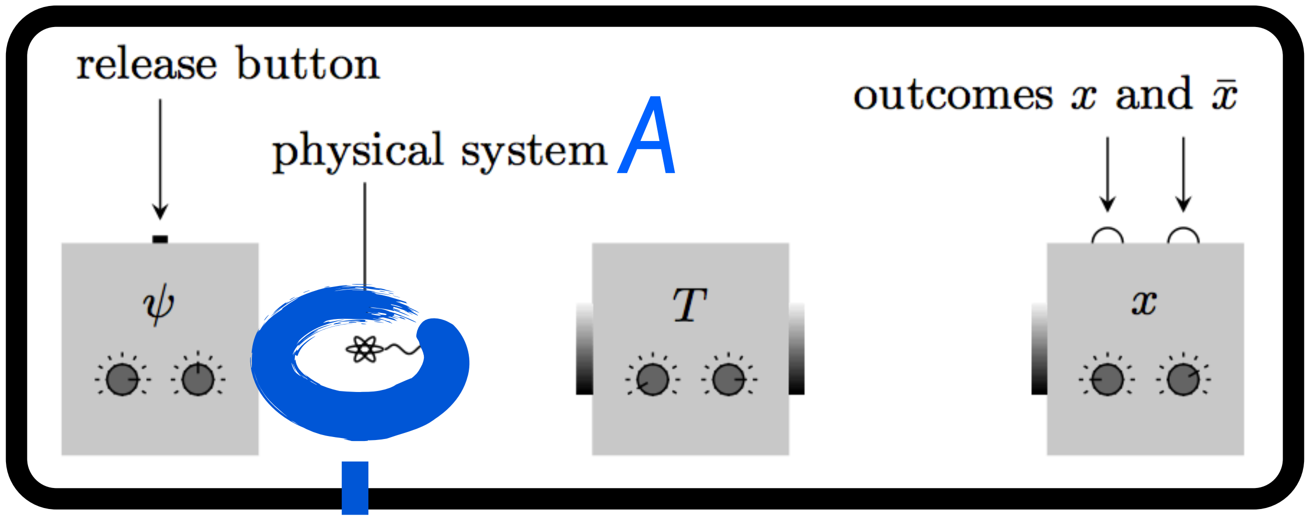
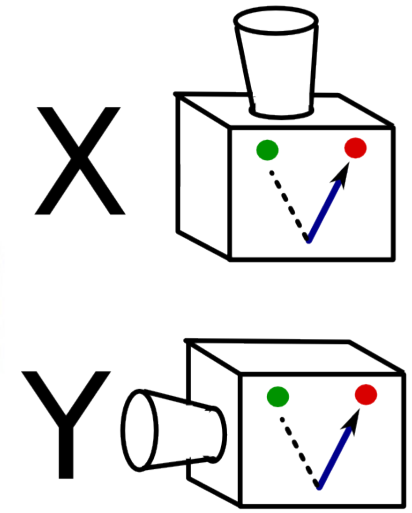
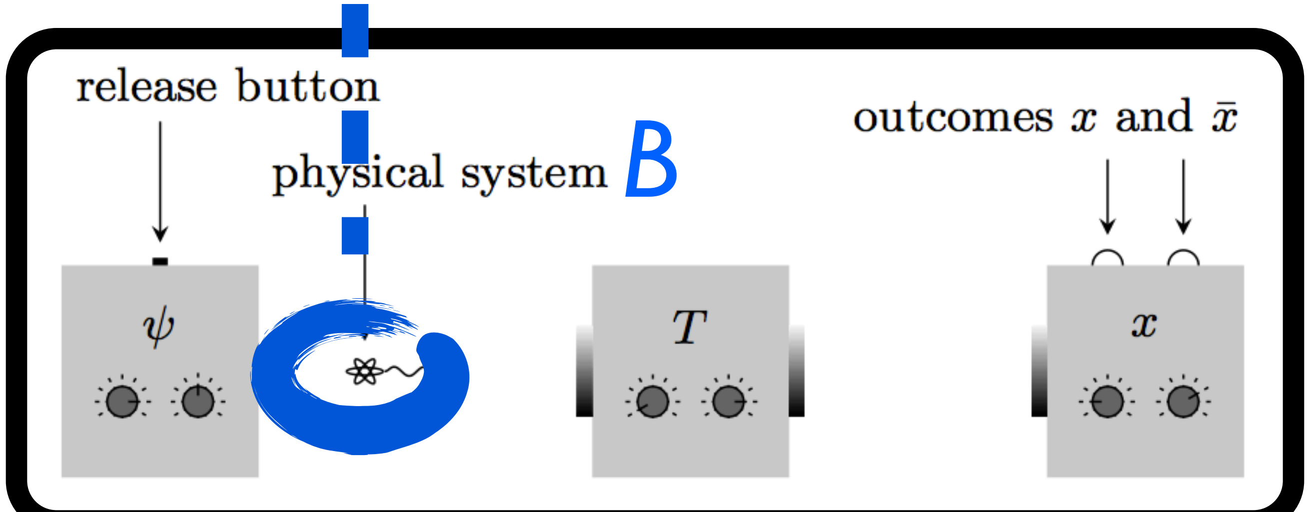
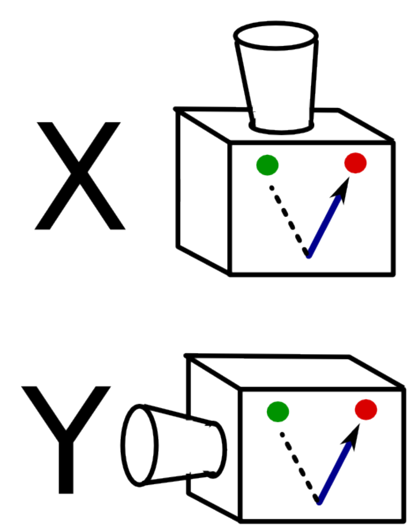




state on AB:
correlations

No-signalling condition:
Alice's probabilities do not depend on Bob's choice of measurement.





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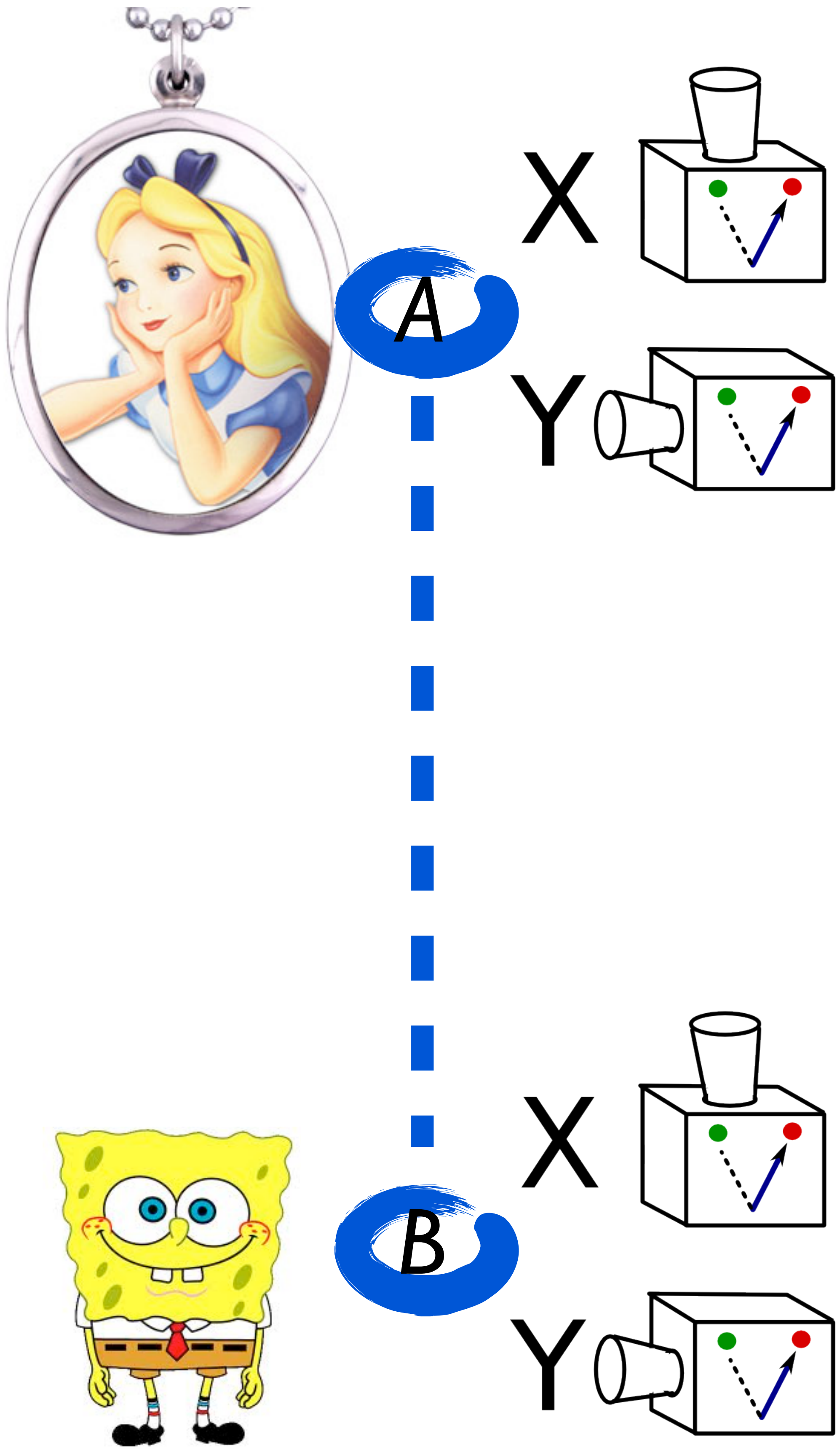

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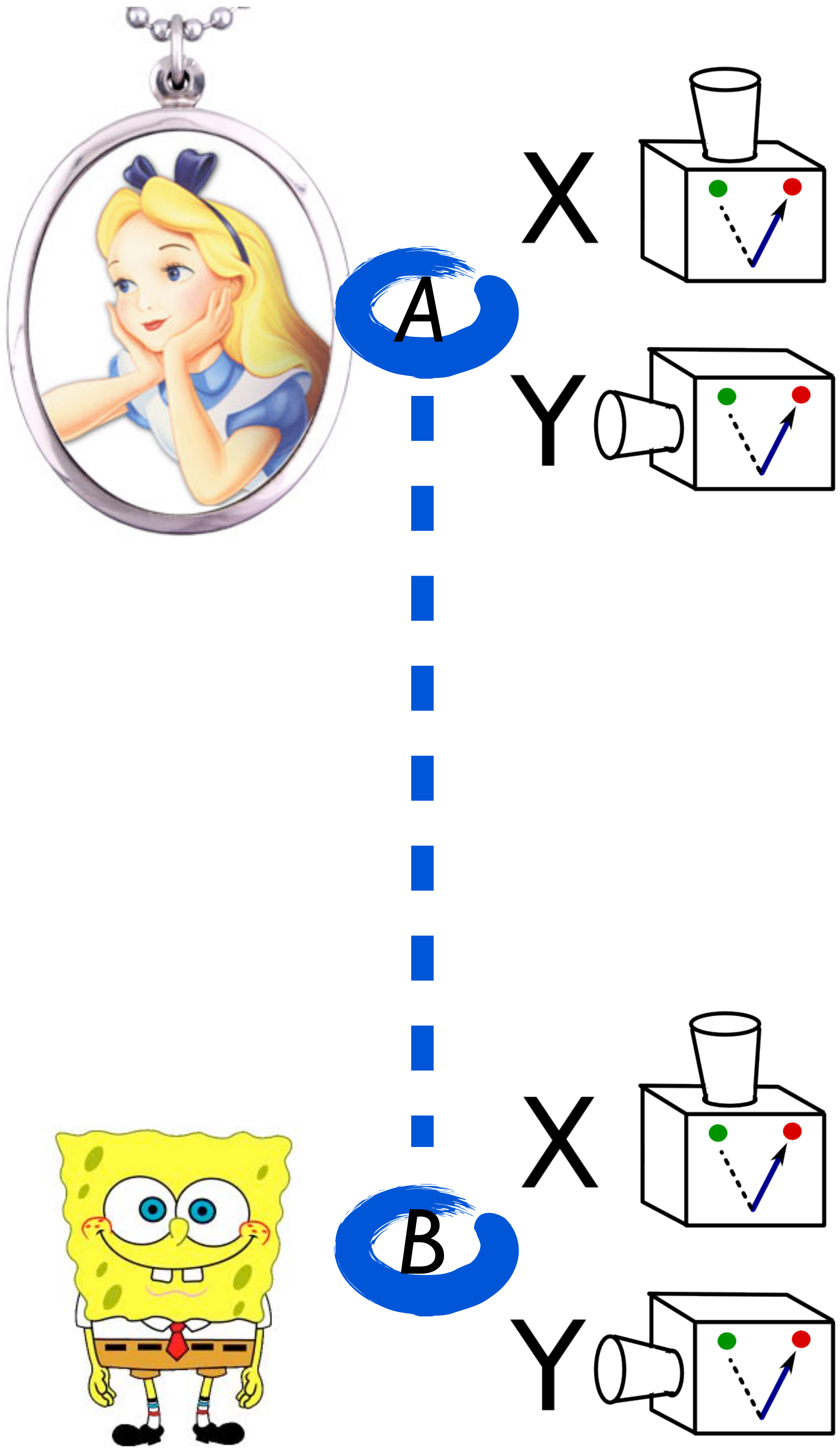

Axiom I: States on AB are uniquely determined by correlations of local measurements on A,B.

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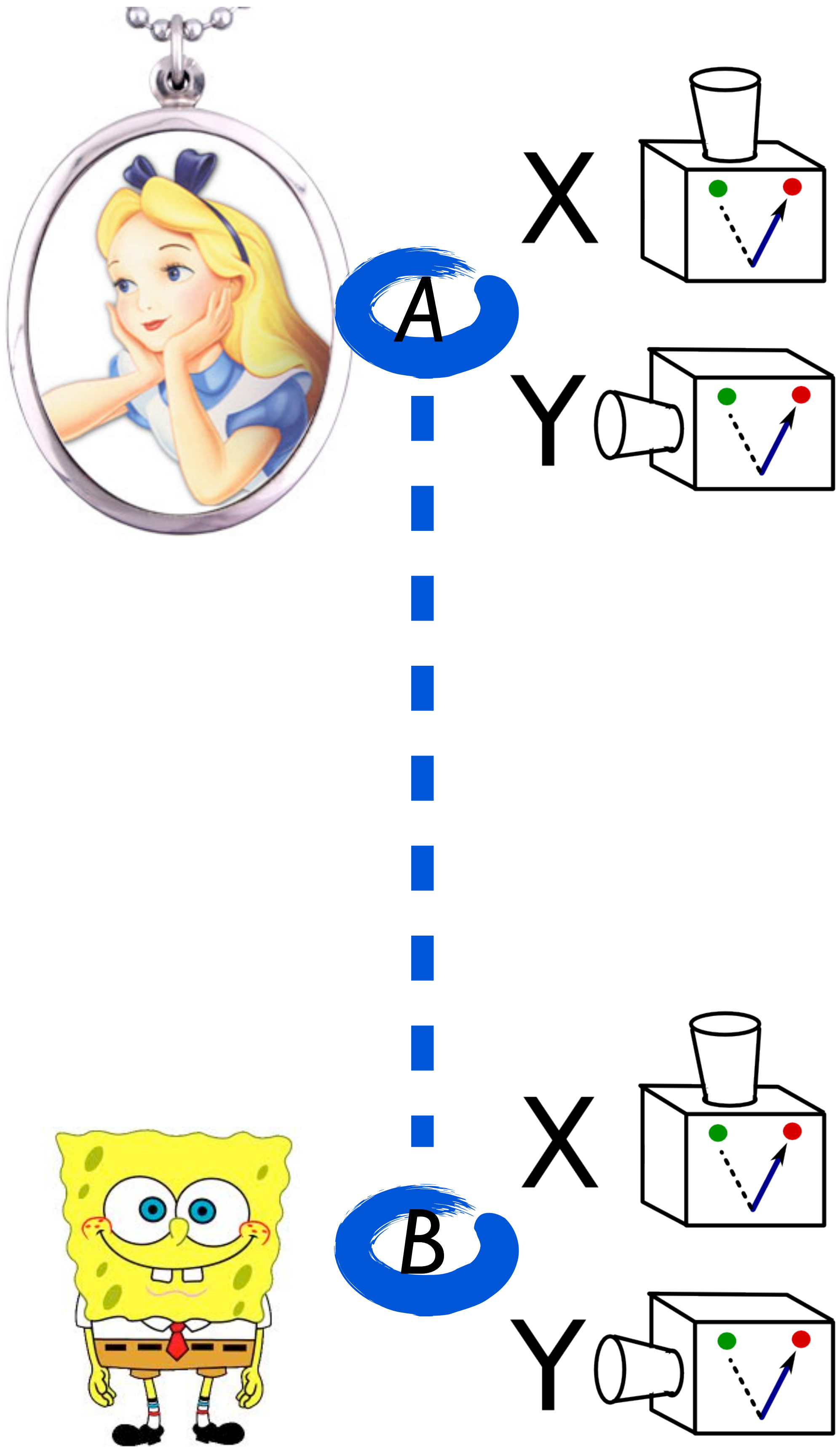

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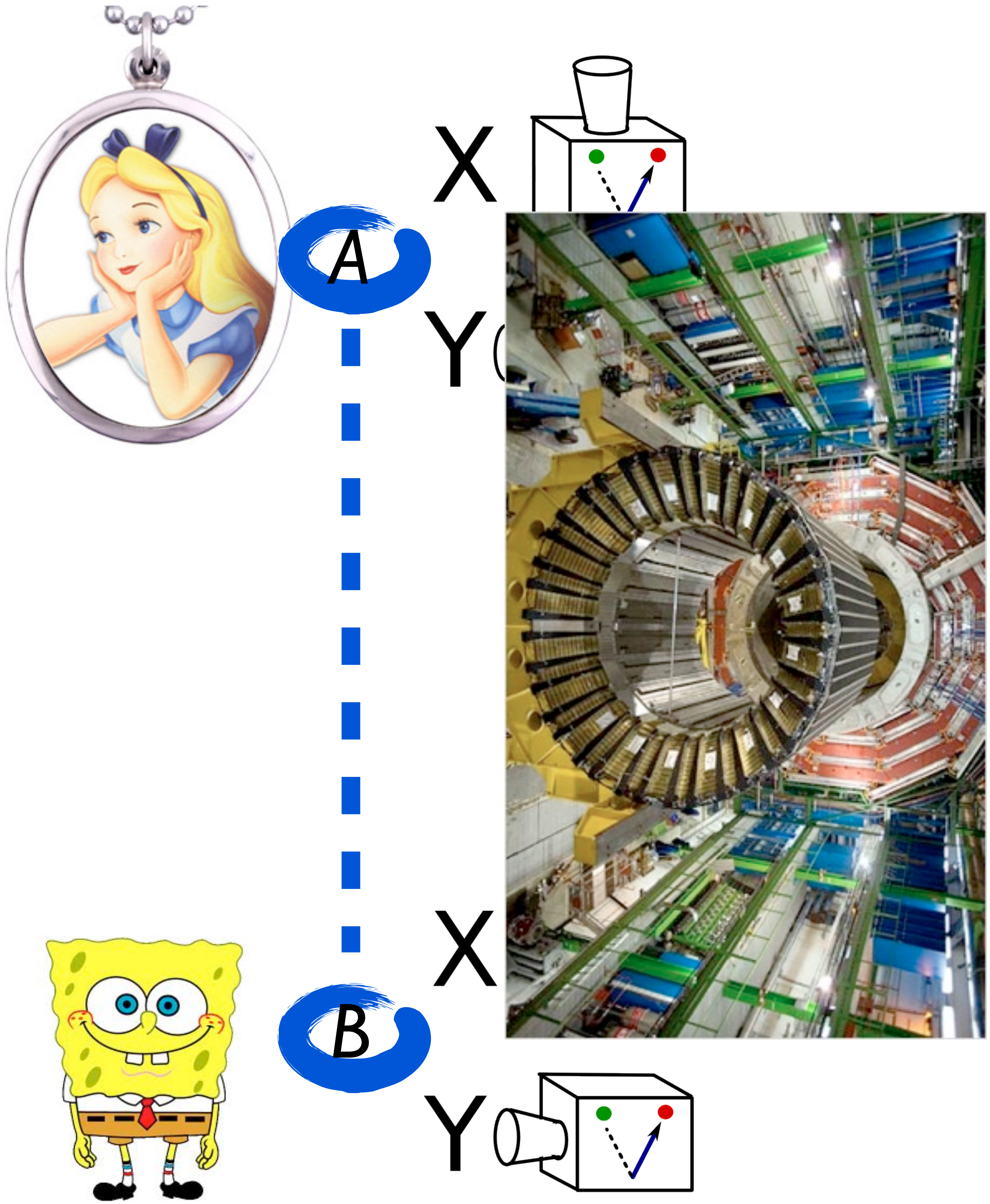

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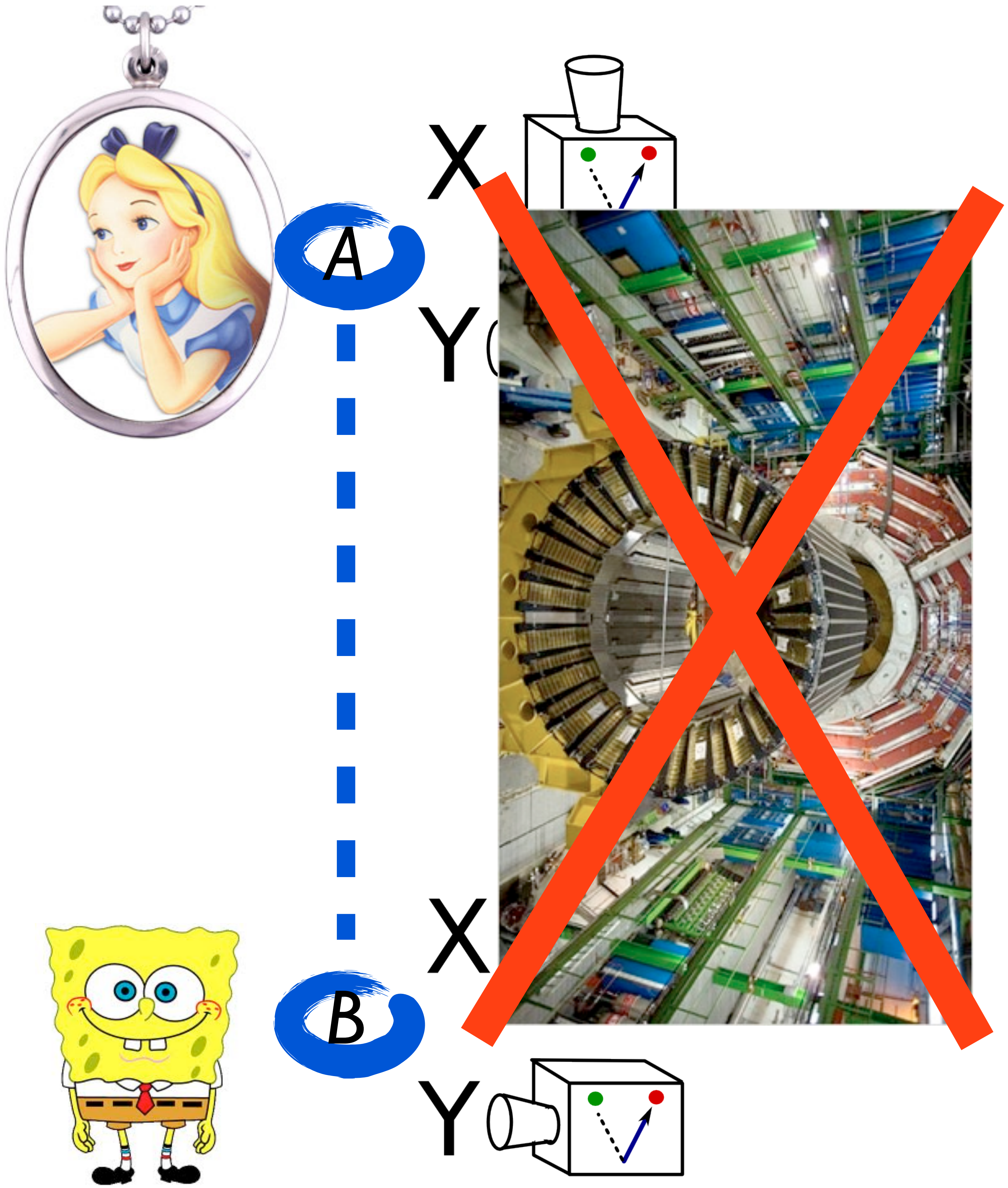

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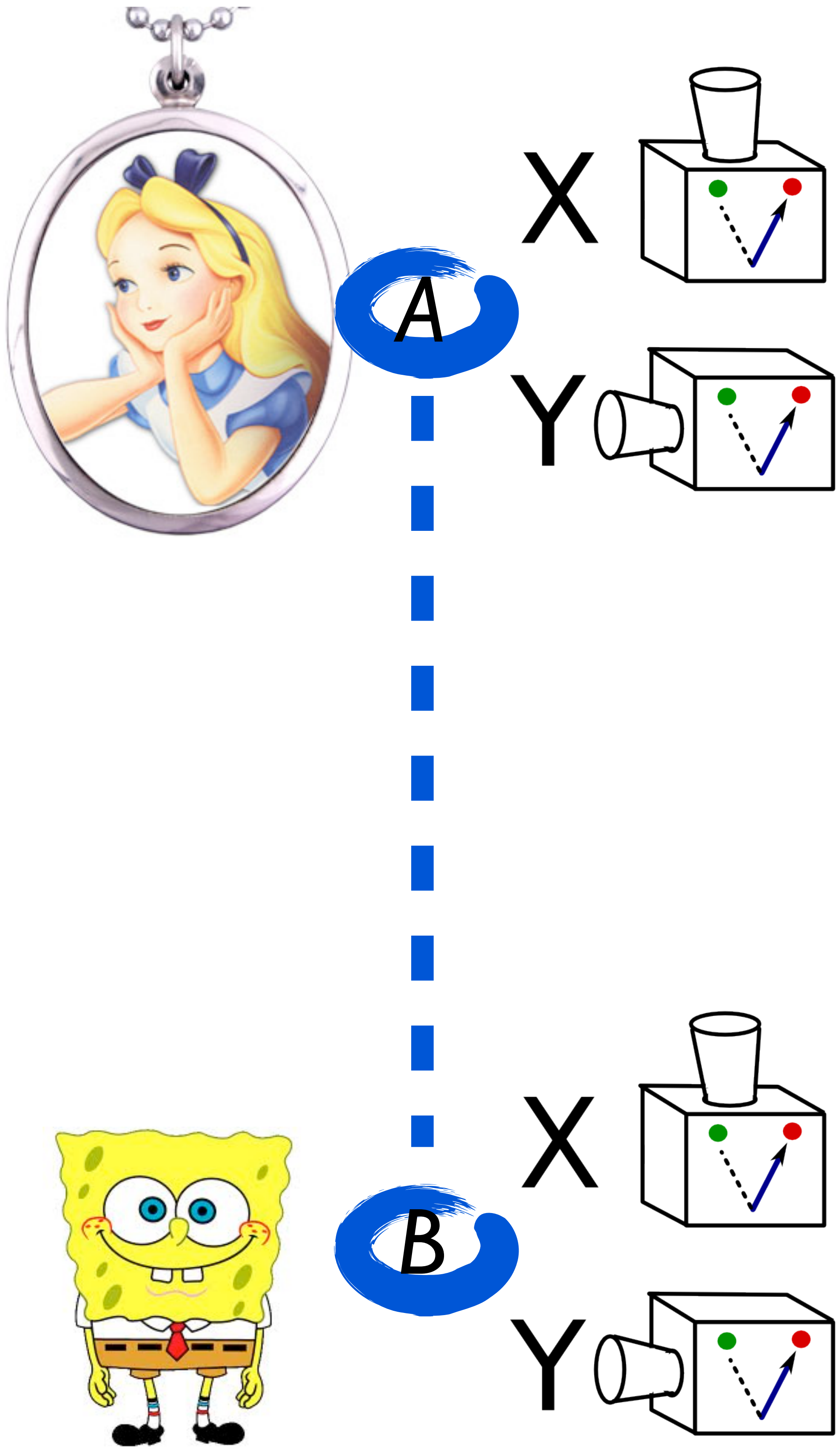

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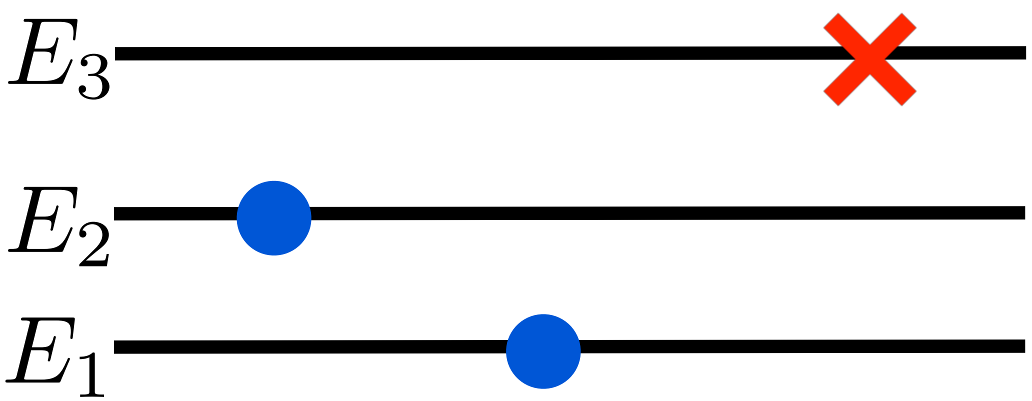
Global state space $\Omega_{AB} \subset A \otimes B$ but not uniquely fixed!

3. The Subspace Axiom

Some 3-level system:



Impossible to have system in 3rd level
$\Rightarrow$ find particle there with probab. 0

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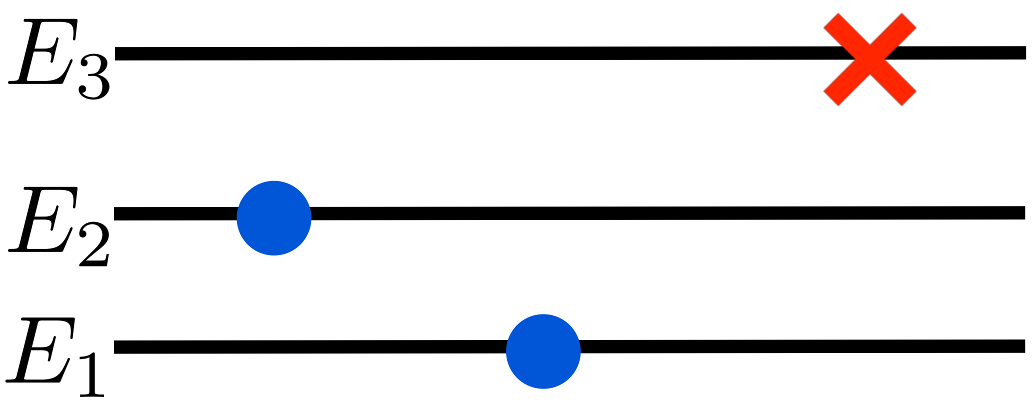

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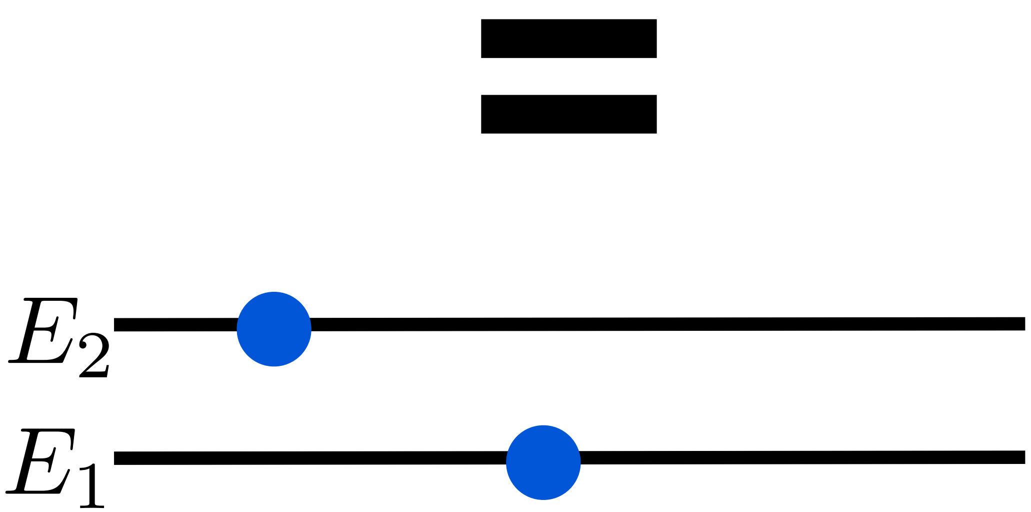

2-level system.

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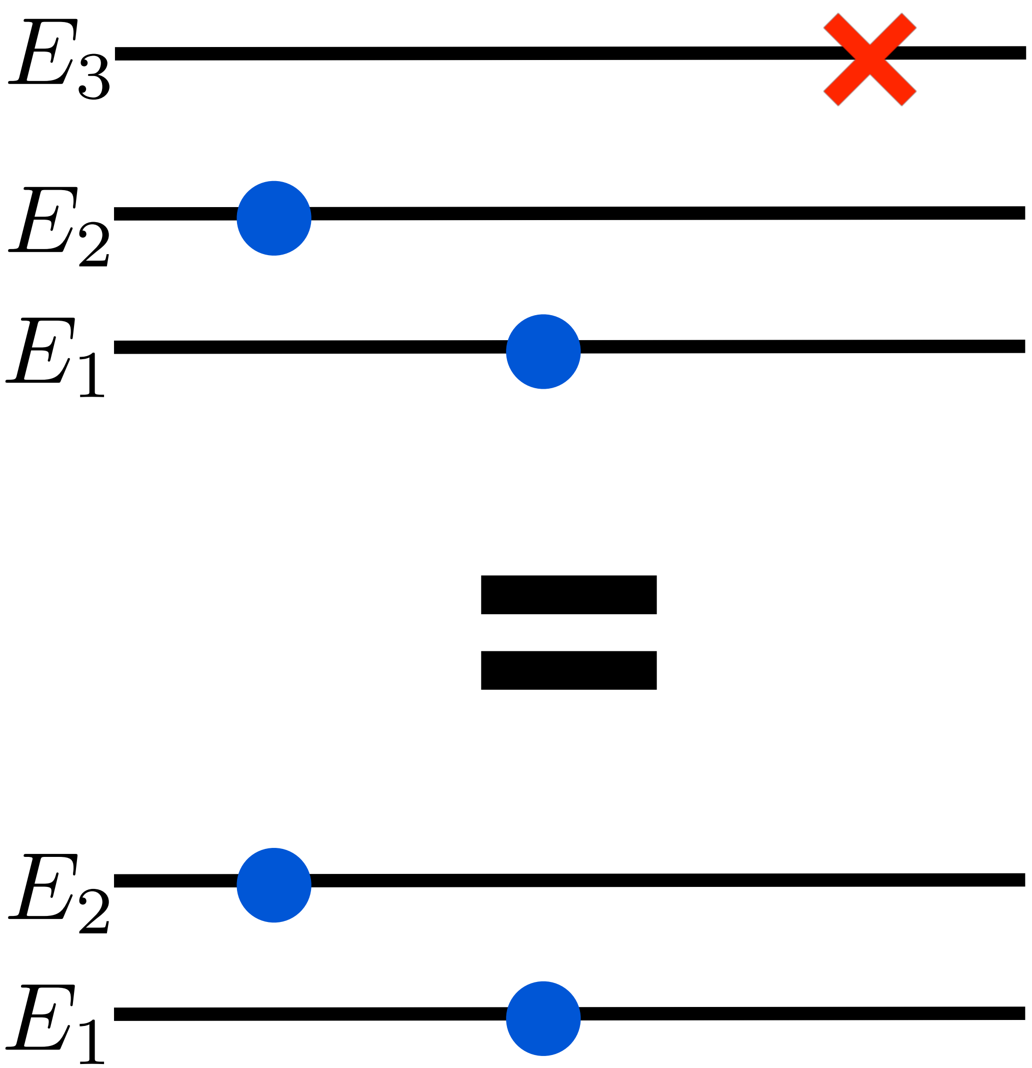

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QT: $\rho^{(3)} = \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \rho^{(2)} = \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}$

CPT: $P^{(3)} = (P_1, P_2, 0) \longrightarrow P^{(2)} = (P_1, P_2)$

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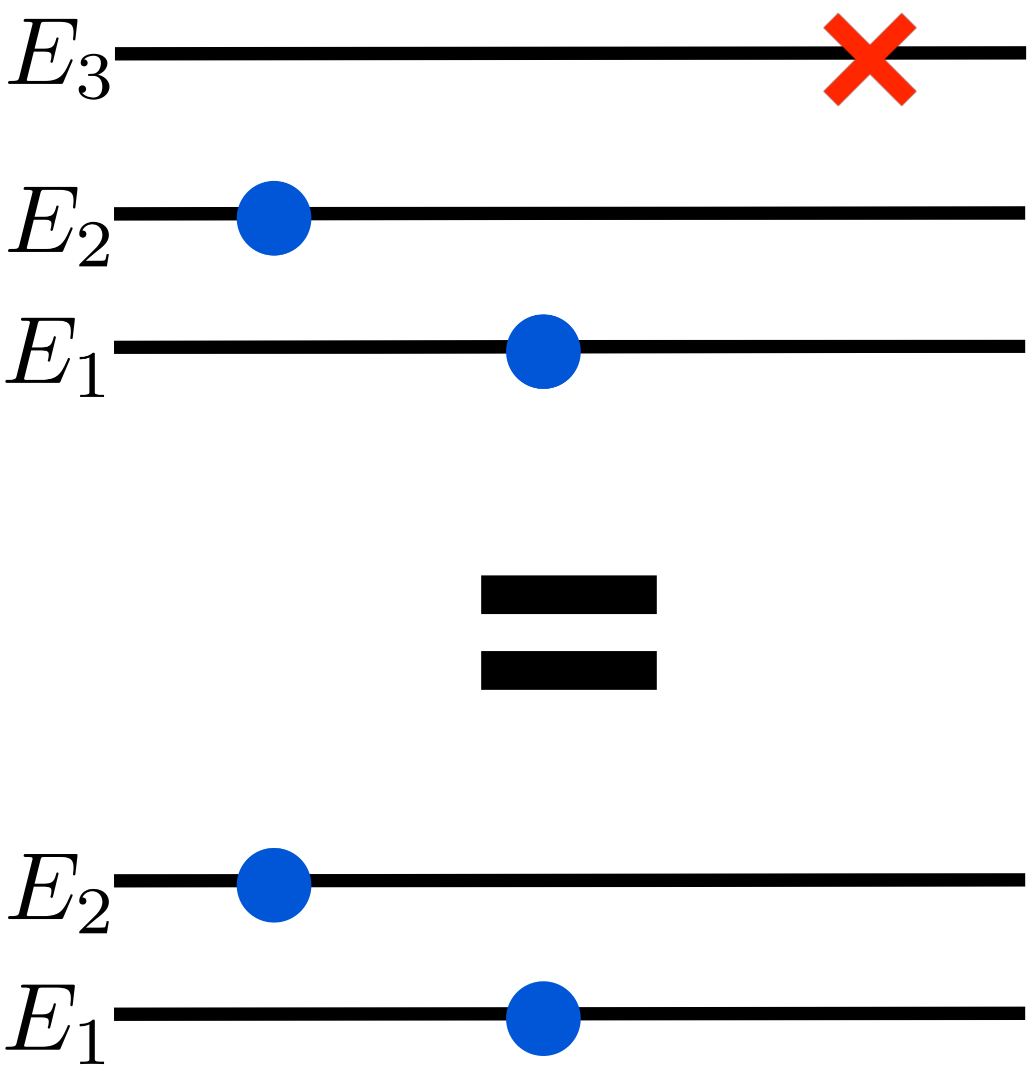

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Otherwise, physics would be affected by impossible potentialities.

3. The Subspace Axiom

Axiom III: Let Ω_N and Ω_{N-1} be systems with capacities *N* and *N-1*. If $(E_1, \ldots, E_N)$ is a complete measurement on Ω_N, then the set of states ω with $E_N(\omega) = 0$ is equivalent to Ω_{N-1}.

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Capacity N of *Ω* = maximal # of perfectly distinguishable states.

$(\omega_1, \ldots, \omega_n)$ perfectly distinguishable, if there is a measurement $(E_1, \ldots, E_n)$ such that $E_i(\omega_j) = \delta_{ij}$.

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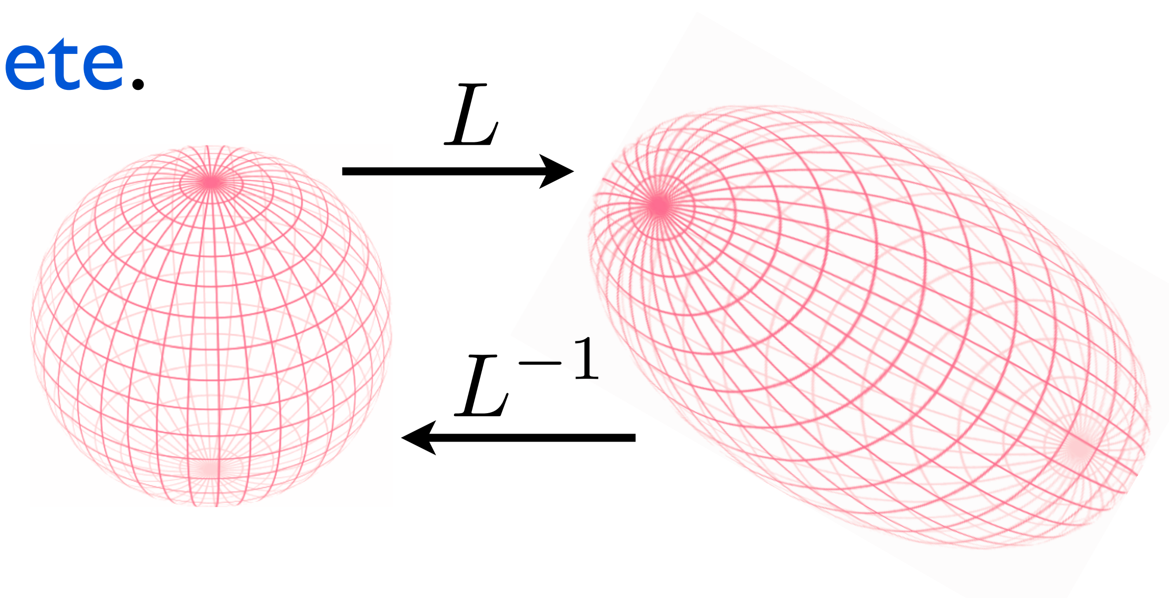
Equivalent = same state spaces up to a linear map (physically the same!)



4. Derivation of the Hilbert space formalism

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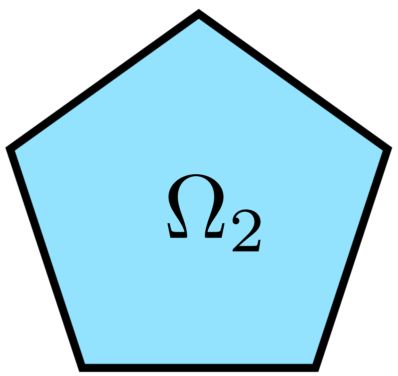

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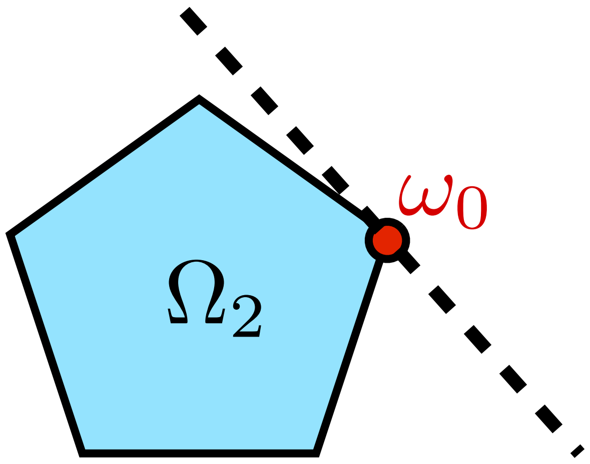

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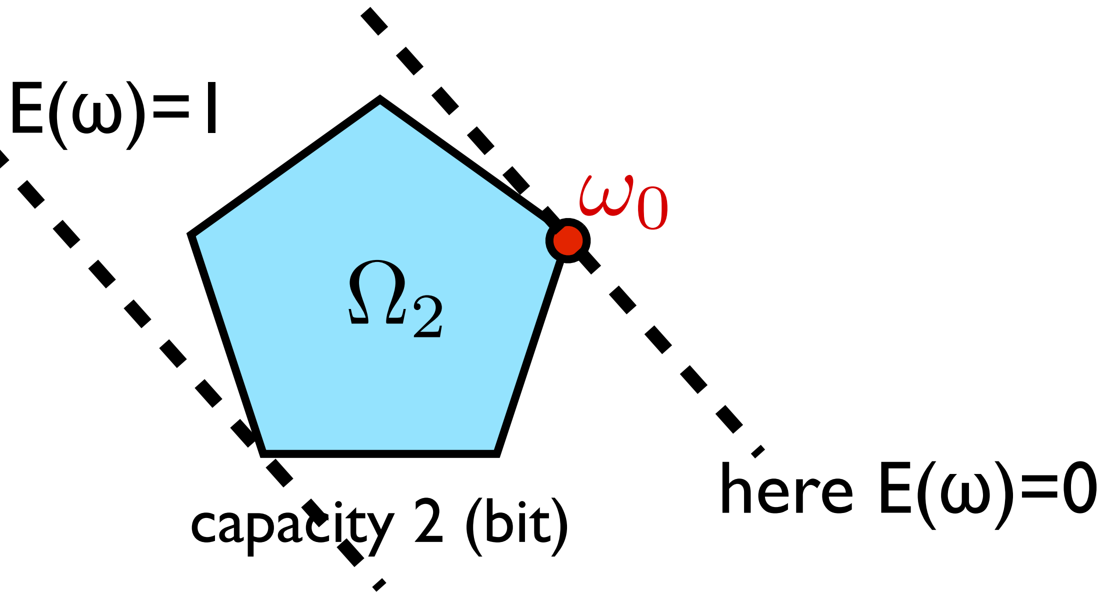

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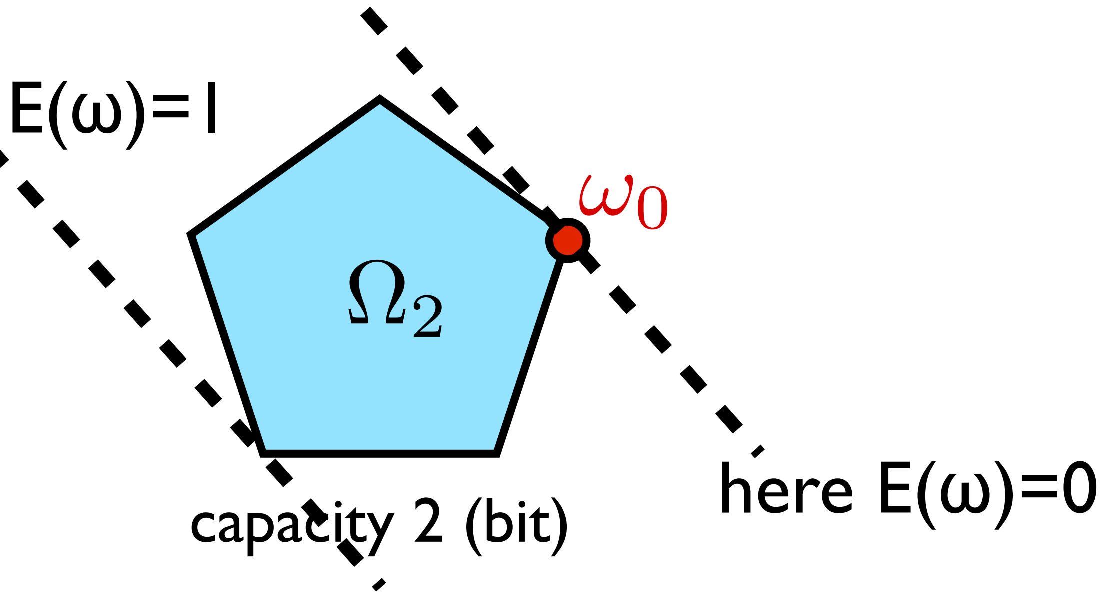

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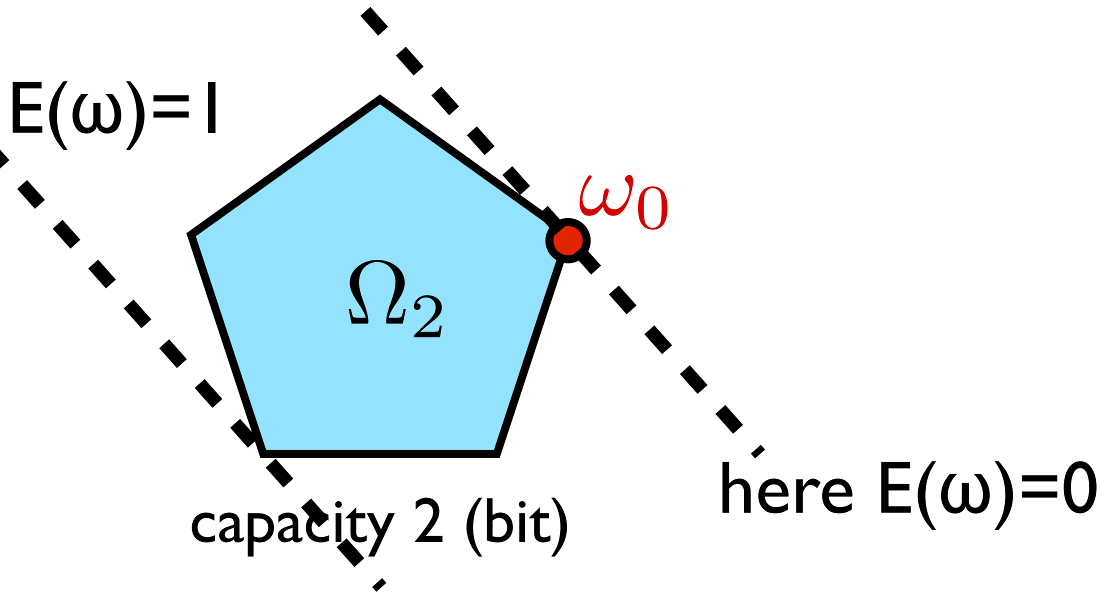

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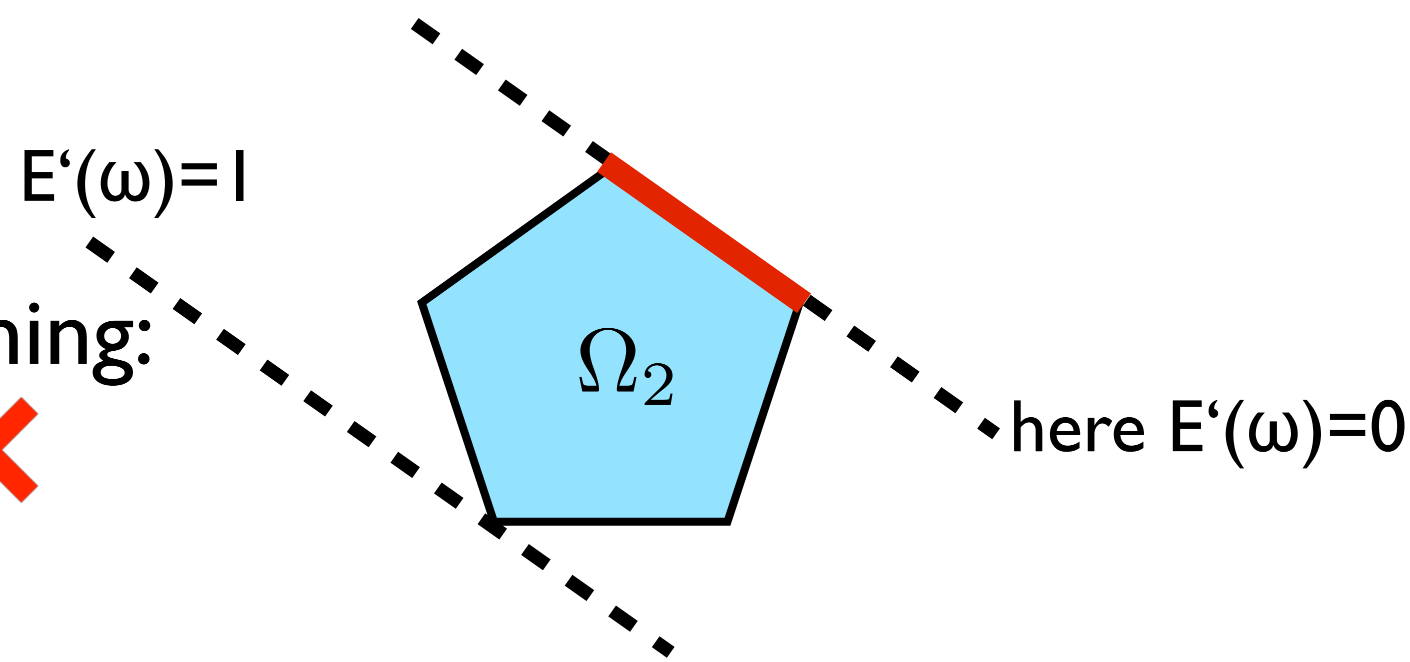

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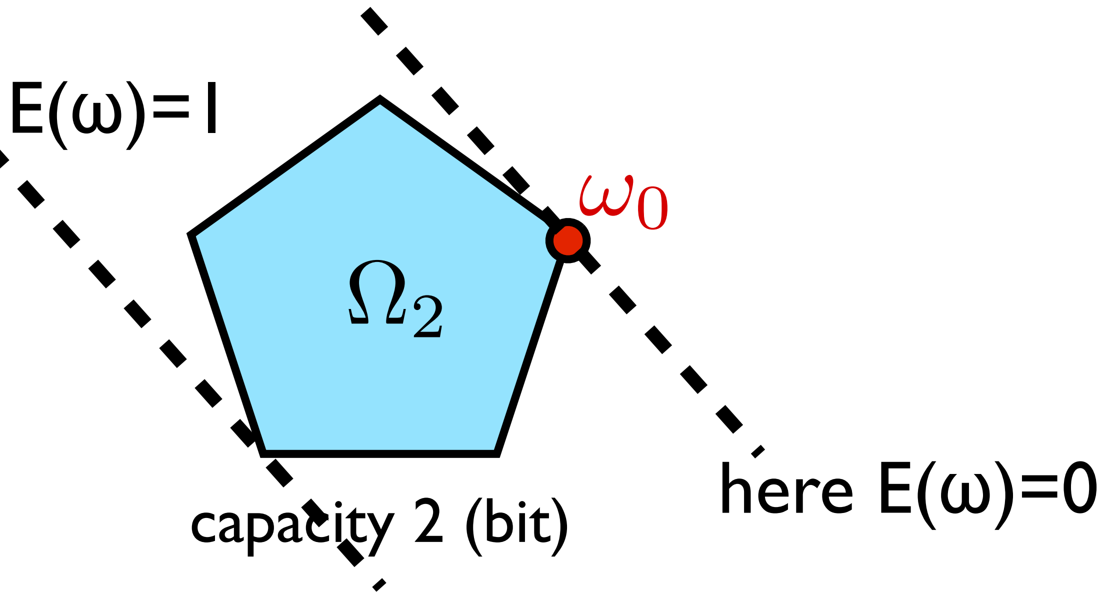

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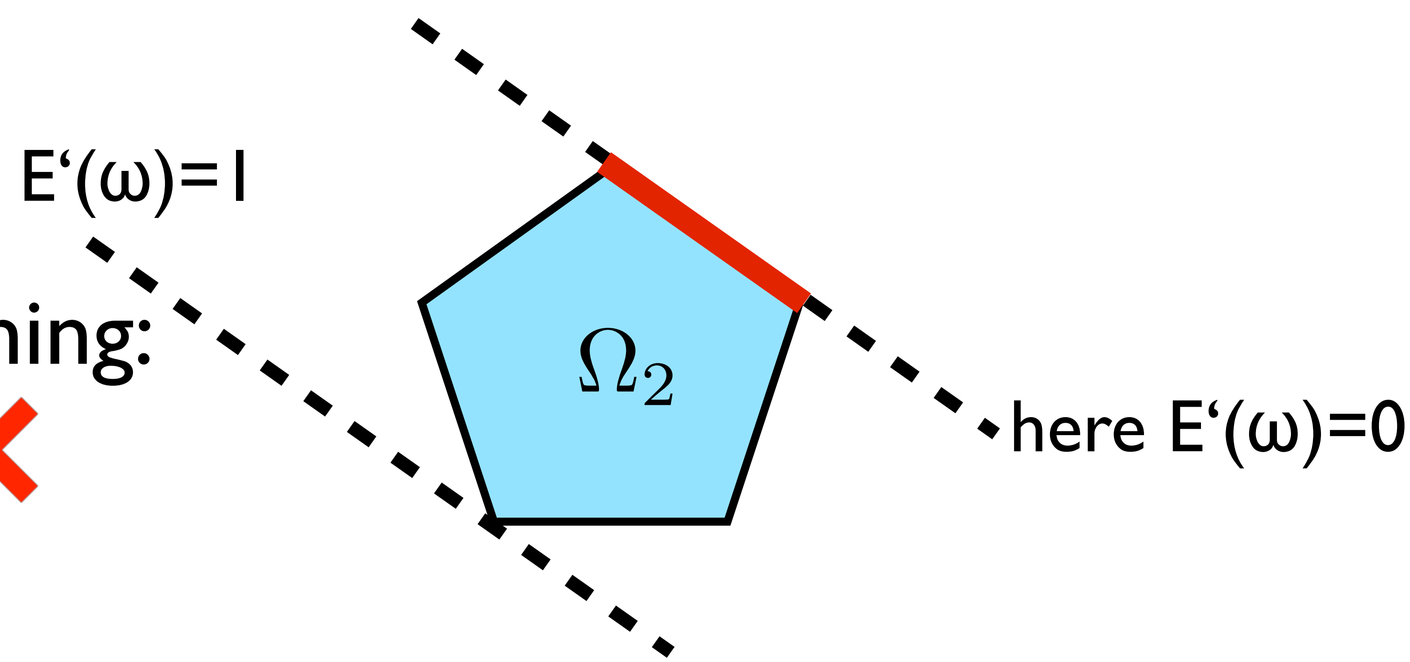

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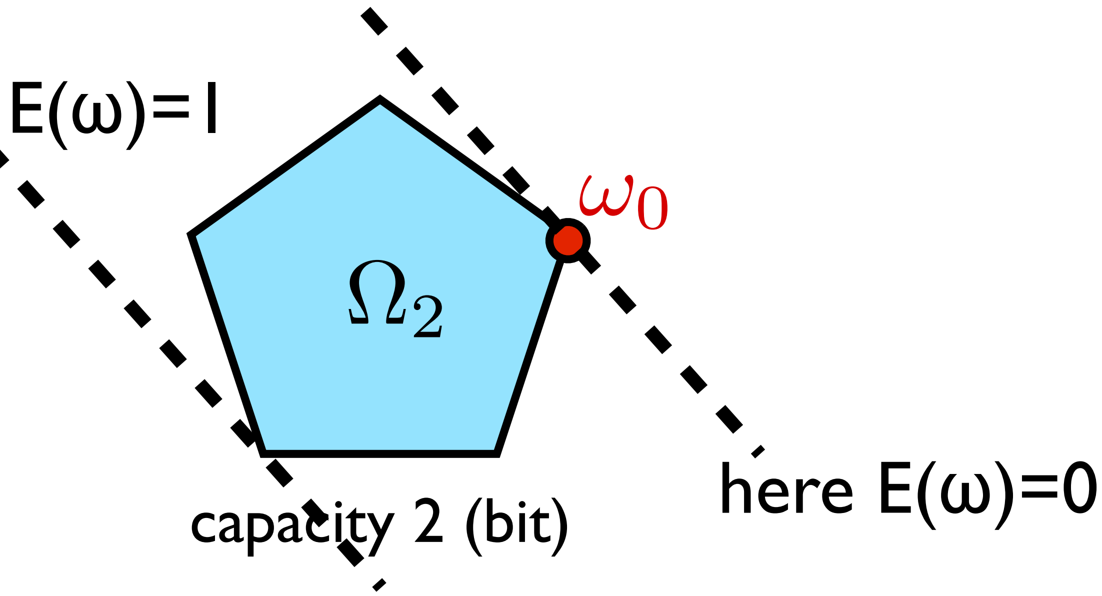

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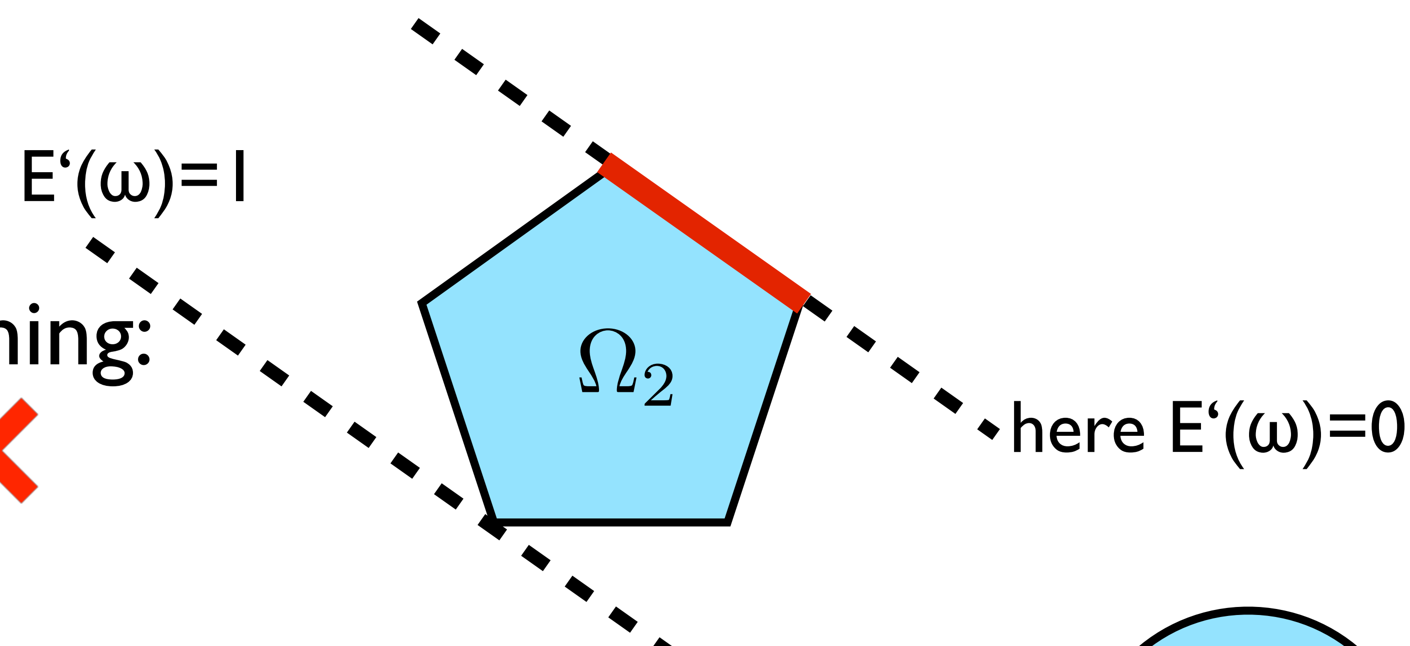

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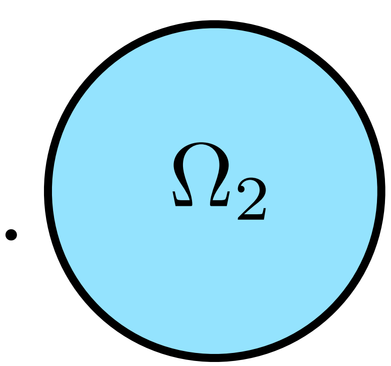
Reversibility axiom $\Rightarrow$ Ω_2 is a ball.



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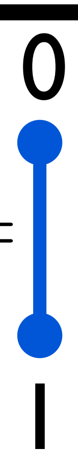
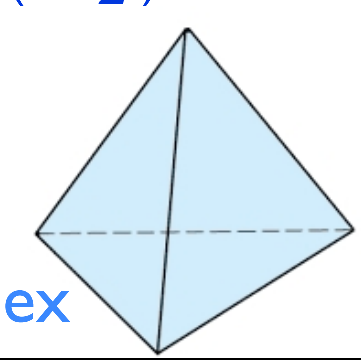


If $\dim(\Omega_2) = 1$ then the theory is CPT (easy):

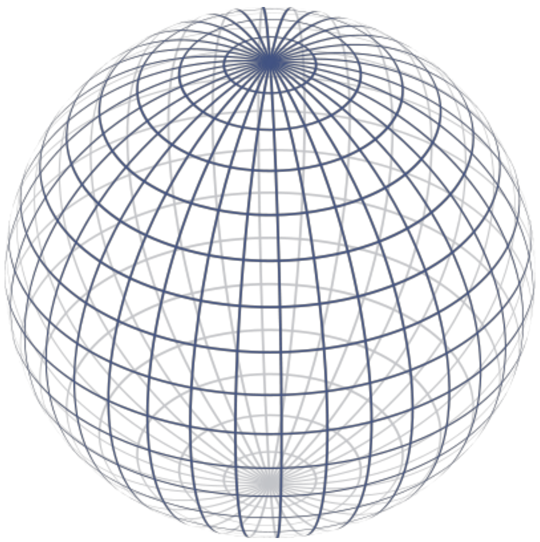
$\Omega_N =$



N-simplex

$\mathcal{G}_N$=permutation group.

4. Derivation of the Hilbert space formalism



Generalized bit Ω_2

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CPT

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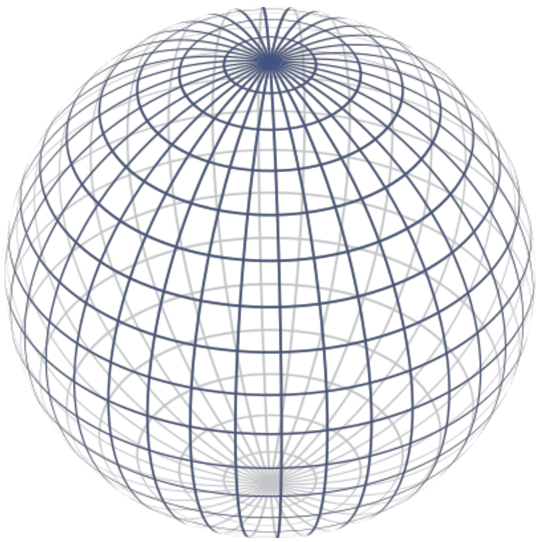

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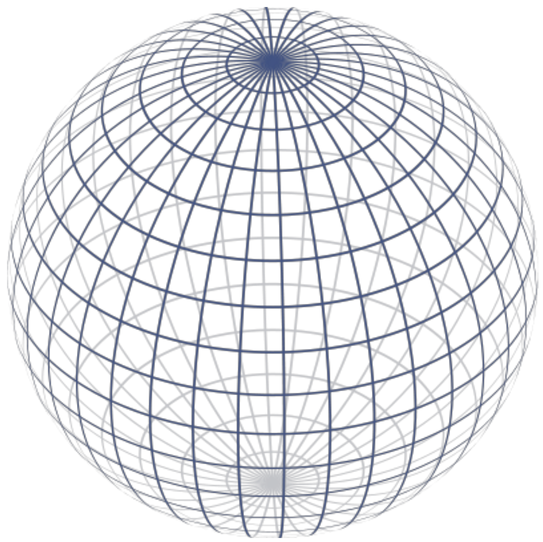

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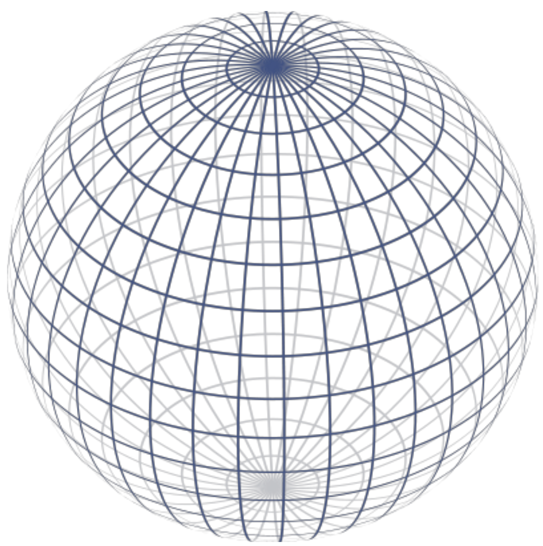

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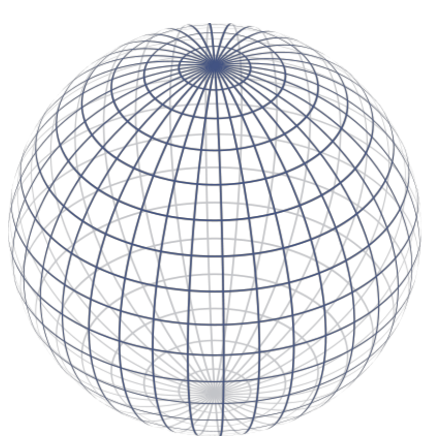
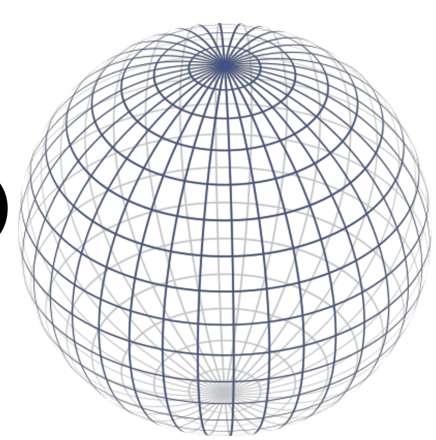
Two bits:  $\otimes$ 

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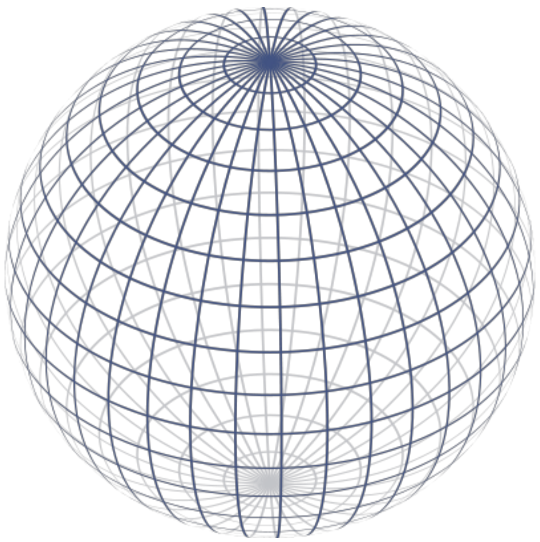

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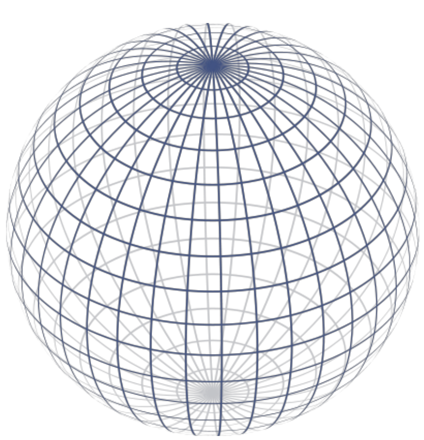
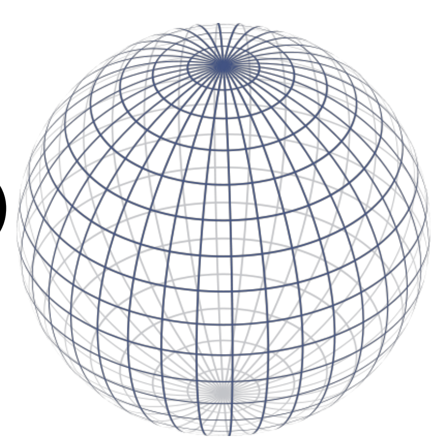
Two bits:  $\otimes$ 

Local transformations contain $\mathcal{G}_2 \otimes \mathcal{G}_2$.

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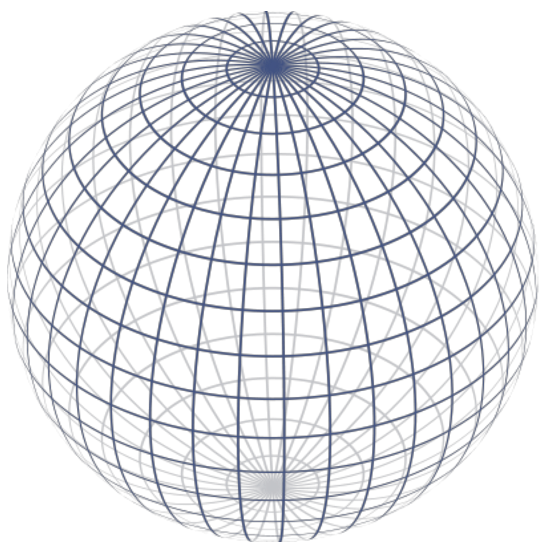

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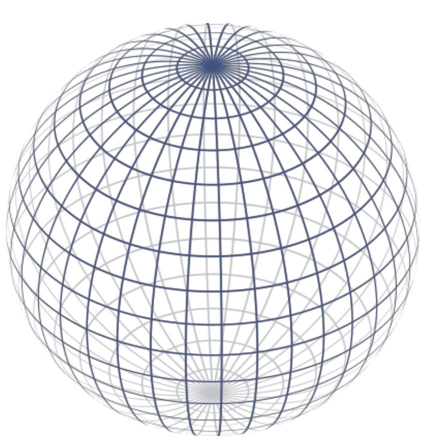
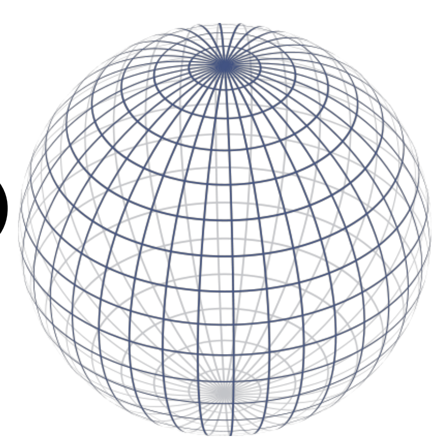
Two bits:  $\otimes$ 

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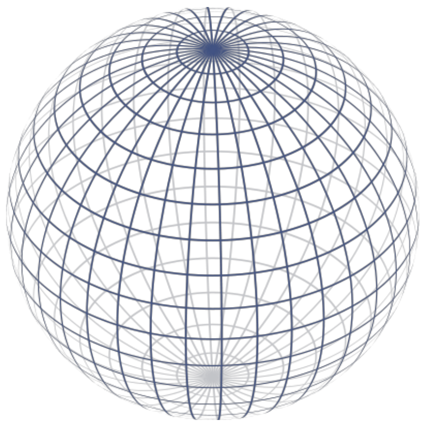
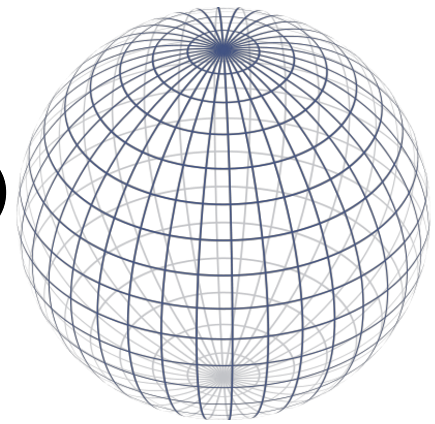
Two bits:  $\otimes$ 

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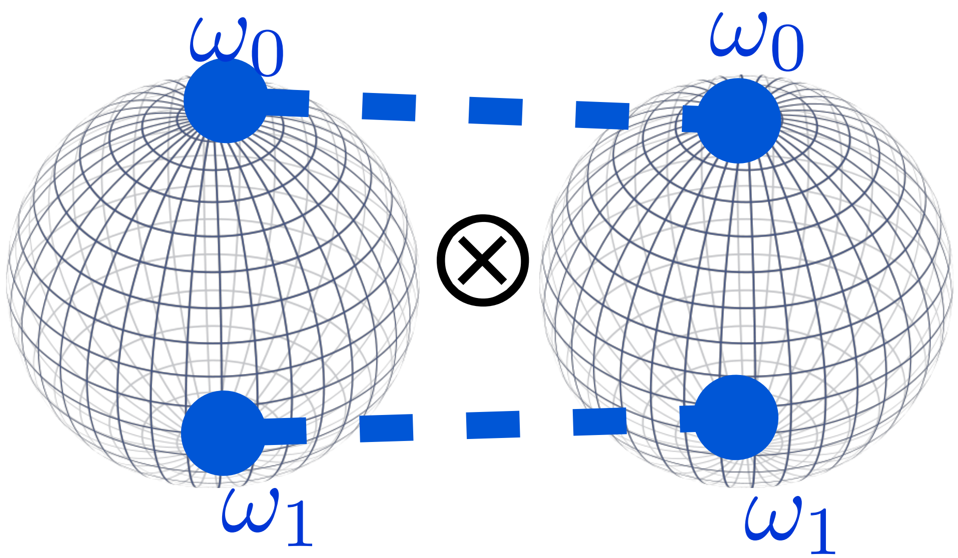

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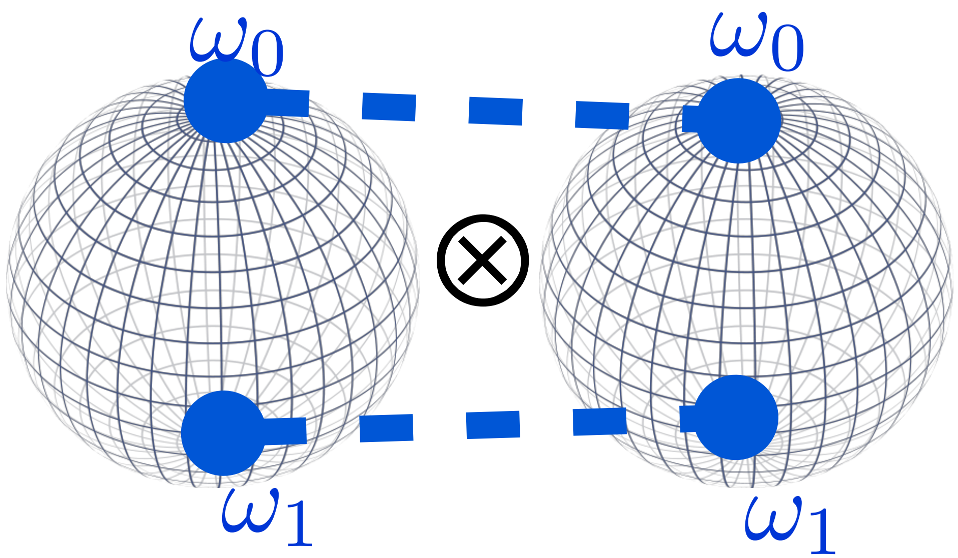

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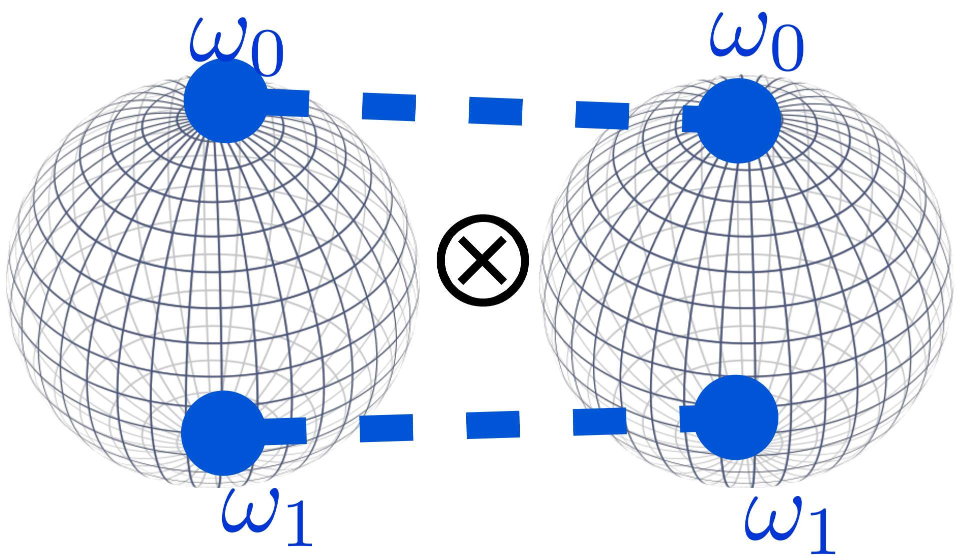

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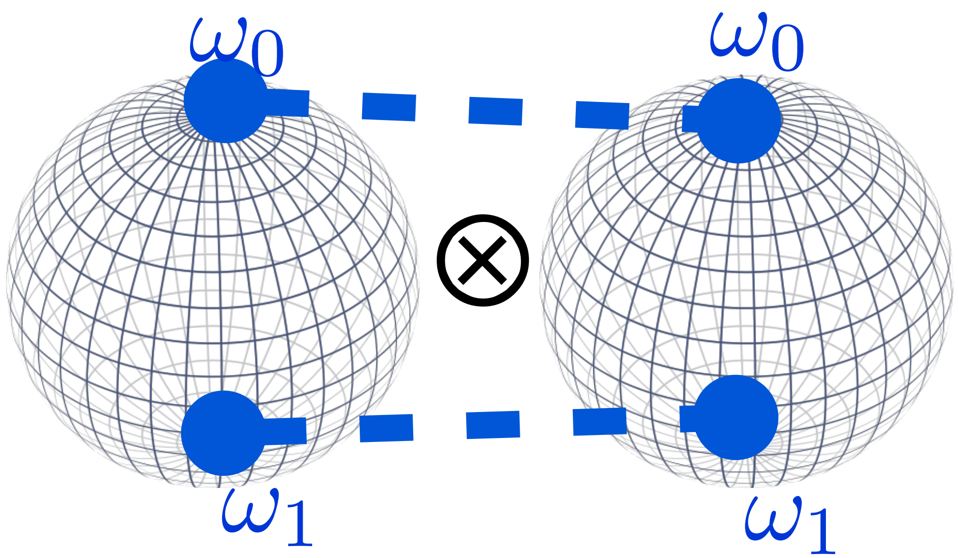

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Map 3-vectors to Hermitian matrices: $L(\omega) := \frac{1}{2}\left(\mathbf{1} + \sum_{i=1}^{3} \omega_i \sigma_i\right)$

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prove:

Theorem: Every theory satisfying Axioms I-V (rather than CPT) is equivalent to $(\Omega_N, \mathcal{G}_N)$, where

- Ω_N are the density matrices on $\mathbb{C}^N$,
- $\mathcal{G}_N$ is the group of unitaries, acting by conjugation,
- the measurements are exactly the POVMs.